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ON SOME ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS

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1. INTRODUCTION

In the papers [3], [4] and [5] are studied nonlinear equations having the from:

$$f(x) + g(x) = 0, \tag{1}$$

where, f, $g : X \to X$, X is a Banach space, f is a differentiable operator and g is continuous but nondifferentiable. For this reason the Newton's method, i.e. the approximation of the solution x^* of the equation (1) by the sequence $(x_n)_{n\geq 0}$ given by

$$x_{n+1} = x_n - \big(f'(x_n)+g'(x_n)\big)^{-1}\big(f(x_n)+g(x_n)\big), \qquad n = 1, 2, \ldots, \ x_0 \in X, \tag{2}$$

cannot be applied.

In the mentioned papers the following Newton-like methods are then considered:

$$x_{n+1} = x_n - f'(x_n)^{-1}\big(f(x_n) + g(x_n)\big), \qquad n = 1, 2, \ldots, \ x_0 \in X, \tag{3}$$

or

$$x_{n+1} = x_n - A(x_n)^{-1}\big(f(x_n) + g(x_n)\big), \qquad n = 1, 2, \ldots, \ x_0 \in X, \tag{3'}$$

where A is a linear operator approximating f'. It is shown that, under certain conditions, these sequences are converging to the solution of (1).

In the present paper, for solving equation (1), we propose the following method:

$$x_{n+1} = x_n - \big(f'(x_n) + [x_{n-1}, x_n; g]\big)^{-1}\big(f(x_n) + g(x_n)\big), \quad n = 1, 2, \ldots, \quad x_0, x_1 \in X \tag{4}$$

where by $[x, y; g]$ we have denoted the first order divided difference of g at the points $x, y \in X$.

So, the proposed method is obtained by combining the Newton's method with the method of chord. The r-convergence order of this method, denoted by p, is the same as for the method of chord (where $p = \frac{1+\sqrt{5}}{2} \approx 1.618$), which is greater than the r-order of the methods (3) and (3′) (see also the numerical example), but is less than the r-order of Newton's method (where usually $p = 2$).

But, unlike the method of chord, the proposed method has a better rate of convergence, because the use of $f'(x_n)$ instead of $[x_{n-1},\ x_n; f]$, as it is in the method of chord, does not affect the coefficient c_2 from the inequalities of the type:

$$\|x_{n+1} - x_n\| \leq c_1 \|x_n - x_{n-1}\|^2 + c_2 \|x_n - x_{n-1}\| \|x_{n-1} - x_{n-2}\|,$$

which we shall obtain in the following.

2. THE CONVERGENCE OF THE METHOD

We shall use, as in [1] and [2] the known definitions for the divided differences of an operator.

DEFINITION 1. *An operator belonging to the space $\mathcal{L}(X,\ X)$ (the Banach space of the linear and bounded operators from X to X) is called the first order divided difference of the operator $g : X \to X$ at the points $x_0,\ y_0 \in X$ if the following properties hold:*

a) $[x_0, y_0; g](y_0 - x_0) = g(y_0) - g(x_0)$, *for* $x_0 \neq y_0$;

b) *if g is Fréchet differentiable at $x_0 \in X$, then*

$$[x_0, x_0; g] = g'(x_0).$$

DEFINITION 2. *An operator belonging to the space $\mathcal{L}(X, \mathcal{L}(X, X))$, denoted by $[x_0, y_0, z_0; g]$ is called the second order divided difference of the operator $g : X \to X$ at the points $x_0, y_0, z_0 \in X$ if the following properties hold:*

a′) $[x_0, y_0, z_0; g]\ (z_0 - x_0) = [y_0, z_0; g] - [x_0, y_0; g]$ *for the distinct points* $x_0, y_0, z_0 \in X$;

b′) *if g is two times differentiable at $x_0 \in X$, then*

$$[x_0, x_0, x_0; g] = \tfrac{1}{2} g''(x_0).$$

We shall denote by $B_r(x_1) = \{x \in X|\ \ \|x - x_1\| < r\}$ the ball having the center at $x_1 \in X$ and the radius $r > 0$.

Concerning the convergence of the iterative process (4) we shall prove the following result.

THEOREM 3. *If there exist the elements $x_0, x_1 \in X$ and the positive real numbers r, l, M, K and ε such that the conditions*

i) *the operator f is Fréchet differentiable on $B_r(x_1)$ and f' satisfies*

$$\left\|f'(x) - f'(y)\right\| \leq l \|x - y\|, \quad \forall x, y \in B_r(x_1);$$

ii) *the operator g is continuous on $B_r(x_1)$,*

iii) *for any distinct points $x, y \in B_r(x_1)$ there exists the application $\left(f'(y) + [x, y;\ g]\right)^{-1}$ and the inequality*

$$\left\|\left(f'(y) + [x, y;\ g]\right)^{-1}\right\| \leq M$$

is true;

iv) *for any distinct points* $x,\ y,\ z \in B_r(x_1)$ *we have the inequality*

$$\left\|[x, y, z;\ g]\right\| \leq K;$$

v) *the elements* $x_0,\ x_1$ *satisfy*

$$\|x_1 - x_0\| \leq M\varepsilon, \quad \textit{where} \ \ \varepsilon = \|f(x_1) + g(x_1)\|\,;$$

vi) *the following relations hold:*

$$\begin{aligned} \|x_2 - x_1\| \leq \|x_1 - x_0\|\,, \quad & \textit{with} \ \ x_2 \textit{ given by } (4) \textit{ for } n = 1, \\ q =& M^2\varepsilon\left(\tfrac{l}{2} + 2K\right) < 1, \ \textit{and} \\ r =& \tfrac{M\varepsilon}{q}\sum_{k=1}^{\infty} q^{u_k}, \end{aligned}$$

where $(u_k)_{k\geq 0}$ *is the Fibonacci's sequence* $u_{k+1} = u_k + u_{k-1},\ k \geq 1,$ $u_0 = u_1 = 1;$

are fulfilled, then the sequence $(x_n)_{n\geq 0}$ *generated by* (4) *is well defined, all its terms belonging to* $B_r(x_1)$.

Moreover, the following properties are true:

j) *the sequence* $(x_n)_{n\geq 0}$ *is convergent;*

jj) *let* $x^* = \lim_{n\to\infty} x_n$. *Then* x^* *is a solution of the equation* (1);

jjj) *we have the a priori error estimates:*

$$\|x^* - x_n\| \leq \frac{M\varepsilon}{q\left(1-q^{\frac{p^n(p-1)}{\sqrt{5}}}\right)}\left(q^{\frac{1}{\sqrt{5}}}\right)^{p^n}, \qquad n \geq 1, p = \tfrac{1+\sqrt{5}}{2}.$$

Proof. We shall prove first by induction that, for any $n \geq 2$,

$$x_n \in B_r(x_1), \tag{5}$$

$$\|x_n - x_{n-1}\| \leq \|x_{n-1} - x_{n-2}\|\,, \text{ and} \tag{6}$$

$$\|x_n - x_{n-1}\| \leq q^{u_{n-1}-1}M\varepsilon. \tag{7}$$

For $n = 2$, from $v)$ and $vi)$ we infer the above relations.

Let us suppose now that relations (5), (6) and (7) hold for $n = 2, 3, \ldots, k$, where $k \geq 2$. Since $x_k, x_{k-1} \in B_r(x_1)$, we can construct x_{k+1} from (4), whence, using $iii)$, we have

$$\|x_{k+1} - x_k\| = \left\|\left(f'(x_k)+[x_{k-1}, x_k;\ g]\right)^{-1}\left(f(x_k)+g(x_k)\right)\right\| \leq M\,\|f(x_k) + g(x_k)\|\,.$$

For the estimation of $\|f(x_k) + g(x_k)\|$ we shall rely on the equality

$$\begin{aligned} &g(x_k) - g(x_{k-1}) - [x_{k-2}, x_{k-1};\ g](x_k - x_{k-1}) = \\ &= [x_{k-2}, x_{k-1}, x_k;\ g](x_k - x_{k-1})(x_k - x_{k-2}) \end{aligned}$$

(easily obtained from Definition 1 and Definition 2), which imply, using $iv)$,

$$\begin{aligned} &\left\|g(x_k) - g(x_{k-1}) - [x_{k-2}, x_{k-1};\ g](x_k - x_{k-1})\right\| \leq \\ &\leq K\,\|x_k - x_{k-1}\|\left(\,\|x_k - x_{k-1}\| + \|x_{k-1} - x_{k-2}\|\,\right) \end{aligned} \tag{8}$$

and on the inequality

$$\left\| f(x_k) - f(x_{k-1}) - f'(x_{k-1})(x_k - x_{k-1}) \right\| \le \tfrac{l}{2} \left\| x_k - x_{k-1} \right\|^2, \tag{9}$$

valid because of the assumptions $i)$ concerning f.

For $n = k-1$, by (4), we get

$$-\big(f'(x_{k-1}) + [x_{k-2}, x_{k-1}; g]\big)(x_k - x_{k-1}) - f(x_{k-1}) - g(x_{k-1}) = 0,$$

whence

$$\begin{aligned} f(x_k) + g(x_k) =& f(x_k) - f(x_{k-1}) - f'(x_{k-1})(x_k - x_{k-1}) + g(x_k) - g(x_{k-1}) - \\ & - [x_{k-2}, x_{k-1}; g](x_k - x_{k-1}). \end{aligned}$$

The above relation, together with (8), (9) and (6) for $n = k$ imply

$$\begin{aligned} &\|x_{k+1} - x_k\| \le \\ &\le M \|f(x_k) + g(x_k)\| \\ &\le \tfrac{Ml}{2} \|x_k - x_{k-1}\|^2 + MK \|x_k - x_{k-1}\| \big(\|x_k - x_{k-1}\| + \|x_{k-1} - x_{k-2}\| \big) \\ &\le M \|x_k - x_{k-1}\| \left(\tfrac{l}{2} \|x_{k-1} - x_{k-2}\| + 2K \|x_{k-1} - x_{k-2}\| \right) \\ &= M \left(\tfrac{l}{2} + 2K \right) \|x_k - x_{k-1}\| \, \|x_{k-1} - x_{k-2}\| . \end{aligned}$$

From the hypothesis of the induction we have on one hand that

$$\begin{aligned} \|x_{k+1} - x_k\| \le& M \left(\tfrac{l}{2} + 2K \right) q^{u_{k-2}-1} M\varepsilon \|x_k - x_{k-1}\| \\ =& q^{u_{k-2}} \|x_k - x_{k-1}\| \\ <& \|x_k - x_{k-1}\| , \end{aligned}$$

that is, (6) for $n = k+1$, and, on the other hand

$$\|x_{k+1} - x_k\| \le q^{u_{k-2}} \|x_k - x_{k-1}\| \le q^{u_{k-2}} q^{u_{k-1}} M\varepsilon = q^{u_k} M\varepsilon,$$

that is, (7) for $n = k+1$.

The fact that $x_{k+1} \in B_r(x_1)$ results from:

$$\begin{aligned} \|x_{k+1} - x_1\| &\le \|x_2 - x_1\| + \|x_3 - x_2\| + \cdots + \|x_{k+1} - x_k\| \\ &\le \tfrac{M\varepsilon}{q} (q^{u_1} + q^{u_2} + \cdots + q^{u_k}) < r. \end{aligned}$$

Now we shall prove that $(x_n)_{n \ge 0}$ is a Cauchy sequence, whence $j)$ follows.

It is obvious that

$$u_k = \tfrac{1}{\sqrt{5}} \left(\left(\tfrac{1+\sqrt{5}}{2} \right)^{k+1} - \left(\tfrac{1-\sqrt{5}}{2} \right)^{k+1} \right) \ge \tfrac{1}{\sqrt{5}} \left(\tfrac{1+\sqrt{5}}{2} \right)^{k} = \tfrac{p^k}{\sqrt{5}},$$

for $k \ge 1$.

So, for any $k \geq 1$, $m \geq 1$ we have

$$\begin{aligned}\|x_{k+m} - x_k\| &\leq \|x_{k+1} - x_k\| + \|x_{k+2} - x_{k+1}\| + \cdots + \|x_{k+m} - x_{k+m-1}\| \\ &\leq \tfrac{M\varepsilon}{q}(q^{u_k} + q^{u_{k+1}} + \cdots + q^{u_{k+m-1}}) \\ &\leq \tfrac{M\varepsilon}{q}\Big(q^{\frac{p^k}{\sqrt{5}}} + q^{\frac{p^{k+1}}{\sqrt{5}}} + \cdots + q^{\frac{p^{k+m-1}}{\sqrt{5}}}\Big).\end{aligned}$$

Using Bernoulli's inequality, it follows

$$\begin{aligned}\|x_{k+m} - x_k\| &\leq \tfrac{M\varepsilon}{q} q^{\frac{p^k}{\sqrt{5}}}\Big(1 + q^{\frac{p^{k+1}-p^k}{\sqrt{5}}} + q^{\frac{p^{k+2}-p^k}{\sqrt{5}}} + \cdots + q^{\frac{p^{k+m-1}-p^k}{\sqrt{5}}}\Big) \\ &= \tfrac{M\varepsilon}{q} q^{\frac{p^k}{\sqrt{5}}}\Big(1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + q^{\frac{p^k(p^2-1)}{\sqrt{5}}} + \cdots + q^{\frac{p^k(p^{m-1}-1)}{\sqrt{5}}}\Big) \\ &\leq \tfrac{M\varepsilon}{q} q^{\frac{p^k}{\sqrt{5}}}\Big(1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + q^{\frac{p^k(1+2(p-1)-1)}{\sqrt{5}}} + \cdots + q^{\frac{p^k(1+(m-1)(p-1)-1)}{\sqrt{5}}}\Big) \\ &= \tfrac{M\varepsilon}{q} q^{\frac{p^k}{\sqrt{5}}}\Big[1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + \Big(q^{\frac{p^k(p-1)}{\sqrt{5}}}\Big)^2 + \cdots + \Big(q^{\frac{p^k(p-1)}{\sqrt{5}}}\Big)^{m-1}\Big] \\ &= \tfrac{M\varepsilon}{q} q^{\frac{p^k}{\sqrt{5}}} \frac{1 - q^{\frac{p^k(p-1)}{\sqrt{5}}m}}{1 - q^{\frac{p^k(p-1)}{\sqrt{5}}}}.\end{aligned}$$

Hence

$$\|x_{k+m} - x_k\| \leq \frac{M\varepsilon q^{\frac{p^k}{\sqrt{5}}}\Big(1 - q^{\frac{p^k(p-1)}{\sqrt{5}}m}\Big)}{q\big(1 - q^{\frac{p^k(p-1)}{\sqrt{5}}}\big)}, \quad k \geq 1,$$

and $(x_n)_{n\geq 0}$ is a Cauchy sequence.

It follows that $(x_n)_{n\geq 0}$ is convergent, and let $x^* = \lim_{n\to\infty} x_n$. For $n \to \infty$ in (4) we get that x^* is a solution of (1). For $m \to \infty$ in the above n equality we obtain the very relation *jjj*).

The theorem is proved. □

3. NUMERICAL EXAMPLE

Given the system

$$\begin{cases} 3x^2y + y^2 - 1 + |x - 1| = 0 \\ x^4 + xy^3 - 1 + |y| = 0, \end{cases}$$

we shall consider $X + (\mathbb{R}^2, \|\cdot\|_\infty)$, $\|x\|_\infty = \|(x', x'')\|_\infty = \max\{|x'|,\ |x''|\}$, $f = (f_1, f_2)$, $g = (g_1, g_2)$. For $x = (x', x'') \in \mathbb{R}^2$ we take $f_1(x', x'') = 3(x')^2x'' + (x'')^2 -1$, $f_2(x', x'') = (x')^4 + x'(x'')^3 - 1$, $g_1(x', x'') = |x' - 1|$, $g_2(x', x'') = |x''|$.

We shall take $[x, y;\, g] \in M_{2\times 2}(\mathbb{R})$ as

$$\begin{aligned}[x, y;\, g]_{i,1} &= \tfrac{g_i(y', y'') - g_i(x', y'')}{y' - x'}, \\ [x, y;\, g]_{i,2} &= \tfrac{g_i(x', y'') - g_i(x', x'')}{y'' - x''}, \quad i = 1, 2.\end{aligned}$$

Using method (3) with $x_0 = (1, 0)$ we obtain

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	1	0	
1	1	0.333333333333333	$3.333 \cdot 10^{-1}$
2	0.906550218340611	0.354002911208151	$9.344 \cdot 10^{-2}$
3	0.885328400663412	0.338027276361332	$2.122 \cdot 10^{-2}$
4	0.891329556832800	0.326613976593566	$1.141 \cdot 10^{-2}$
5	0.895238815463844	0.326406852843625	$3.909 \cdot 10^{-3}$
6	0.895154671372635	0.327730334045043	$1.323 \cdot 10^{-3}$
7	0.894673743471137	0.327979154372032	$4.809 \cdot 10^{-4}$
8	0.894598908977448	0.327865059348755	$1.140 \cdot 10^{-4}$
9	0.894643228355865	0.327815039208286	$5.002 \cdot 10^{-5}$
10	0.894659993615645	0.327819889264891	$1.676 \cdot 10^{-5}$
11	0.894657640195329	0.327826728208560	$6.838 \cdot 10^{-6}$
12	0.894655219565091	0.327827351826856	$2.420 \cdot 10^{-6}$
13	0.894655074977661	0.327826643198819	$7.086 \cdot 10^{-7}$
...			
39	0.894655373334687	0.327826521746298	$5.149 \cdot 10^{-19}$

Using the method of chord with $x_0 = (5, 5)$, $x_1 = (1, 0)$, we obtain

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	5	5	
1	1	0	$5.000 \cdot 10^{+00}$
2	0.989800874210782	0.012627489072365	$1.262 \cdot 10^{-02}$
3	0.921814765493287	0.307939916152262	$2.953 \cdot 10^{-01}$
4	0.900073765669214	0.325927010697792	$2.174 \cdot 10^{-02}$
5	0.894939851624105	0.327725437396226	$5.133 \cdot 10^{-03}$
6	0.894658420586013	0.327825363500783	$2.814 \cdot 10^{-04}$
7	0.894655375077418	0.327826521051833	$3.045 \cdot 10^{-06}$
8	0.894655373334698	0.327826521746293	$1.742 \cdot 10^{-09}$
9	0.894655373334687	0.327826521746298	$1.076 \cdot 10^{-14}$
10	0.894655373334687	0.327826521746298	$5.421 \cdot 10^{-20}$

Using method (4) with $x_0 = (5, 5)$, $x_1 = (1, 0)$, we obtain

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	5	5	
1	1	0	$5.000 \cdot 10^{+00}$
2	0.909090909090909	0.363636363636364	$3.636 \cdot 10^{-01}$
3	0.894886945874111	0.329098638203090	$3.453 \cdot 10^{-02}$
4	0.894655531991499	0.327827544745569	$1.271 \cdot 10^{-03}$
5	0.894655373334793	0.327826521746906	$1.022 \cdot 10^{-06}$
6	0.894655373334687	0.327826521746298	$6.089 \cdot 10^{-13}$
7	0.894655373334687	0.327826521746298	$2.710 \cdot 10^{-20}$

It can be easily seen that, given these data, method (4) is converging faster than (3) and than the method of chord.

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