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Sufficient convergence conditions for certain accelerated successive approximations

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Abstract

We have recently characterized the q-quadratic convergence of the perturbed successive approximations. For a particular choice of the parameters, these sequences resulted as accelerated iterations toward a fixed point.

We give here a Kantorovich-type result, which provides sufficient conditions ensuring the convergence of the accelerated iterates.

1 Introduction

Let $(X, \|\cdot\|)$ be a Banach space and $G : \Omega \subseteq X \to \Omega$ a nonlinear mapping having $x^* \in \operatorname{int}\Omega$ as fixed point:

$$x^* = G(x^*).$$

We are interested in the q-quadratic convergence toward x^* of the sequences of successive approximation type. Recall that an arbitrary sequence $(y_k)_{k\geq 0} \subset X$ converges (q-)quadratically to its limit $\bar{y} \in X$ if [11], [12], [13]

$$\inf \left\{ \alpha \in [1, +\infty) : Q_\alpha\{y_k\} = +\infty \right\} = 2,$$

where

$$Q_\alpha\{y_k\} = \begin{cases} 0, & \text{if } y_k = \bar{y}, \text{ for all but finitely many } k, \\ \limsup\limits_{k\to\infty} \dfrac{\|y_{k+1} - \bar{y}\|}{\|y_k - \bar{y}\|^\alpha}, & \text{if } y_k \neq \bar{y}, \text{ for all but finitely many } k, \\ +\infty, & \text{otherwise.} \end{cases}$$

In the case when $0 < Q_2\{y_k\} < +\infty$, one obtains the well known estimate of the form

$$\|y_{k+1} - \bar{y}\| \leq \left(Q_p\{y_k\} + \varepsilon\right) \|y_k - \bar{y}\|^2, \quad \text{for all } k \geq k_0$$

(in the sense that for all $\varepsilon > 0$ there exists $k_0 \geq 0$ such the above inequalities hold).

The successive approximations converging quadratically to x^* are characterized by the following result.

Theorem 1.1 [6] *Assume that G is differentiable on a neighborhood D of x^*, with the derivative G' Lipschitz continuous:*

$$\|G'(x) - G'(y)\| \leq L \|x - y\|, \quad \forall x, y \in D.$$

Suppose further that for a certain initial approximation $x_0 \in D$, the successive approximations

$$x_{k+1} = G(x_k), \quad k \geq 0,$$

converge to x^*, and $I - G'(x_k)$ are invertible starting from a certain step.

Then the convergence is with order 2 if and only if G' has a zero eigenvalue and, starting from a certain step, the corrections $x_{k+1} - x_k$ are corresponding eigenvectors:

$$G'(x^*)(x_{k+1} - x_k) = 0, \quad \forall k \geq k_0.$$

This condition holds equivalently iff the errors $x_k - x^*$ are corresponding eigenvectors:

$$G'(x^*)(x_k - x^*) = 0, \quad \forall k \geq k_0,$$

or iff

$$x_k \in x^* + \operatorname{Ker} G'(x^*), \quad \forall k \geq k_0.$$

This result implies that if G' has no eigenvalue 0, there exists no sequence of successive approximations converging to x^* with order 2. [1] In such a case, one may choose to consider for some $(\delta_k)_{k \geq 0} \subset X$ the perturbed successive approximations

$$x_{k+1} = G(x_k) + \delta_k, \quad k \geq 0. \tag{1}$$

Their quadratic convergence is characterized by the following result, which does not require the existence of the eigenvalue 0.

Theorem 1.2 [6] *Suppose that G satisfies the assumptions of Theorem 1.1, and that the sequence (1) of perturbed successive approximations converges to x^*. Then the convergence is with q-order 2 iff*

$$\|G'(x_k)(x_k - G(x_k)) + (I - G'(x_k))\delta_k\| = O(\|x_k - G(x_k)\|^2), \text{ as } k \to \infty. \tag{2}$$

In [5] we have shown that if we write

$$\delta_k = (I - G'(x_k))^{-1}\big(G'(x_k)(G(x_k) - x_k) + \gamma_k\big)$$

[1] In this case, the successive approximations cannot converge faster than q-linearly [6].

with $(\gamma_k)_{k\geq 0} \subset X$, then condition

$$\gamma_k = O(\|x_k - G(x_k)\|^2), \quad \text{as } k \to \infty,$$

is equivalent to (2).

We have also noticed in [5] that, under the assumption $\|G'(x)\| \leq q < 1$ for all x in a certain neighborhood of x^*, and for a given $K > 0$, a natural choice (implied by the Banach lemma) for δ_k is:

$$\delta_k = \big(I + \cdots + G'(x_k)^{i_k}\big) G'(x_k)(G(x_k) - x_k),$$

with i_k such that

$$\tfrac{q^{i_k+2}}{1-q} \leq K\|x_k - G(x_k)\|. \tag{3}$$

When applying Theorem 1.2 to characterize the quadratic convergence of the resulted sequence

$$x_{k+1} = G(x_k) + \big(I + \cdots + G'(x_k)^{i_k}\big) G'(x_k)(G(x_k) - x_k), \quad k \geq 0, \tag{4}$$

with i_k given by (3), we must assume that this sequence converges to the fixed point x^*. But is this assumption reasonable? The purpose of this note is to show that under certain natural conditions the sequence converges to x^*, so the answer is positive.

2 Main result

First of all, we remark that the fixed point problem is equivalent to solving

$$F(x) = 0, \quad \text{with } F(x) = x - G(x),$$

for which the Newton method generates the iterates

$$\begin{aligned} s_k^N &= -F'(x_k)^{-1} F(x_k) \qquad (5)\\ x_{k+1} &= x_k + s_k^N, \quad k = 0, 1, \ldots . \end{aligned}$$

In this setting, iterations (4) may be rewritten as

$$\begin{aligned} x_{k+1} &= x_k + (I + G'(x_k) + \ldots + G'(x_k)^{i_k+1})(G(x_k) - x_k) \qquad (6)\\ &:= x_k + s_k, \quad k = 0, 1, \ldots, \\ &\text{with } i_k \text{ s.t. } \tfrac{q^{i_k+2}}{1-q} \leq K\|F(x_k)\|, \end{aligned}$$

i.e., as quasi-Newton iterations (see, e.g., [11], [8], [7]).

We obtain the following sufficient Kantorovich-type conditions for the convergence to x^* of these iterates.

Theorem 2.1 *Assume that G is differentiable on the domain Ω, with G' bounded on Ω by*

$$\|G'(x)\| \leq q < 1, \ \ \forall x \in \Omega, \tag{7}$$

and Lipschitz continuous:

$$\|G'(x) - G'(y)\| \leq L \|x - y\|, \ \ \forall x, y \in \Omega.$$

Let $x_0 \in \Omega$ and $K > 0$ be chosen such that

$$\nu = \left(\tfrac{L}{2(1-q)^2} + K(1+q)\right) \|F(x_0)\| < 1, \tag{8}$$

and suppose that $\bar{B}_r(x_0) = \left\{x \in X : \|x - x_0\| \leq r\right\} \subseteq \Omega$ for

$$r = \tfrac{1}{(1-\nu)(1-q)} \|F(x_0)\|.$$

Then the elements of the sequence defined by (6) remain in the ball $\bar{B}_r(x_0)$ and converge to a fixed point x^* of G, which is unique in this ball. According to Theorem 1.2, the convergence is quadratic.

Proof. Recall first [11, 3.2.12] that the Lipschitz hypothesis on G' implies that

$$\|G(y) - G(x) - G'(x)(y - x)\| \leq \tfrac{L}{2} \|y - x\|^2, \ \ \forall x, y \in \Omega,$$

while (7) attracts the existence of $(I - G'(x))^{-1} = I + G'(x) + \ldots + G'(x)^k + \ldots$ and the bound

$$\left\|(I - G'(x))^{-1}\right\| \leq \tfrac{1}{1-q}, \ \ \forall x \in \Omega.$$

Our hypotheses imply the following inequalities:

$$\|s_0\| \leq \tfrac{1}{1-q} \|F(x_0)\|$$

i.e., $x_1 \in \bar{B}_r(x_0)$ and also

$$\begin{aligned}
\|F(x_1)\| &= \left\|F(x_1) - F(x_0) - F'(x_0)s_0^N\right\| \qquad \text{(by (5))} \\
&\leq \|F(x_1) - F(x_0) - F'(x_0)s_0\| + \left\|F'(x_0)(s_0^N - s_0)\right\| \\
&\leq \|G(x_1) - G(x_0) - G'(x_0)s_0\| + (1+q)\left\|s_0^N - s_0\right\| \\
&\leq \tfrac{L}{2}\|s_0\|^2 + (1+q)\left\|G'(x_0)^{i_0+2}(I + G'(x_0) + \ldots)F(x_0)\right\| \\
&\leq \tfrac{L}{2}(1 + q + \ldots + q^{i_0+1})^2 \|F(x_0)\|^2 + \tfrac{q^{i_0+2}}{1-q}(1+q)\|F(x_0)\| \\
&\leq \tfrac{L(1-q^{i_0+2})^2}{2(1-q)^2}\|F(x_0)\|^2 + K(1+q)\|F(x_0)\|^2 \\
&\leq \nu \|F(x_0)\|.
\end{aligned}$$

In an analogous fashion, we obtain by induction that for all $k \geq 2$

$$\begin{aligned} \|F(x_k)\| &\leq \left(\tfrac{L}{2(1-q)^2} + K(1+q)\right) \|F(x_{k-1})\|^2 \\ &\leq \nu \|F(x_{k-1})\| \\ &\vdots \\ &\leq \nu^k \|F(x_0)\|, \end{aligned}$$

$$\|x_k - x_{k-1}\| = \|s_{k-1}\| \leq \tfrac{1}{1-q} \|F(x_{k-1})\| \leq \tfrac{\nu^{k-1}}{1-q} \|F(x_0)\|,$$

$$\|x_k - x_0\| \leq \|x_k - x_{k-1}\| + \ldots + \|x_1 - x_0\| \leq \tfrac{1}{(1-\nu)(1-q)} \|F(x_0)\| = r.$$

It follows that

$$\begin{aligned} \|x_{k+m} - x_k\| &\leq \|x_{k+m} - x_{k+m-1}\| + \ldots + \|x_{k+1} - x_k\| \\ &\leq \tfrac{\nu^{k+m-1} + \ldots + \nu^k}{(1-q)} \|F(x_0)\| \\ &\leq \tfrac{\nu^k}{(1-\nu)(1-q)} \|F(x_0)\|, \end{aligned}$$

which shows that $(x_k)_{k\geq 0}$ is a Cauchy sequence, and therefore converges to a certain $x^* \in \bar{B}_r(x_0)$. By the definition and continuity of F, x^* is a fixed point of G, which is unique in $\bar{B}_r(x_0)$ (and also in Ω) since G is a contraction. ■

We note that condition (8) contains certain natural demands: $\|F(x_0)\|$ is sufficiently small (which holds, e.g., when x_0 is sufficiently close to x^*), q is sufficiently small (in accordance with the results in [6]), the Lipschitz constant L is sufficiently small (the graph of G is close to a constant in case $X = R$) and K is sufficiently small (the linear systems are solved with increasingly precision, the iterates approaching to those given by the Newton method—see the classical results of Dennis and Moré [8]).

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