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AFFINE INVARIANT CONDITIONS FOR THE INEXACT PERTURBED NEWTON METHOD*

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Abstract. The high convergence orders of the inexact Newton iterates were characterized by Ypma in terms of some affine invariant conditions. Using these results, we obtain affine invariant characterizations for the convergence orders of the inexact perturbed Newton iterates.

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1. INTRODUCTION

Given a nonlinear mapping $F:\mathbb{R}^n\to\mathbb{R}^n$, the inexact Newton (IN) method for solving the system $F(x)=0$ is given by the iterations:

$$\begin{aligned} &F'(x_k)\,s_k = -F(x_k)+r_k\\ &x_{k+1}=x_k+s_k,\quad k=0,1,\dots,\quad x_0\in\mathbb{R}^n, \end{aligned}$$

where r_k represent the residuals of the approximate solutions s_k of the linear systems. The local convergence of these iterates to a solution $x^*\in\mathbb{R}^n$ is usually studied under the following assumptions, which we shall implicitly assume throughout this paper:

- the mapping F is Fréchet differentiable on a neighborhood of x^*, with F' continuous at x^*;
- the Jacobian $F'(x^*)$ is invertible.

We recall that, given an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$, a sequence $(x_k)_{k\geq 0}\subset\mathbb{R}^n$ is said that converges (q-)superlinearly to its limit $\bar{x}\in\mathbb{R}^n$ if

$$\lim_{k\to\infty}\frac{\|x_{k+1}-\bar{x}\|}{\|x_k-\bar{x}\|}=0,\qquad \text{(assuming } x_k\neq\bar{x} \text{ for all } k\geq k_0),$$

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also denoted by $\|x_{k+1} - \bar{x}\| = o(\|x_k - \bar{x}\|)$, as $k \to \infty$. For rigorous definitions and results concerning the high convergence orders we refer the reader to [11, ch. 9].

The high convergence orders of the IN iterates were characterized by Dembo, Eisenstat and Steihaug.

Theorem 1. [8]. *Assume that the IN iterates converge to x^*. Then the convergence is superlinear if and only if*

$$\|r_k\| = o\big(\|F(x_k)\|\big), \quad \text{as } k \to \infty.$$

If, additionally, F' is Hölder continuous with exponent $p \in (0,1]$ at x^*, i.e., there exist $L, \varepsilon > 0$ such that

$$\big\|F'(x) - F'(x^*)\big\| \le L\,\|x - x^*\|^p, \quad \text{when } \|x - x^*\| < \varepsilon,$$

then the convergence is with order $1+p$ if and only if

$$\|r_k\| = \mathcal{O}\big(\|F(x_k)\|^{1+p}\big), \quad \text{as} \quad k \to \infty.$$

In the inexact perturbed Newton (IPN) method there is assumed that at each step there appear different errors: the Jacobians are perturbed, the functions evaluations are approximately performed, and the resulting linear systems are only approximately solved:

$$\begin{aligned} &\big(F'(x_k) + \Delta_k\big)s_k = (-F(x_k) + \delta_k) + \hat{r}_k \\ &x_{k+1} = x_k + s_k, \qquad k = 0,1,\ldots, \quad x_0 \in D. \end{aligned}$$

We have obtained the following characterizations for the convergence of these iterates.

Theorem 2. [5]. *Assume that the IPN iterates are uniquely defined (i.e. the perturbations $(\Delta_k)_{k\ge 0}$ are such that the matrices $F'(x_k)+\Delta_k$ are invertible for $k = 0,1,\ldots$) and converge to x^*. Then the convergence is superlinear if and only if*

$$\begin{aligned} &\big\|\Delta_k(F'(x_k) + \Delta_k)^{-1}F(x_k) + \big(I - \Delta_k(F'(x_k) + \Delta_k)^{-1}\big)(\delta_k + \hat{r}_k)\big\| = \\ &= o\big(\|F(x_k)\|\big), \quad \text{as } k \to \infty. \end{aligned}$$

If, additionally, F' is Hölder continuous with exponent $p \in (0,1]$ at x^*, then the convergence is with order $1+p$ if and only if

$$\begin{aligned} &\big\|\Delta_k(F'(x_k) + \Delta_k)^{-1}F(x_k) + \big(I - \Delta_k(F'(x_k) + \Delta_k)^{-1}\big)(\delta_k + \hat{r}_k)\big\| = \\ &= \mathcal{O}\big(\|F(x_k)\|\big), \quad \text{as } k \to \infty. \end{aligned}$$

Theorem 3. [7]. *Assume that the IPN iterates are well defined (i.e. the perturbed linear systems are compatible) and converge to x^*. Then the convergence is superlinear if and only if*

$$\|-\Delta_k s_k + \delta_k + \hat{r}_k\| = o\big(\|F(x_k)\|\big), \quad \text{as } k \to \infty.$$

If, additionally, F' is Hölder continuous with exponent $p \in (0,1]$ at x^*, then the convergence is with order $1+p$ if and only if

$$\|-\Delta_k s_k + \delta_k + \hat{r}_k\| = \mathcal{O}\big(\|F(x_k)\|^{1+p}\big), \quad \text{as } k \to \infty.$$

2. MAIN RESULTS

The affine invariant conditions are those conditions which do not change when considering, instead of the original system, the modified system $CF(x) = 0$, where $C \in \mathbb{R}^{n\times n}$ is nonsingular. The (exact) Newton iterates remain the same under such transformation, such that the conditions on the convergence of the iterates should also remain unchanged.

Ypma has obtained in [13] the following characterizations for the IN iterates, which are affine invariant.

Theorem 4. [13]. *Assume that the IN iterates converge to x^*. Then the convergence is superlinear if and only if*

$$\big\|F'(x_k)^{-1} r_k\big\| = o\big(\|F'(x_k)^{-1} F\,(x_k)\,\|\big), \quad \text{as } k \to \infty.$$

If, additionally, F' is Hölder continuous with exponent $p \in (0,1]$ at x^*, then the convergence is with order $1+p$ if and only if

$$\big\|F'(x_k)^{-1} r_k\big\| = \mathcal{O}\big(\|F'(x_k)^{-1} F\,(x_k)\,\|^{1+p}\big) \quad \text{as} \quad k \to \infty.$$

Using this result, we obtain the following affine invariant results for the inexact perturbed Newton method:

Theorem 5. *Assume that the IPN iterates are uniquely defined and converge to x^*. Then the convergence is superlinear if and only if*

$$\big\|F'(x_k)^{-1}\big(\Delta_k (F'(x_k)+\Delta_k)^{-1} F(x_k) + \big(I - \Delta_k (F'(x_k)+\Delta_k)^{-1}\big)(\delta_k + \hat{r}_k)\big)\big\| =$$
$$= o\big(\|F'(x_k)^{-1} F(x_k)\|\big), \quad \text{as } k \to \infty.$$

If, additionally, F' is Hölder continuous with exponent $p \in (0,1]$ at x^*, then the convergence is with order $1+p$ if and only if

$$\big\|F'(x_k)^{-1}\big(\Delta_k (F'(x_k)+\Delta_k)^{-1} F(x_k) + \big(I - \Delta_k (F'(x_k)+\Delta_k)^{-1}\big)(\delta_k + \hat{r}_k)\big)\big\| =$$
$$= \mathcal{O}\big(\|F'(x_k)^{-1} F(x_k)\|^{1+p}\big), \quad \text{as } k \to \infty.$$

Theorem 6. *Assume that the IPN iterates are well defined and converge to x^*. Then the convergence is superlinear if and only if*

$$\big\|F'(x_k)^{-1}(-\Delta_k s_k + \delta_k + \hat{r}_k)\big\| = o\big(\|F'(x_k)^{-1} F(x_k)\|\big), \quad \text{as } k \to \infty.$$

If, additionally, F' is Hölder continuous with exponent $p \in (0,1]$ at x^*, then the convergence is with order $1+p$ if and only if

$$\big\|F'(x_k)^{-1}(-\Delta_k s_k + \delta_k + \hat{r}_k)\big\| = \mathcal{O}\big(\|F'(x_k)^{-1} F(x_k)\|^{1+p}\big), \quad \text{as } k \to \infty.$$

The proofs are easily obtained by using the fact that the IPN iterates are IN iterates having the residuals

$$\begin{aligned} r_k &= \Delta_k(F'(x_k)+\Delta_k)^{-1}F(x_k) + \big(I - \Delta_k(F'(x_k)+\Delta_k)^{-1}\big)(\delta_k + \hat{r}_k) \\ &= -\Delta_k s_k + \delta_k + \hat{r}_k. \end{aligned}$$

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