ESTIMATING THE RADIUS OF THE ATTRACTION BALLS

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ABSTRACT. Given a nonlinear mapping G differentiable at a fixed point x^*, the Ostrowski theorem offers the sharp sufficient condition

$$\rho(G'(x^*)) < 1$$

for x^* to be an attraction point, where $\rho(G'(x^*))$ is the spectral radius of $G'(x^*)$. However, no estimation for the size of an attraction ball is known.

We show in this note that such an estimate may be readily obtained in terms of $\|G'(x^*)\| < 1$ and of the Hölder (in particular Lipschitz) continuity constant of G'. An elementary example shows that this estimate may be sharp.

Our assumptions do not necessarily require G to be of contractive-type.

ESTIMATION OF THE RADIUS

Let $G : D \subseteq \mathbb{R}^n \to D$ be a nonlinear mapping which has a fixed point $x^* \in \operatorname{int} D$:

$$x^* = G(x^*).$$

This point is an *attraction point* [7, Def. 10.1.1] if, given a norm on $\mathbb{R}^n$, there exists an open ball $B_r := B_r(x^*) = \{x \in \mathbb{R}^n : \|x - x^*\| < r\} \subseteq D$, such that for any initial approximation $x_0 \in B_r$, the successive approximations

$$x_{k+1} = G(x_k), \qquad k = 0, 1, \ldots \tag{1}$$

remain in D and converge to x^*. Note that a finite number of iterates are allowed to lie outside B_r.

The following result is well known.

Theorem 1 (Ostrowski)**.** (*see, e.g.,* [8, Th.22.1]*,* [7, Th.10.1.3] *and* [12, Th.3.5])*. If G is differentiable at the fixed point x^* and the spectral radius satisfies*

$$\sigma := \rho\big(G'(x^*)\big) = \max\big\{|\lambda| : \lambda \in \mathbb{C}, \lambda \text{ eigenvalue of } G'(x^*)\big\} < 1 \tag{2}$$

then x^* is an attraction point.

Remark 1. According to [7, N.R.10.1-2], this result holds also when instead of $\mathbb{R}^n$ one considers an arbitrary Banach space X, with the remark that in defining the spectral radius of a linear continuous operator from X to X one takes its resolvent and spectrum (see, e.g., [14, p. 795]).

Condition (2) is sharp:

Example 1. [7, E.10.1.2]. The function $G : \mathbb{R} \to \mathbb{R}$, $G(x) = x + x^3$, is differentiable on $\mathbb{R}$, and $\sigma = 1$ at the fixed point $x^* = 0$, which is not an attraction point.

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The spectral radius also offers some global information regarding the convergence rate toward x^*, while the spectral elements of $G'(x^*)$ characterize the convergence rate of each sequence. Indeed, σ yields the worst convergence rate among the sequences converging to x^* (see [7, Th.10.1.4] and [12, Th.3.5]), while the zero eigenvalue and its corresponding eigenvectors characterize the high convergence rates (more precisely, the q-superlinear convergence and with q-orders $1+p$, $p \in (0,1]$) of a single such sequence [3].[1]

It is interesting to note that in the case of the Newton method

$$\begin{aligned} F'(x_k)s_k &= -F(x_k) \\ x_{k+1} &= x_k + s_k, \quad k = 0, 1, \ldots \end{aligned} \tag{3}$$

for solving nonlinear systems $F(x) = 0$, $F : \mathbb{R}^n \to \mathbb{R}^n$, there exist several results which estimate the radius of the convergence ball (see [5], [11], [6], [1]). Unfortunately, these results cannot be applied to the successive approximations since iterates (1) cannot be framed in (3); usually it is the other way around: the Newton iterates are regarded as successive approximations for $G(x) = x - F'(x)^{-1}F(x)$.

However, we have shown in [3] that under additional smoothness assumptions on G, the successive approximations represent particular instances of inexact Newton iterates

$$\begin{aligned} F'(x_k)s_k &= -F(x_k) + r_k \\ x_{k+1} &= x_k + s_k, \quad k = 0, 1, \ldots, \end{aligned}$$

when $F(x) = x - G(x)$. Ypma [13] has obtained estimates for the radius of convergence of this method, but when applying them to our context we obtain results which are far from being sharp.

We note that the inexact Newton iterates may also be regarded as quasi-Newton iterates (which assume perturbed Jacobians at each step) or as inexact perturbed Newton iterates (which assume perturbed Jacobians and function evaluations at each step, and then approximate solving of the resulted perturbed linear systems) [4], but these models do not lead to better estimations in our case. To this end, we can proceed in a straightforward manner, as shown in the main result of this note.

Theorem 2. *Suppose there exist $r_1 > 0$, $p \in (0,1]$, $K_p > 0$ and a norm $\|\cdot\|$ in $\mathbb{R}^n$ such that G is differentiable on B_{r_1}, with G' Hölder continuous at exponent p:*

$$\|G'(x) - G'(y)\| \le K_p \|x - y\|^p, \qquad \forall x, y \in B_{r_1}.$$

Moreover, assume that

$$\|G'(x^*)\| \le q < 1 \tag{4}$$

and denote

$$\begin{aligned} r_2 &= \left(2\tfrac{1-q}{K_p}\right)^{\frac{1}{p}}, \\ r &= \min\{r_1, r_2\}. \end{aligned}$$

Then, for any initial approximation $x_0 \in B_r$, the successive approximations remain in B_r and converge (q-)linearly:

$$\|x_{k+1} - x^*\| \le t \|x_k - x^*\|, \qquad k = 0, 1, \ldots, \tag{5}$$

[1]For the rigorous definitions of the convergence rates and for different relating results we refer the reader to [7, ch.9] and [9] (see also [12, ch.3] and [10]).

where $t = \frac{K_p}{2}\|x_0 - x^*\|^p + q < 1.$

Therefore,

$$\|x_k - x^*\| \le t^k \|x_0 - x^*\|, \qquad k = 1, 2, \dots.$$

Proof. The continuity hypothesis on G' implies (see, e.g., [7, 3.2.12]):

$$\big\|G(x) - G(x^*) - G'(x^*)(x - x^*)\big\| \le \tfrac{K_p}{2}\|x - x^*\|^{1+p}, \qquad \forall x \in B_{r_1}.$$

Next,

$$\begin{aligned}\|x_1 - x^*\| &\le \|G(x_0) - G(x^*) - G'(x^*)(x_0 - x^*)\| + \|G'(x^*)(x_0 - x^*)\|.\\ &\le \big(\tfrac{K_p}{2}\|x_0 - x^*\|^p + q\big)\|x_0 - x^*\| := t\|x_0 - x^*\|.\end{aligned}$$

The key condition is $t < 1$, which yields $\|x_0 - x^*\| < r_2$. The rest of the proof follows by induction. □

Remark 2. a) The relationship between condition (2) and the existence of a norm such that (4) holds is the following. Condition (4) implies (2), since the spectral radius satisfies

$$\rho\big(G'(x^*)\big) \le \|G'(x^*)\|, \quad \text{for any norm } \|\cdot\| \text{ on } \mathbb{R}^n. \tag{6}$$

Condition (2) does not imply (4) in any norm, but for any $\varepsilon > 0$ there exists a norm $\|\cdot\|_\varepsilon$ on $\mathbb{R}^n$ such that (see, e.g., [7, 2.2.8], [8, Th.19.3]):

$$\sigma \le \|G'(x^*)\|_\varepsilon \le \sigma + \varepsilon. \tag{7}$$

In the case of a Banach space instead of $\mathbb{R}^n$, the statements regarding relations (6) and (7) remain true, with the remark that the involved norms are equivalent to the initial one (see, e.g., [14, p. 795]). Therefore, the relationship between (2) and (4) remains the same.

b) When G' is Lipschitz continuous, i.e., Hölder continuous with $p = 1$ and $L := K_1$, we get that

$$r_2 = 2\frac{1-q}{L}$$

and $t = \frac{L}{2}\|x_0 - x^*\| + q$.

c) Our result implies the necessary condition

$$\|G'(x)\| < 2 - q, \qquad \forall x \in B_r, \tag{8}$$

regardless of the value of the Hölder or Lipschitz continuity constant. This is obtained by taking into account the triangle inequality

$$\|G'(x)\| \le \|G'(x) - G'(x^*)\| + \|G'(x^*)\|.$$

Example 2 below shows a concrete situation when $\|G'(x)\| = 2-q = 2$ for $\|x-x^*\| = r$, and therefore that condition (8) is sharp. Consequently, the assumptions we have considered do not necessarily require contractive-type nonlinear mappings.

d) We also note that the assumptions of our result imply decreasing values for $\|x_k - x^*\|$, $k = 0, 1, \dots$, which is not required in the definition of an attraction point.

The following example shows that our estimation may in certain cases be sharp.

Example 2. Let $G : \mathbb{R} \to \mathbb{R}$, $G(x) = x^2$, having $x^* = 0$ as attraction point; $r_1 > 0$ may be chosen arbitrarily large, the derivative G' is Lipschitz continuous on $\mathbb{R}$, with $L = 2$. Since $G'(0) = 0$, one obtains

$$r = r_2 = 1,$$

which is sharp.

Remark 3. We notice that the predicted radius r_2 may vary inverse proportionally with r_1, since the Hölder continuity constant may increase with r_1. In the previous example, if we take $r_1 = \frac{1}{2}$ we get $L = 1$ and then $r_2 = 2$, which is too large. Analogously, we can obtain too small values for r_2 if we take for instance $G(x) = x^3$ and large values for r_1.

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