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Solving polynomial operator equations of degree 2 by Steffensen-type iterations with approximate inverses

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Abstract

We give a semilocal convergence theorem for a Steffensen-type method when the nonlinear equation to solve is a polynomial operators of degree 2. We consider a sequence of linear operators approximating the inverses of the first order divided difference appearing at each iteration step.

We present some numerical results applying the studied method for solving matrix eigenproblems.

1 Introduction

Let X be a Banach space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $F : X \to X$ a polynomial operator of degree 2, and consider the equation

$$F(x) = 0. \tag{1.1}$$

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Such equations often appear in different problems, as eigenproblems for linear operators, Chandrasekhar integral equations, etc. This motivates the study of the iterative methods for solving (1.1), by enhancing the particularities appearing in the case of the second degree polynomials. The second order derivative is a constant bilinear operator, i.e. its expression does not depend on x. This allows the obtaining of some simpler convergence conditions, compared to the general case (see also [19], [26], [36], [10]–[13]).

The solving of a linear equation at each iteration step is an important aspect when analyzing the iterative methods. From this standpoint, the methods which approximate at each step the inverse of the Frechét derivative of the first order divided differences are important to analyze.

Given $x_0 \in X$ and the linear continuous operator $\Gamma_0 \in \mathcal{L}(X)$, we shall consider the sequences $(x_k)_{k\geq 0}$ and $(\Gamma_k)_{k\geq 0}$ given by

$$\begin{aligned} x_{k+1} &= x_k - \Gamma_k F(x_k) \\ \Gamma_{k+1} &= \Gamma_k \left(2I - [x_{k+1}, G(x_{k+1}); F]\, \Gamma_k\right), \quad k = 0, 1, \ldots, \end{aligned} \tag{1.2}$$

where $[x, y, F]$ denotes the first order divided difference of F on x and y, and $G : X \to X$ is an operator such that equation (1.1) is equivalent to

$$x - G(x) = 0. \tag{1.3}$$

As it can be seen, the above method is a Steffensen type one, with the approximation of the inverse of the first order divided difference.

We shall consider in the following the application G given by

$$G(x) = x - \gamma F(x), \tag{1.4}$$

with some $\gamma \in \mathbb{K}$ ($\gamma \neq 0$) fixed.

Since F is a polynomial of degree 2, one has

$$[t, u, v, w; F] = \theta_3, \qquad \forall t, u, v, w \in X, \tag{1.5}$$

where θ_3 denotes the trilinear null mapping, and $[t, u, v, w; F]$ denotes the divided difference of order 3 of F on the nodes $t, u, v, w \in X$. It is then obvious that the second order divided difference is a constant bilinear operator (it does not depend on the nodes). Denote

$$D = [u, v, w; F], \qquad \forall u, v, w \in X.$$

2 A semilocal convergence result

We shall need the following auxiliary result.

Lemma 1 *Let* $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 0$ *and* $(\delta_k)_{k \geq 0}$, $(\eta_k)_{k \geq 0} \subset \mathbb{R}_+$ *obeying*

$$\begin{aligned} &\delta_{k+1} \leq \delta_k \eta_k + \alpha \delta_k^2 \\ &\eta_{k+1} \leq (\eta_k + \beta \delta_k)^2, \quad k = 0, 1, \ldots \end{aligned} \tag{2.1}$$

If $\delta_0, \eta_0 \leq uq$ *for some* $0 < q < 1$ *and* $u = \min\left\{\frac{1}{1+\alpha}, \frac{1}{(1+\beta)^2}\right\}$, *then*

$$\delta_k, \eta_k \leq uq^{2^k}. \tag{2.2}$$

Proof. Obviously, $u(1+\alpha) \leq 1$ and $u(1+\beta)^2 \leq 1$. Relations (2.2) for $k = 0$ are verified by the assumption. Supposing now that (2.2) are satisfied for $k = s$, then by (2.1) we obtain for $k = s+1$

$$\begin{aligned} \delta_{s+1} &\leq \left(u^2 + \alpha u^2\right) q^{2^{s+1}} = u^2 (1+\alpha) q^{2^{s+1}} \leq uq^{2^{s+1}} \\ \eta_{s+1} &\leq (u + \beta u) q^{2^{s+1}} = u^2 (1+\beta)^2 q^{2^{s+1}} \leq uq^{2^{s+1}}, \end{aligned}$$

which ends the proof. ■

We shall assume that

$$\|D\| = a, \tag{2.3}$$

and that, given $x_0 \in X$, there exists $r > 0$ such that

$$\|[x, y; F]\| \leq c, \qquad \forall x, y \in \overline{B}_r(x_0), \tag{2.4}$$

where $\overline{B}_r(x_0) = \{x \in X : \|x - x_0\| \leq r\}$.

We obtain the following result.

Theorem 2 *Assume* (1.5), (2.3), (2.4), *and that the operator* $\Gamma_0 \in \mathcal{L}(X)$ *is invertible, obeying*

$$\|I - [x_0, G(x_0); F]\Gamma_0\| = \eta_0 \leq uq, \tag{2.5}$$

where

$0 \leq q < 1,$
$b_0 = \left\| [x_0, G(x_0); F]^{-1} \right\|,$
$\rho = \frac{b_0}{1-q},$
$b = \rho\eta_0 + 1,$
$\beta = ab^2(2 + |\gamma|\, c),$
$\alpha = ab(b + |\gamma|),$
$u = \min\left\{ \frac{1}{1+\alpha}, \frac{1}{(1+\beta)^2} \right\}$ *and*
$p = b_0 a(2 + |\gamma|\, c)r.$
If $x_0, G(x_0) \in B_r(x_0)$, $\|F(x_0)\| = \delta_0 \leq uq$, *and*

$$|\gamma|\, \|F(x_0)\| + \tfrac{buq}{1-q} \leq r,$$
$$p < 1,$$

then

- *the elements* $x_k, G(x_k)$ *remain in* $\bar{B}_r(x_0)$,
- *the sequences* $(x_k)_{k\geq 0}, (\Gamma_k)_{k\geq 0}$ *are Cauchy,*
- *denoting* $x^* = \lim x_k$, *one has the estimates*

$$\|x^* - x_k\| \leq \frac{buq^{2^k}}{1 - q^{2^k}}, \quad k = 0, 1, \ldots,$$

- $\lim_{k\to\infty} \Gamma_k = [x^*, G(x^*); F]^{-1}$, *and*

$$\left\| [x^*, G(x^*); F]^{-1} - \Gamma_k \right\| \leq u(1 + ab\,|\gamma|)q^{2^k}, \quad k = 0, 1, \ldots$$

Proof. Hypothesis $p < 1$ and the existence of $[x_0, G(x_0); F]^{-1}$ imply the existence of $[x, G(x); F]^{-1}$ for all $x \in \overline{B}_r(x_0)$, and moreover

$$\left\| [x, G(x); F]^{-1} \right\| \leq \frac{b_0}{1-p} = \rho, \quad \forall x \in \overline{B}_r(x_0) \tag{2.6}$$

and

$$\|\Gamma_k\| \leq \rho\eta_0 + 1. \tag{2.7}$$

Denoting $\eta_k = \|I - [x_k, G(x_k); F]\Gamma_k\|$ and $\delta_k = \|F(x_k)\|$, $k = 0, 1, \dots,$ for $x_k, G(x_k) \in \overline{B}_r(x_0)$, we will show that the elements η_k and δ_k satisfy the assumptions of Lemma 2.1, which will imply

$$\delta_k, \eta_k \leq uq^{2^k}. \tag{2.8}$$

Indeed, since F is a polynomial operator of degree 2 we have

$$\begin{aligned} F(x_{k+1}) =& F(x_k) + [x_k, G(x_k); F](x_{k+1} - x_k) + \\ &+ [x_{k+1}, x_k, G(x_k); F](x_{k+1} - x_k)(x_{k+1} - G(x_k)) \end{aligned}$$

whence, by (1.2)

$$\begin{aligned} F(x_{k+1}) =& F(x_k) - [x_k, G(x_k); F]\Gamma_k F(x_k) - \\ &- D\Gamma_k F(x_k)(-\Gamma_k F(x_k) + \gamma F(x_k)), \end{aligned}$$

and therefore

$$\begin{aligned} \|F(x_{k+1})\| \leq & \|I - [x_k, G(x_k); F]\Gamma_k\| \cdot \|F(x_k)\| + \\ &+ \|D\| \left(\|\Gamma_k\|^2 + |\gamma| \|\Gamma_k\|\right) \|F(x_k)\|^2, \end{aligned}$$

i.e. $\delta_{k+1} \leq \eta_k \delta_k + \alpha \delta_k^2$, with $\alpha = ab(b + |\gamma|)$.

From (1.2) we also get

$$\|I - [x_{k+1}, G(x_{k+1}); F]\Gamma_{k+1}\| \leq \|I - [x_{k+1}, G(x_{k+1}); F]\Gamma_k\|^2.$$

One can easily verify that

$$\begin{aligned} &\|I - [x_{k+1}, G(x_{k+1}); F]\Gamma_k\| \leq \\ &\leq \|I - [x_k, G(x_k); F]\Gamma_k\| \\ &\quad + \|D\| \cdot \|\Gamma_k\|^2 \left(2 + |\gamma| \|[x_{k+1}, x_k; F]\| \cdot \|F(x_k)\|\right), \end{aligned}$$

whence, taking into account the notations we made, it follows

$$\eta_{k+1} \leq (\eta_k + \beta\delta_k)^2,$$

with $\beta = ab^2 \left(2 + |\gamma|\, c\right).$

Now we show that $(x_k)_{k\geq 0}, (G(x_k))_{k\geq 0} \in \overline{B}_r(x_0)$.

Assume that for some $k \geq 0$, $x_s \in \overline{B}_r(x_0)$, $s = \overline{0,k}$.

We have that

$$x_{s+1} - x_s = -\Gamma_s F(x_s),$$

$$\|x_{s+1} - x_s\| \leq \|\Gamma_s\| \cdot \|F(x_s)\| \leq b\delta_s \leq buq^{2^s}, \qquad s = \overline{0,k}.$$

Then

$$\|x_{k+1} - x_0\| \leq \sum_{s=0}^{k} \|x_{s+1} - x_s\| \leq \sum_{s=0}^{k} buq^{2^s} \leq \frac{buq}{1-q} \leq r,$$

and

$$\begin{aligned}\|G(x_{k+1}) - x_0\| &\leq \|G(x_{k+1}) - x_{k+1}\| + \|x_{k+1} - x_0\| \leq \\ &\leq \frac{buq}{1-q} + |\gamma|\, \|F(x_k)\| \leq \frac{buq}{1-q} + |\gamma|\, \|F(x_0)\| \leq r.\end{aligned}$$

Using inequality (2.8) we shall show that the sequences $(\Gamma_k)_{k\geq 0}$ and $(x_k)_{k\geq 0}$ are Cauchy, and therefore they converge.

Indeed,

$$\|x_{k+1} - x_k\| \leq buq^{2^k}, \qquad k = 0, 1, \ldots$$

Then,

$$\begin{aligned}\|x_{k+m} - x_k\| &\leq \sum_{s=0}^{m-1} \|x_{k+s+1} - x_{k+s}\| \leq bu \sum_{s=0}^{m-1} q^{2^{k+s}} \\ &\leq buq^{2^k} \sum_{s=0}^{m-1} q^{2^{k+s}-2^k} = bu2^k \sum_{s=0}^{m-1} q^{2^k(2^s-1)} \\ &\leq buq^{2^k}\left(1 + q^{2^k} + q^{3\cdot 2^k} + \cdots\right) = \frac{buq^{2^k}}{1-q^{2^k}},\end{aligned}$$

which shows that $(x_k)_{k\geq 0}$ is fundamental. Consequently, there exists $x^* = \lim_{k\to\infty} x_k$ and

$$\|x^* - x_k\| \leq \frac{buq^{2^k}}{1-q^{2^k}}, \qquad k = 0, 1, \ldots$$

Regarding the approximations Γ_k we have

$$\begin{aligned}\left\|[x^*, G(x^*); F]^{-1} - \Gamma_k\right\| &\leq \left\|[x^*, G(x^*); F]^{-1}\right\| \cdot \|I - [x^*, G(x^*); F]\,\Gamma_k\| \\ &\leq \rho \, \|I - [x^*, G(x^*); F]\,\Gamma_k\| .\end{aligned}$$

But

$$\begin{aligned}&\|I - [x^*, G(x^*); F]\,\Gamma_k\| \leq \\ \leq &\|I - [x_k, G(x_k); F]\,\Gamma_k\| + \\ &+ \|\Gamma_k\| \cdot \|[x_k, G(x_k); F] - [x^*, G(x^*); F]\| \\ \leq &\eta_k + \|\Gamma_k\| \, \|D\| \, |\gamma| \, \delta_k,\end{aligned}$$

whence

$$\left\|[x^*, G(x^*); F]^{-1} - \Gamma_k\right\| \leq \eta_k + ab\,|\gamma|\,\delta_k,$$

which shows that $\lim \Gamma_k = [x^*, G(x^*); F]^{-1}$, and

$$\left\|[x^*, G(x^*); F]^{-1} - \Gamma_k\right\| \leq (u + ab\,|\gamma|\,u)\, q^{2^k}, \qquad k = 0, 1, \ldots$$

■

3 Numerical examples: the approximation of the eigenpairs of matrices.

Denote $V = \mathbb{K}^n$ and let $A \in \mathbb{K}^{n\times n}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

For computing the eigenpairs of A one may consider a mapping $H : V \to \mathbb{K}$ with $H(0) \neq 1$.

The eigenvalues and eigenvectors of A are the solutions of

$$F(x) = \begin{pmatrix} Av - \lambda v \\ H(v) - 1 \end{pmatrix} = 0,$$

where $x = \binom{v}{\lambda} \in V \times \mathbb{K} = \mathbb{K}^{n+1}$.

Denoting $v = \left(x^{(1)}, x^{(2)}, \ldots, x^{(n)}\right)$ and $\lambda = x^{(n+1)}$ then the first n components of F are

$$F_i(x) =$$
$$a_{i1}x^{(1)} + \cdots + a_{i,i-1}x^{(i-1)} + \left(a_{ii} - x^{(n+1)}\right)x^{(i)} + a_{i,i+1}x^{(i+1)} + \cdots + a_{in}x^{(n)}.$$

If we take H as

$$H(v) = \alpha \|v\|_2$$

for some fixed $\alpha \in \mathbb{R}$ then

$$F_{n+1}(x) = \alpha\left(\left(x^{(1)}\right)^2 + \cdots + \left(x^{(n)}\right)^2\right) - 1.$$

The matrices associated to the first order divided differences of F at the points $x_1, x_2 \in \mathbb{K}^{n+1}$ are

$$[x_1, x_2; F] = \begin{pmatrix} b_{11} & a_{12} & \cdots & a_{1n} & a^1_{1,n+1} \\ a_{21} & b_{22} & \cdots & a_{2n} & a^1_{2,n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_{nn} & a^1_{n,n+1} \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} & 0 \end{pmatrix}$$

where $b_{ii} = a_{ii} - \frac{1}{2}\left(x_2^{(n+1)} + x_1^{(n+1)}\right)$, $a_{i,n+1} = -\frac{1}{2}\left(x_2^{(i)} + x_1^{(i)}\right)$ and $a_{n+1,i} = \alpha(x_1^{(i)} + x_2^{(i)})$, for $i = 1, \ldots, n$.

The second order divided differences of F at x_1, x_2, x_3 are given by

$$[x_1, x_2, x_3; F]\, hk =$$

$$= \begin{pmatrix} -\frac{1}{2}k^{(n+1)} & 0 & \cdots & 0 & -\frac{1}{2}k^{(1)} \\ 0 & -\frac{1}{2}k^{(n+1)} & \cdots & 0 & -\frac{1}{2}k^{(2)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{1}{2}k^{(n+1)} & -\frac{1}{2}k^{(n)} \\ \alpha k^{(1)} & \alpha k^{(2)} & \cdots & \alpha k^{(n)} & 0 \end{pmatrix} \begin{pmatrix} h^{(1)} \\ h^{(2)} \\ \vdots \\ h^{(n)} \\ h^{(n+1)} \end{pmatrix}$$

for all $h, k \in \mathbb{K}^{n+1}$.

We shall consider the max norm on V and taking $\|x\| = \max\{\|v\|_\infty, |\lambda|\}$ for all $x = \binom{v}{\lambda} \in \mathbb{K}^{n+1}$ we are led to the max norm on $\mathbb{K}^{n+1}$. It can be easily verified that $\|[x_1, x_2, x_3; F]\|_\infty \leq \max\{1, n|\alpha|\}$.

The classical choice for α is $\alpha = 1/2$. We have proposed in [11] the choice $\alpha = 1/(2n)$, which may lead to smaller bounds for the norm of the second order finite differences.

We shall consider the Fidap002 test matrix from the Harwell Boeing collection[1] in order to study the behavior of method (1.2) for approximating the eigenpairs. The program was written in Matlab; we took $\gamma = 0.05$ and $\Gamma_0 = [x_0, G(x_0); F]^{-1}$, computed with MatLab.

This real symmetric matrix of dimension $n = 441$ arise from finite element modeling. Its eigenvalues are all simple and range from $-7 \cdot 10^8$ to $3 \cdot 10^6$. We have chosen to study the smallest eigenvalue, which is well separated. The initial approximations were taken $\lambda_0 = \lambda^* + 1 \cdot 10^1$ and for the initial vector v_0 we perturbed the solution v^* with random vectors having the components uniformly distributed on (-ε,ε), $\varepsilon = 1 \cdot 10^{-4}$; this value for ε had to be taken smaller than for other methods in order to obtain the attraction of the iterates (1.2); also, these iterates converged slower than others (see, e.g. [13]). The following results are typical for the runs made (we have considered a common vector ε).

[1] These matrices are available from MatrixMarket at the following address: `http://math.nist.gov/MatrixMarket/`.

	Choice I		Choice II	
k	$\|x^* - x_k\|$	$\|F(x_k)\|$	$\|x^* - x_k\|$	$\|F(x_k)\|$
0	$1.0000 \cdot 10^{+01}$	$4.4922 \cdot 10^{+4}$	$1.0000 \cdot 10^{+01}$	$4.4923 \cdot 10^{+4}$
1	$1.3522 \cdot 10^{+00}$	$9.7175 \cdot 10^{+3}$	$1.9985 \cdot 10^{+00}$	$8.9863 \cdot 10^{+3}$
2	$6.3276 \cdot 10^{-01}$	$1.1103 \cdot 10^{+2}$	$1.4514 \cdot 10^{-01}$	$6.4562 \cdot 10^{+2}$
3	$1.5029 \cdot 10^{-01}$	$8.4619 \cdot 10^{+0}$	$1.5287 \cdot 10^{-03}$	$5.3715 \cdot 10^{+0}$
4	$6.9239 \cdot 10^{-03}$	$1.0478 \cdot 10^{-1}$	$5.4049 \cdot 10^{-06}$	$6.8680 \cdot 10^{-4}$
5	$6.7966 \cdot 10^{-05}$	$1.0432 \cdot 10^{-4}$	$1.3970 \cdot 10^{-09}$	$6.7701 \cdot 10^{-8}$
6	$3.0734 \cdot 10^{-08}$	$4.2982 \cdot 10^{-8}$	$4.6567 \cdot 10^{-10}$	$4.4450 \cdot 10^{-8}$
7	$4.6566 \cdot 10^{-10}$	$3.5186 \cdot 10^{-9}$	$2.7909 \cdot 10^{-12}$	$5.4399 \cdot 10^{-8}$

Table 1. Method (1.2) for Fidap002.

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