ON AN AITKEN TYPE METHOD∗

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Abstract. In this note we study the convergence of a generalized Aitken type method for approximating the solutions of nonlinear equations in \( \mathbb{R} \). We obtain conditions which assure monotone convergence of the generated sequences. We also obtain a posteriori estimations for the errors.

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1. INTRODUCTION

Consider the equation

\[ f(x) = 0 \]

where \( f : [a, b] \to \mathbb{R}, a, b \in \mathbb{R}, a < b. \)

Consider also the following two equations, both equivalent to (1):

\[ x - g_i(x) = 0, \quad g_i : [a, b] \to [a, b], \quad i = 1, 2. \]

In order to approximate a root \( x^* \) of (1), we consider the sequence \((x_n)_{n \geq 0}\), generated by the following relations:

\[ x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g_1(x_n); f]} - \frac{[x_n, g_1(x_n), g_2(x_n); f] f(x_n) f(g_1(x_n))}{[x_n, g_1(x_n); f][x_n, g_2(x_n); f][g_1(x_n), g_2(x_n); f]} \]

\( x \in [a, b], \ n = 0, 1, 2, ..., \)

where \([x, y; f]\) denotes the first order divided difference of \( f \) on \( x \) and \( y \).

Relation (3) suggests an Aitken type method. If we assume that \( f \in C^3[a, b] \) and \( f'(x) \neq 0 \) for all \( x \in [a, b] \), denoting \( F = f([a, b]) \), it is known that there exists \( f^{-1} : F \to [a, b] \) and \( f^{-1} \in C^3([a, b]) \). Moreover, the following relation holds

\[ (f^{-1}(y))^" = \frac{3(f'(x))^2 - f'(x)f"(x)}{(f'(x))^3} \]

where \( y = f(x) \) and \( x \in [a, b] \) (see [1], [2], [4], [7]).

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In [3] was studied the convergence of a Steffensen type method, analogous to [3]. In order to obtain a control of the error at each iteration step, we have studied in [6] the cases when the generalized Steffensen type method leads to monotone approximations. Under some reasonable conditions of monotonicity and convexity (concavity) on $f$, in [6] were obtained monotone sequences which yield bilateral approximations to the solution $x^*$ of (1). A condition such that the Steffensen type method studied in [6] to lead to monotone approximations is that the sign of the expression

$$E_f(x) = 3(f''(x))^2 - f'(x)f'''(x)$$

is negative: $E_f(x) < 0$, $\forall x \in [a, b]$.

In this note we shall show that if $E_f(x) > 0$ $\forall x \in [a, b]$, one may construct functions $g_1$ and $g_2$, and on may choose the initial approximations $x_0 \in [a, b]$ such that, under some reasonable hypotheses on $f$, the sequence (3) converges monotonically.

In an analogous fashion as in [3], we can easily show that for a given $x_n \in [a, b]$, then exists $\xi_n \in]c_n, d_n[$, where $[c_n, d_n]$ is the smallest interval containing the points $x_n, g_1(x_n), g_2(x_n), x^*$, such that

$$x^* - x_{n+1} = \frac{E_f(\xi_n)f(x_n)f(g_1(x_n))f(g_2(x_n))}{6[f'(\xi_n)]^5}, \quad n = 0, 1, \ldots$$

2. THE CONVERGENCE OF THE AITKEN METHOD

We consider the following hypotheses on the functions $f_1, g_1, g_2$, and on the initial approximation $x_0 \in [a, b]$

i. $f \in C^5[a, b]$ and $E_f(x) > 0$, $\forall x \in [a, b]$ where $E_f$ is given by (5);

ii. $f(x_0) < 0$;

iii. $f'(x) > 0$, $\forall x \in [a, b]$;

iv. $f''(x) \geq 0$, $x \in [a, b]$;

v. $g_1$ and $g_2$ are continuous functions, decreasing on $[a, b]$;

vi. equation (1) has a solution $x^* \in [a, b]$;

vii. $g_1(x_0) \leq b, g_2(x_0) \leq b$;

viii. equations (1) and (2) are equivalent.

The following result holds:

**Theorem 1.** If functions $f_1, g_1, g_2$ and the initial approximation $x_0 \in [a, b]$ verify hypotheses i.–viii., then the sequences $(x_n)_{n \geq 0}$, $(g_1(x_n)), (g_2(x_n))$ generated by (3) have the following properties:

j. the sequence $(x_n)_{n \geq 0}$ is increasing;

jj. the sequences $(g_1(x_n)), (g_2(x_n))_{n \geq 0}$ are decreasing;

jjj. $\lim x_n = \lim g_1(x_n) = \lim g_2(x_n) = x^*$;

jv. the following relations hold:

$$x^* - x_n \leq \min\{g_1(x_n) - x_n, g_2(x_n) - x_n\}, \quad n = 0, 1, \ldots$$
Proof. Let \( x_n \in [a, b] \) be an approximation of the solution \( x^* \) such that \( f(x_n) < 0, g_1(x_n) \leq b, g_2(x_n) \leq b \). From ii. and iii. it follows \( x_n < x^* \).

By v. we have \( g_1(x_n) > x^*, g_2(x_n) > x^* \). Taking into account the fact that \( f(x_n) < 0, f(g_1(x_n)) > 0 \) and using hypothesis iii. and iv., from (3) it follows that \( x_n+1 > x_n \). Since \( f(g_2(x_n)) > 0 \), by iii. i. and (3) we get \( x_{n+1} < x^* \).

By v. and \( x_n < x_{n+1} \) it follows \( g_1(x_n) > g_1(x_{n+1}) \) and \( g_2(x_n) > g_2(x_{n+1}) \). By v. it also follows that \( g_1(x_{n+1}) > x^* \) and \( g_2(x_{n+1}) > x^* \). These relations imply j. and jj.

In order to prove jjj., let \( \ell = \lim x_n \). For \( n \to \infty \), from (3) we obtain \( \ell = x^* \). Obviously, from the continuity of \( g_1 \) and \( g_2 \) it follows that

\[
\lim g_1(x_n) = g_1(x^*) = x^* \quad \text{and} \quad \lim g_2(x_n) = g_2(x^*) = x^*.
\]

Relations jv. are obvious, and allow us to evaluate the a posteriori error at each iteration step.

\[ \square \]

3. CONSTRUCTION OF FUNCTIONS \( g_1 \) AND \( g_2 \)

The basic conditions on \( g_1 \) and \( g_2 \) on v. and vii.

We shall consider a function \( g : [a, b] \to \mathbb{R} \) given by

\[
g(x) = x - \lambda f(x)
\]

where \( \lambda \in \mathbb{R} \).

If \( g'(x) \leq 0 \), then \( g \) is decreasing. This attracts

\[
1 - \lambda f'(x) < 0
\]

i.e.,

\[
\lambda > \frac{1}{f'(x)}
\]

But since \( f'(x) > 0 \) and \( f'' \geq 0 \), we obtain

\[
f'(a) \leq f'(x) \leq f'(b)
\]

or

\[
\frac{1}{f'(a)} \geq \frac{1}{f'(x)} \geq \frac{1}{f'(b)}.
\]

By [2], it is obvious that if \( \lambda > \frac{1}{f'(a)} \) then (8) is verified for every \( x \in [a, b] \), from which \( g \) is decreasing.

In order to obtain \( g(x_0) \leq b \), we need that \( x_0 - \lambda(x_0) \leq b \), which leads to

\[
\lambda \leq \frac{x_0 - b}{f(x_0)}
\]

which in turn holds if \( x_0 \) is sufficiently close to \( x^* \).

Let \( \lambda_i, i = 1, 2, \lambda_1 \neq \lambda_2 \) be two numbers such that

\[
\frac{1}{f'(a)} \leq \lambda_i \leq \frac{x_0 - b}{f(x_0)};
\]

then functions \( g_i, i = 1, 2 \) given by

\[
g_i(x) = x - \lambda_i f(x)
\]

verify hypotheses v. and vii.
4. NUMERICAL EXAMPLE

Consider the equation
\[ f(x) = x - 2 \cos x = 0, \]
with \( x \in \left[ \frac{\pi}{6}, \frac{\pi}{2} \right] \).

Obviously, \( f'(x) > 0 \), for all \( x \in \left[ \frac{\pi}{6}, \frac{\pi}{2} \right] \), and \( f''(x) > 0 \), \( x \in \left[ \frac{\pi}{6}, \frac{\pi}{2} \right] \). If in (7) we take \( \lambda_1 = 0.5 \), \( \lambda_2 = 0.6 \) and \( x_0 = \frac{\pi}{6} \), then \( g_1 \) and \( g_2 \) verify v. and vii.

We have:
\[
\begin{align*}
g_1(x) &= \cos x + \frac{x}{2}, \\
g_2(x) &= \frac{6 \cos x + 2x}{5}. 
\end{align*}
\]

Obviously, \( g_1(\frac{\pi}{6}) < \frac{\pi}{2} \) and \( g_2(\frac{\pi}{6}) < \frac{\pi}{7} \). Function \( E_f \) is given by
\[
E_f(x) = 4 + 8 \cos^2 x + 2 \sin x > 0, \quad \forall x \in \left[ \frac{\pi}{6}, \frac{\pi}{2} \right].
\]

In the table below we have obtained the following results.

<table>
<thead>
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<th>( n )</th>
<th>( x_n )</th>
<th>( g_1(x_n) )</th>
<th>( g_2(x_n) )</th>
<th>( f(x_n) )</th>
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<td>0.5235987755982988</td>
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</tr>
<tr>
<td>2</td>
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<td>(-5.830220460833369 \cdot 10^{-9})</td>
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<td>1.029866529322259</td>
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<td>0</td>
</tr>
</tbody>
</table>

The numerical results confirm that inequalities
\[ 0 \leq x^* - x_4 < 10^{-15} \]
hold.

REFERENCES


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