ON THE SUPERLINEAR CONVERGENCE OF THE SUCCESSIVE APPROXIMATIONS METHOD *

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Abstract

The Ostrowski theorem is a classical result which ensures the attraction of all the successive approximations $x_{k+1} = G(x_k)$ near a fixed point x^*. Different conditions (ultimately on the magnitude of $G'(x^*)$) provide lower bounds for the convergence order of the process as a whole.

In this note we consider only one such sequence and we characterize its high convergence orders in terms of spectral elements of $G'(x^*)$; we obtain that the set of trajectories with high convergence orders is restricted to some affine subspaces, regardless of the nonlinearity of G.

We also analyze the stability of the successive approximations under perturbation assumptions.

Keywords. Successive approximations, convergence orders, inexact Newton iterates.

MSC Classification: 47H10, 65F10, 65H10.

1 Introduction

Consider a subset $D \subseteq \mathbb{R}^n$ and a mapping $G : D \to D$ which has a fixed point $x^* \in \operatorname{int}(D)$:

$$G(x^*) = x^*.$$

We are interested in the convergence to x^* of the successive approximations $(x_k)_{k\geq 0}$, given for some $x_0 \in D$ by

$$x_{k+1} = G(x_k), \quad k = 0, 1, \ldots. \tag{1}$$

First we briefly remind the definitions of the convergence orders. The symbol $\|\cdot\|$ will stand for a given norm in $\mathbb{R}^n$ and for its induced operator norm.

Definition 1 [1, ch. 9]. *Let $(x_k)_{k\geq 0} \subset \mathbb{R}^n$ be an arbitrary sequence converging to some $x^* \in \mathbb{R}^n$. The quotient and the root convergence factors are defined for*

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each $\alpha \in [1, +\infty)$ *as*

$$Q_\alpha\{x_k\} = \begin{cases} 0, & \text{if } x_k = x^*, \text{ for all but finitely many } k, \\ \limsup\limits_{k\to\infty} \dfrac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^\alpha}, & \text{if } x_k \neq x^*, \text{ for all but finitely many } k, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$R_\alpha\{x_k\} = \begin{cases} \limsup\limits_{k\to\infty} \|x_k - x^*\|^{1/\alpha^k}, & \text{when } \alpha > 1, \\ \limsup\limits_{k\to\infty} \|x_k - x^*\|^{1/k}, & \text{when } \alpha = 1. \end{cases}$$

The q*- and* r*-convergence orders are defined by*

$$O_Q\{x_k\} = \begin{cases} +\infty, & \text{if } \quad Q_\alpha\{x_k\} = 0, \quad \forall \alpha \in [1, +\infty), \\ \inf\big\{\alpha \in [1, +\infty) : Q_\alpha\{x_k\} = +\infty\big\}, & \text{otherwise} \end{cases}$$

$$O_R\{x_k\} = \begin{cases} +\infty, & \text{if } \quad R_\alpha\{x_k\} = 0, \quad \forall \alpha \in [1, +\infty), \\ \inf\big\{\alpha \in [1, +\infty) : R_\alpha\{x_k\} = 1\big\}, & \text{otherwise.} \end{cases}$$

Remark 1 *When* $Q_1\{x_k\} = 0$*, it is said that the sequence converges q-superlinearly; this may be written as*

$$\|x_{k+1} - x^*\| = o(\|x_k - x^*\|), \quad \text{as } k \to \infty.$$

When $Q_{\alpha_0}\{x_k\} < +\infty$ *for some* $\alpha_0 > 1$*, one may write*

$$\|x_{k+1} - x^*\| = \mathcal{O}(\|x_k - x^*\|^{\alpha_0}), \quad \text{as } k \to \infty.$$

We also remind that q-convergence with a certain order implies r-convergence with at least the same order, the converse being false; for other different relating results we refer the reader to [1, ch. 9] *and* [2] (*see also* [3, ch. 3] *and* [4]).

When considering a whole iterative process, its convergence order measures the worst convergence among the sequences with a same limit. In this note we shall deal with conditions ensuring that x^* is an attraction point, i.e. there exists an open ball with center at x^* such that all the sequences given by (1), with the initial approximation x_0 from that ball, converge to x^*. The set of all such sequences will be denoted by $\mathcal{S}$. The q- and r-factors of the iterative process $\mathcal{S}$ are then defined as

$$\begin{aligned} Q_\alpha(\mathcal{S}) &= \sup\big\{Q_\alpha\{x_k\} : (x_k)_{k\geq 0} \in \mathcal{S}\big\}, \\ R_\alpha(\mathcal{S}) &= \sup\big\{R_\alpha\{x_k\} : (x_k)_{k\geq 0} \in \mathcal{S}\big\}, \end{aligned}$$

the convergence orders being defined in the same fashion as for a single sequence.

The following attraction theorem is well known; see also [3, Theorem 3.5].

Theorem 1 (Ostrowski) ([5, Th. 22.1] and [1, Thms. 10.1.3 and 10.1.4]). *Assume that the mapping* G *is differentiable at the fixed point* $x^* \in \mathrm{int}(D)$*. If the spectral radius of* $G'(x^*)$ *satisfies*

$$\rho(G'(x^*)) = \sigma < 1,$$

then x^* is an attraction point for the successive approximations. Moreover, $R_1(\mathcal{S}) = \sigma$, and if $\sigma > 0$ then $O_R(\mathcal{S}) = O_Q(\mathcal{S}) = 1$.

The condition $\sigma < 1$ is sharp:

Example 1 [1, Exercise 10.1-2]. *For $G : \mathbb{R} \to \mathbb{R}$,*

$$G(x) = x - x^3,$$

$x^* = 0$ is an attraction point, while for

$$G(x) = x + x^3,$$

the same fixed point is no longer an attraction point; in both cases $\sigma = 1$.

It is also worth noting that r-superlinear convergence of $\mathcal{S}$ does not generally imply q-superlinear convergence (see also [3, p. 30]):

Example 2 [1, Exercise 10.1-6]. *For $G : \mathbb{R}^2 \to \mathbb{R}^2$,*

$$G(u, v) = (u^2 - v, v^2),$$

with $x^* = 0$, one obtains $R_1(\mathcal{S}) = 0$, but $Q_1(\mathcal{S}) > 0$ in any norm.

A sufficient condition for $R_1(\mathcal{S}) = Q_1(\mathcal{S}) = \sigma \in [0, 1)$ is that $G'(x^*)$ is an M-matrix (see [3, p. 30]), i.e. there exists a norm $\|\cdot\|$ in $\mathbb{R}^n$ such that $\|G'(x^*)\| = \sigma$ (equivalently, for any eigenvalue λ of $G'(x^*)$ with $|\lambda| = \sigma$, all Jordan blocks containing λ are one-dimensional; see, e.g., [6, p. 46]). As a limiting situation we are led to the following result, which was proved in a direct manner in [1] (see also [3, p. 30]).

Theorem 2 [1, Th. 10.1.6]. *Under the assumptions of the Ostrowski theorem, if $G'(x^*) = 0$ then $R_1(\mathcal{S}) = Q_1(\mathcal{S}) = 0$, i.e. $\mathcal{S}$ has q- and r-superlinear convergence.*

The convergence orders in the above theorem are actually higher if G is smoother (see also [3, Th. 3.6]):

Theorem 3 [1, Th. 10.1.7]. *Assume that the mapping G is continuously differentiable on an open neighborhood of the fixed point $x^* \in \operatorname{int}(D)$. If $G'(x^*) = 0$ and G is twice differentiable at x^*, then*

$$O_R(\mathcal{S}) \geq O_Q(\mathcal{S}) \geq 2,$$

i.e. the process has the q- and r-convergence orders at least two. If, additionally,

$$G''(x^*)(h, h) \neq 0, \qquad \text{for all } h \neq 0 \text{ in } \mathbb{R}^n,$$

then the convergence orders are exactly two: $O_R(\mathcal{S}) = O_Q(\mathcal{S}) = 2$.

Ortega and Rheinboldt have also noticed that the conditions in the previous two results are not also necessary:

Example 3 [1, Exercise 10.1-12]. *For* $G : \mathbb{R}^2 \to \mathbb{R}^2$,

$$G(u, v) = (0, u + uv + v^\alpha),$$

arbitrarily high q-convergence orders $\alpha > 1$ *may be attained at* $x^* = 0$, *even if* $G'(x^*) \neq 0$.

The above results on sufficiency are global, in the sense that they provide lower bounds for the convergence orders of all the sequences from $\mathcal{S}$. However, it is possible for some sequences to exhibit higher convergence orders than the lowest bound ensured for some $\alpha_0 \geq 1$ by $Q_{\alpha_0}(\mathcal{S})$ or $R_{\alpha_0}(\mathcal{S})$:

Example 4 *Consider some* $\alpha > 1$ *and* $G : \mathbb{R}^2 \to \mathbb{R}^2$,

$$G(u, v) = (\tfrac{1}{2}u, v^\alpha),$$

with $x^* = 0$ *and* $\sigma = 1/2$. *Then*

(a) for $x_0 = (u_0, v_0),\ u_0 \neq 0,\ 0 \leq |v_0| < 1,\ (x_k)_{k\geq 0}$ *converges only linearly;*

(b) for $x_0 = (0, v_0),\ 0 < |v_0| < 1,\ (x_k)_{k\geq 0}$ *converges with q-order* $\alpha > 1$.

The aim of this note is to characterize the high convergence orders of the sequence (1). This will be done in the following section, while in §3 we shall analyze the stability of these iterations under some perturbation assumptions.

2 Convergence Orders of the Successive Approximations

Given a subset $D' \subseteq \mathbb{R}^n$ and a nonlinear mapping $F : D' \to \mathbb{R}^n$, the Newton method for approximating a solution of the nonlinear system $F(x) = 0$ is given by

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k), \quad k = 0, 1, \ldots, \quad x_0 \in D'.$$

Several results have revealed the local convergence properties of this method and of other Newton-type iterations (see, e.g., [1], [3], [5] and [7]–[36]). We shall remind the results of Dembo, Eisenstat and Steihaug on inexact Newton methods, which will allow us to analyze the local behavior of the successive approximations.

Consider the following (standard) assumptions on F:

(a) there exists $x^* \in D'$ such that $F(x^*) = 0$;

(b) the mapping F is differentiable on an open neighborhood of x^*, with F' continuous at x^*;

(c) the Jacobian $F'(x^*)$ is nonsingular.

The derivative F' is said to be Hölder continuous at x^* with exponent p, $p \in (0,1]$, if there exist $L, \varepsilon > 0$ such that

$$\|F'(x) - F'(x^*)\| \le L\|x - x^*\|^p, \quad \text{when } \|x - x^*\| < \varepsilon.$$

Given an initial approximation $x_0 \in D'$, the inexact Newton (IN) method for approximating the solution x^* is given by the following iterations:

For $k = 0, 1, \ldots,$ until convergence do the following steps.
Step 1. Find s_k such that $F'(x_k)s_k = -F(x_k) + r_k$.
Step 2. Set $x_{k+1} = x_k + s_k$.

The residuals r_k are the amounts by which the approximate solutions s_k fail to satisfy the exact linear systems.

The following result was obtained.

Theorem 4 [16]. *Assume that F satisfies the standard assumptions and for an initial approximation $x_0 \in D'$, the IN iterates converge to x^*. Then the convergence is q-superlinear iff*

$$\|r_k\| = o\big(\|F(x_k)\|\big), \quad \text{as } k \to \infty.$$

If, additionally, F' is Hölder continuous at x^* with exponent p, $p \in (0,1]$, then the convergence is with q-order $1+p$ iff

$$\|r_k\| = \mathcal{O}\big(\|F(x_k)\|^{1+p}\big), \quad \text{as } k \to \infty,$$

while it has r-order $1+p$ iff

$$r_k \to 0 \quad \text{with r-order } 1+p, \text{ as } k \to \infty.$$

We obtain the following result concerning the successive approximations.

Theorem 5 *Assume that the mapping G is differentiable on an open neighborhood of the fixed point x^*, with G' continuous at x^*, and for which $\rho(G'(x^*)) = \sigma < 1$. Let $x_0 \in D$ be an initial approximation such that the sequence of successive approximations converges to x^*. Then $(x_k)_{k\ge 0}$ converges q-superlinearly iff*

$$\big\|G'(x_k)(x_k - G(x_k))\big\| = o\big(\|x_k - G(x_k)\|\big), \quad \text{as } k \to \infty. \tag{2}$$

Suppose that, additionally, G' is Hölder continuous at x^* with exponent p, $p \in (0,1]$. Then $(x_k)_{k\ge 0}$ converges with q-order $1+p$ iff

$$\big\|G'(x_k)(x_k - G(x_k))\big\| = \mathcal{O}\big(\|x_k - G(x_k)\|^{1+p}\big), \quad \text{as } k \to \infty, \tag{3}$$

while the convergence is with r-order $1+p$ iff

$$G'(x_k)\big(x_k - G(x_k)\big) \to 0 \quad \text{with r-order } 1+p, \text{ as } k \to \infty.$$

Proof. The successive approximations may be regarded as IN iterates for solving $F(x) = x - G(x) = 0$:

$$(I - G'(x_k))(G(x_k) - x_k) = -(x_k - G(x_k)) + G'(x_k)(x_k - G(x_k)).$$

The standard assumptions on $F = I - G$ are obviously satisfied, the invertibility of $F'(x^*) = I - G'(x^*)$ being ensured by hypothesis $\sigma < 1$. Next, the Hölder continuity of G' at x^* implies the same property for F':

$$\|F'(x) - F'(x^*)\| \leq (1 + L)\|x - x^*\|^p, \quad \text{when } \|x - x^*\| < \varepsilon < 1.$$

Denoting $r_k = G'(x_k)(x_k - G(x_k))$, the conclusions are now straightforward from the previous theorem. □

Remark 2 *(a) The superlinear convergence of $\mathcal{S}$ when $G'(x^*) = 0$, assured by Theorem 2, is retrieved under the hypotheses of this result:*

$$\begin{aligned} \|G'(x_k)(x_k - G(x_k))\| &\leq \|G'(x_k)\|\|x_k - G(x_k)\| \\ &= o(\|x_k - G(x_k)\|), \quad \text{as } k \to \infty. \end{aligned}$$

(b) The conclusions of Theorem 3 may be obtained in the same fashion:

$$\begin{aligned} \|G'(x_k)(x_k - G(x_k))\| &= \|(G'(x_k) - G'(x^*))(x_k - G(x_k))\| \\ &\leq \big(\|G''(x^*)\|\|x_k - x^*\| + o(\|x_k - x^*\|)\big)\|x_k - G(x_k)\| \\ &= \mathcal{O}(\|x_k - G(x_k)\|^2), \quad \text{as } k \to \infty, \end{aligned}$$

since the standard hypotheses on F ensure the existence of $\alpha, \varepsilon > 0$ such that (see [16])

$$\tfrac{1}{\alpha}\|x - x^*\| \leq \|F(x)\| \leq \alpha\|x - x^*\|, \quad \text{for } \|x - x^*\| < \varepsilon.$$

(c) The same conclusions could be obtained in the above theorem by writing the successive approximations either as quasi-Newton iterates or as inexact perturbed Newton iterates, cf. [14], [11] and [15].

(d) Instead of $\sigma < 1$, one may assume a more general condition, namely that $I - G'(x^*)$ is invertible (which holds iff $G'(x^*)$ has no eigenvalue equal to one) and $(x_k)_{k\geq 0}$ converges to x^*.

(e) The Ostrowski theorem holds also in Banach spaces (see, e.g., [37, Th. 4C]) *as well as the corresponding characterizations for the IN iterations, so our result may be restated in this more general frame.*

Theorem 5 characterizes the convergence orders of the successive approximations in terms of the iterates alone. In the following two results we shall show that the convergence orders are intimately related to some spectral elements of $G'(x^*)$.

Theorem 6 *Under the assumptions of Theorem 5, the sequence $(x_k)_{k\geq 0}$ converges q-superlinearly if and only if $G'(x^*)$ has a zero eigenvalue and, starting from a certain step, the corrections $x_{k+1} - x_k$ are corresponding eigenvectors:*

$$G'(x^*)(x_{k+1} - x_k) = 0, \quad \forall k \geq k_0. \tag{4}$$

Provided that G' is Hölder continuous at x^* with exponent p, $p \in (0,1]$, the above condition characterizes in fact the q-convergence orders $1+p$.

Proof. The residuals in the IN iterates may be written as sums of two terms:

$$-r_k = (G'(x_k) - G'(x^*))(x_{k+1} - x_k) + G'(x^*)(x_{k+1} - x_k), \quad k = 0, 1, \ldots$$

so the sufficiency of condition (4) is obvious. For necessity we notice that the residuals and their first terms converge to zero with rate at least $o(\|x_k - G(x_k)\|)$ as $k \to \infty$, which requires (4). $\square$

Remark 3 *(a) This result implies that no q-superlinear convergence to x^* of any sequence of successive approximations may occur when all the eigenvalues of $G'(x^*)$ are nonzero.*

(b) As Example 3 shows, $\mathcal{S}$ may attain q-superlinear convergence even if not all the nonzero vectors in $\mathbb{R}^n$ are eigenvectors of the eigenvalue 0, i.e. when $G'(x^*)$ has only the zero eigenvalue but is defective. However, in such a case the set of the possible trajectories is restricted.

(c) We also notice that when $O_Q(\mathcal{S}) = 1$, the eventual sequences with q-superlinear convergence are highly sensitive to perturbations, which is not good news for the floating point arithmetic context. We shall analyze the convergence of the perturbed sequences in the following section.

The other characterization may be stated in terms of errors:

Theorem 7 *Under the assumptions of Theorem 5, $(x_k)_{k\geq 0}$ converges q-superlinearly if and only if $G'(x^*)$ has a zero eigenvalue and, starting from a certain step, the errors $x_k - x^*$ are corresponding eigenvectors:*

$$G'(x^*)(x_k - x^*) = 0, \quad \forall k \geq k_0. \tag{5}$$

Provided that G' is Hölder continuous at x^* with exponent p, $p \in (0,1]$, the above condition characterizes in fact the q-convergence orders $1+p$.

Proof. The sufficiency is again obvious. For necessity, using (4) we get that

$$G'(x^*)(x_k - x^*) = G'(x^*)(x_{k_0} - x^*), \quad \forall k \geq k_0 + 1,$$

and therefore the conclusions, since these constant terms have the limit zero. $\square$

It is interesting to note that when $(x_k)_{k\geq 0}$ converge q-superlinearly to x^* and the zero eigenvalue of $G'(x^*)$ is simple, then its trajectory is restricted from a certain step to a line containing x^* (regardless of the nonlinearity of G); when the eigenvalue is double, the trajectory is restricted from a certain step to a plane containing x^*, etc. The trajectory may theoretically be arbitrary only when $G'(x^*) = 0$.

Consider now the affine mapping

$$G(x) = Bx + c, \quad B \in \mathbb{R}^{n \times n}, \ c \in \mathbb{R}^n \text{ given},$$

and for some initial approximation $x_0 \in \mathbb{R}^n$ the iterations

$$x_{k+1} = Bx_k + c, \quad k = 0, 1, \ldots \tag{6}$$

The condition $\rho(B) < 1$ in the Ostrowski theorem becomes necessary and sufficient for these iterates to converge for any initial approximation $x_0 \in \mathbb{R}^n$ to the unique fixed point x^* in $\mathbb{R}^n$ (see, e.g., [1, Th. 10.1.5]). Our results may also be refined in this case:

Theorem 8 *If $\rho(B) = 0$ then the sequence given by (6) converges to x^* in less than n steps, for any initial approximation $x_0 \in \mathbb{R}^n$. If $0 < \rho(B) < 1$ then $(x_k)_{k\geq 0}$ converge q-superlinearly if and only if there exists $k_0 \in \mathbb{N}$ such that*

$$B^{k+1}((I - B)x_0 - c) = 0, \quad \forall k \geq k_0, \tag{7}$$

in which case $x_{k_0+1} = x^*$.

Proof. It is known that a matrix has spectral radius zero iff is nilpotent, i.e., there exists $l_0 \in \mathbb{N}$ such that $B^{l_0} = 0$, in which case l_0 may be taken smaller than n (see [38, Pb. 159]). Relation

$$x_{k+1} - x_k = B^k(x_1 - x_0), \quad k = 2, 3, \ldots, \quad \forall x_0 \in \mathbb{R}^n, \tag{8}$$

completes the proof of the first affirmation.

Relation (7) is immediately obtained from (4) and (8). □

When the initial approximation is taken zero we obtain the following result:

Corollary 1 *If $\rho(B) < 1$ and $x_0 = 0$, then the sequence given by (6) converges in a finite number of steps if and only if $\rho(B) = 0$ or there exists $k_0 \in \mathbb{N}$ such that*

$$B^{k_0}c = 0.$$

It is worth noting that in the affine case the q-superlinear convergence reduces to convergence in a finite number of steps, i.e., to convergence with infinite order.

3 Stability of the Successive Approximations

Assume that the evaluation of G at each step is only approximately performed:

$$x_{k+1} = G(x_k) + \delta_k, \quad k = 0, 1, \ldots \tag{9}$$

These iterates may be viewed again as IN iterations for solving

$$F(x) = x - G(x) = 0,$$

since

$$\begin{aligned}(I - G'(x_k))(G(x_k) + \delta_k - x_k) = \quad & -(x_k - G(x_k)) + G'(x_k)(x_k - G(x_k)) \\ & +(I - G'(x_k))\delta_k.\end{aligned}$$

We obtain the following result.

Theorem 9 *Assume that G satisfies the assumptions of Theorem 5, and the sequence (9) of perturbed successive approximations converges to x^*. Then the convergence is q-superlinear iff*

$$\|G'(x_k)(x_k - G(x_k)) + (I - G'(x_k))\delta_k\| = o(\|x_k - G(x_k)\|), \text{ as } k \to \infty.$$

If, additionally, G' is Hölder continuous at x^* with exponent p, the convergence is with q-order $1+p$ iff

$$\|G'(x_k)(x_k - G(x_k)) + (I - G'(x_k))\delta_k\| = \mathcal{O}(\|x_k - G(x_k)\|^{1+p}), \text{ as } k \to \infty,$$

and with r-order $1+p$ iff

$$G'(x_k)(x_k - G(x_k)) + (I - G'(x_k))\delta_k \to 0 \text{ with } r\text{-order } 1+p, \text{ as } k \to \infty.$$

Remark 4 *We notice that sufficient conditions for the high convergence orders of the perturbed successive approximations are obtained when $x_k - G(x_k)$ and δ_k are eigenvectors corresponding to the eigenvalue 0 of $G'(x^*)$ and δ_k converge to zero with certain speed. In such a case the trajectory remains in the set of the high convergence trajectories corresponding to the unperturbed iterations.*

4 Conclusions

The condition $\rho(G'(x^*)) < 1$ in the Ostrowski theorem ensures that the fixed point x^* is an attraction point and yields the lowest r-convergence order attained by all the sequences of successive approximations. Our results characterize the high q-convergence orders of a single such sequence (which may be attained even when $\rho(G'(x^*)) \neq 0$), indicating the set of all possible trajectories with convergence orders up to two (in this sense the study of the convergence orders higher than two may consist a direction of future research). The Ostrowski theorem requires the knowledge of the spectral radius of $G'(x^*)$, while ours require the spectral structure of $G'(x^*)$.

The obtained results may be applied to the study of the iterative methods used in practice and which may be written as fixed point problems with known mappings G and G', and as well to the study of the fixed point problems in the more abstract setting of Banach spaces (differential and integral equations, dynamical systems, etc).

We shall also study further the existence and estimates for the radius of the attraction balls for the successive approximations with high convergence orders.

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