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ON AN AITKEN TYPE METHOD*

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Abstract. In this note we study the convergence of a generalized Aitken type method for approximating the solutions of nonlinear equations in $\mathbb{R}$. We obtain conditions which assure monotone convergence of the generated sequences. We also obtain a posteriori estimations for the errors.

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1. INTRODUCTION

Consider the equation

$$f(x) = 0 \tag{1}$$

where $f : [a, b] \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$.

Consider also the following two equations, both equivalent to (1):

$$x - g_i(x) = 0, \ g_i : [a, b] \to [a, b], \ i = 1, 2. \tag{2}$$

In order to approximate a root x^* of (1), we consider the sequence $(x_n)_{n\geq 0}$, generated by the following relations:

(3)
$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g_1(x_n); f]} - \frac{[x_n, g_1(x_n), g_2(x_n); f]f(x_n)f(g_1(x_n))}{[x_n, g_1(x_n); f][x_n, g_2(x_n); f][g_1(x_n), g_2(x_n); f]}$$
$$x \in [a, b], n = 0, 1, 2, ...,$$

where $[x, y; f]$ denotes the first order divided difference of f on x and y.

Relation (3) suggests an Aitken type method. If we assume that $f \in C^3[a, b]$ and $f'(x) \neq 0$ for all $x \in [a, b]$, denoting $F = f([a, b])$, it is known that there exists $f^{-1} : F \to [a, b]$ and $f^{-1} \in C^3([a, b])$. Moreover, the following relation holds

$$(f^{-1}(y))''' = \frac{3(f''(x))^2 - f'(x)f'''(x)}{(f'(x))^5} \tag{4}$$

where $y = f(x)$ and $x \in [a, b]$ (see [1], [2], [4], [7]).

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In [3] was studied the convergence of a Steffensen type method, analogous to (3). In order to obtain a control of the error at each iteration step, we have studied in [6] the cases when the generalized Steffensen type method leads to monotone approximations. Under some reasonable conditions of monotonicity and convexity (concavity) on f, in [6] were obtained monotone sequences which yield bilateral approximations to the solution x^* of (1). A condition such that the Steffensen type method studied in [6] to lead to monotone approximations is that the sign of the expression

$$E_f(x) = 3(f''(x))^2 - f'(x)f'''(x) \tag{5}$$

to be negative: $E_f(x) < 0$, $\forall x \in [a,b]$.

In this note we shall show that if $E_f(x) > 0$ $\forall x \in [a,b]$, one may construct functions g_1 and g_2, and on may choose the initial approximations $x_0 \in [a,b]$ such that, under some reasonable hypotheses on f, the sequence (3) converges monotonically.

In an analogous fashion as in [6], we can easily show that for a given $x_n \in [a,b]$, then exists $\xi_n \in]c_n, d_n[$, where $[c_n, d_n]$ is the smallest interval containing the points x_n, $g_1(x_n)$, $g_2(x_n)$, x^*, such that

$$x^* - x_{n+1} = -\frac{E_f(\xi_n) f(x_n) f(g_1(x_n)) f(g_2(x_n))}{6[f'(\xi_n)]^5}, \ n = 0, 1, ... \tag{6}$$

2. THE CONVERGENCE OF THE AITKEN METHOD

We consider the following hypotheses on the functions f_1, g_1, g_2, and on the initial approximation $x_0 \in [a,b]$:

i. $f \in C^3[a,b]$ and $E_f(x) > 0$, $\forall x \in [a,b]$ where E_f is given by (5);
ii. $f(x_0) < 0$;
iii. $f'(x) > 0$, $\forall x \in [a,b]$;
iv. $f''(x) \geq 0$, $x \in [a,b]$;
v. g_1 and g_2 are continuous functions, decreasing on $[a,b]$;
vi. equation (1) has a solution $x^* \in]a,b[$;
vii. $g_1(x_0) \leq b, g_2(x_0) \leq b$;
viii. equations (1) and (2) are equivalent.

The following result holds:

THEOREM 1. *If functions f_1, g_1, g_2 and the initial approximation $x_0 \in [a,b]$ verify hypotheses* i.–viii., *then the sequences $(x_n)_{n\geq 0}$, $(g_1(x_n))$, $(g_2(x_n))$ generated by (3) have the following properties:*

j. *the sequence $(x_n)_{n\geq 0}$ is increasing;*
jj. *the sequences $(g_1(x_n))$, $(g_2(x_n))_{n\geq 0}$ are decreasing;*
jjj. $\lim x_n = \lim g_1(x_n) = \lim g_2(x_n) = x^*$;
jv. *the following relations hold:*

$$x^* - x_n \leq \min\{g_1(x_n) - x_n, g_2(x_n) - x_n\}, \quad n = 0, 1, ...$$

Proof. Let $x_n \in [a,b]$ be an approximation of the solution x^* such that $f(x_n) < 0$, $g_1(x_n) \le b$, $g_2(x_n) \le b$. From ii. and iii. it follows $x_n < x^*$. By v. we have $g_1(x_n) > x^*$, $g_2(x_n) > x^*$. Taking into account the fact that $f(x_n) < 0$, $f(g_1(x_n)) > 0$ and using hypothesis iii. and iv., from (3) it follows that $x_{n+1} > x_n$. Since $f(g_2(x_n)) > 0$, by iii., i. and (6) we get $x_{n+1} < x^*$.

By v. and $x_n < x_{n+1}$ it follows $g_1(x_n) > g_1(x_{n+1}))$ and $g_2(x_n) > g_2(x_{n+1})$. By v. it also follows that $g_1(x_{n+1}) > x^*$ and $g_2(x_{n+1}) > x^*$. These relations imply j. and jj.

In order to prove jjj., let $\ell = \lim x_n$. For $n \to \infty$, from (3) we obtain $f(\ell) = 0$ and hence $\ell = x^*$. Obviously, from the continuity of g_1 and g_2 it follows that $\lim g_1(x_n) = g_1(x^*) = x^*$ and $\lim g_2(x_n) = g_2(x^*) = x^*$.

Relations jv. are obvious, and allow us to evaluate the a posteriori error at each iteration step. □

3. CONSTRUCTION OF FUNCTIONS g_1 AND g_2

The basic conditions on g_1 and g_2 on v. and vii.

We shall consider a function $g : [a,b] \to \mathbb{R}$ given by

$$g(x) = x - \lambda f(x) \tag{7}$$

where $\lambda \in \mathbb{R}$.

If $g'(x) \le 0$, then g is decreasing. This attracts

$$1 - \lambda f'(x) < 0$$

i.e.,

$$\lambda > \tfrac{1}{f'(x)} \tag{8}$$

But since $f'(x) > 0$ and $f'' \ge 0$, we obtain

$$f'(a) \le f'(x) \le f'(b)$$

or

$$\tfrac{1}{f'(a)} \ge \tfrac{1}{f'(x)} \ge \tfrac{1}{f'(b)}. \tag{9}$$

By (9), it is obvious that if $\lambda > \frac{1}{f'(a)}$ then (8) is verified for every $x \in [a,b]$, from which g is decreasing.

In order to obtain $g(x_0) \le b$, we need that$x_0 - \lambda(x_0) \le b$, which leads to $\lambda \le \frac{x_0 - b}{f(x_0)}$ which in turn holds if x_0 is sufficiently close to x^*.

Let λ_i, $i = 1,2$, $\lambda_1 \ne \lambda_2$ be two numbers such that

$$\tfrac{1}{f'(a)} \le \lambda_i \le \tfrac{x_0 - b}{f(x_0)};$$

then functions g_i, $i = 1,2$ given by

$$g_i(x) = x - \lambda_i f(x)$$

verify hypotheses v. and vii.

4. NUMERICAL EXAMPLE

Consider the equation

$$f(x) = x - 2\cos x = 0,$$

with $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Obviously, $f'(x) > 0$, for all $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$, and $f''(x) > 0$, $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$. If in (7) we take $\lambda_1 = 0.5$, $\lambda_2 = 0.6$ and $x_0 = \frac{\pi}{6}$, then g_1 and g_2 verify v. and vii.

We have:

$$g_1(x) = \cos x + \tfrac{x}{2},\ g_2(x) = \tfrac{6\cos x + 2x}{5}.$$

Obviously, $g_1(\frac{\pi}{6}) < \frac{\pi}{2}$ and $g_2(\frac{\pi}{6}) < \frac{\pi}{2}$. Function E_f is given by

$$E_f(x) = 4 + 8\cos^2 x + 2\sin x > 0,\ \forall x \in [\tfrac{\pi}{6}, \tfrac{\pi}{2}].$$

In the table below we have obtained the following results.

Table 1. Numerical results

n	x_n	$g_1(x_n)$	$g_2(x_n)$	$f(x_n)$
1	0.5235987755982988	1.127824791583588	1.248669994780646	$-1.208\,452\,031\,970\,579 \cdot 10^{+0}$
2	1.027717814817341	1.030632925047758	1.031215947093841	$-5.830\,220\,460\,833\,369 \cdot 10^{-3}$
3	1.029866528928396	1.029866529462959	1.029866529569871	$-1.069\,125\,232\,788\,792 \cdot 10^{-9}$
4	1.029866529322259	1.029866529322259	1.029866529322259	0

The numerical results confirm that inequalities

$$0 \le x^* - x_4 < 10^{-15}$$

hold.

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