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ON AN APPROXIMATION OPERATOR AND ITS LIPSCHITZ CONSTANT

MARIA CRĂCIUN*

Abstract. In this note we consider an approximation operator of Kantorovich type in which expression appears a basic sequence for a delta operator and a Sheffer sequence for the same delta operator. We give a convergence theorem for this operator and we find its Lipschitz constant.

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1. INTRODUCTION

In this section we will remind some basic notions and results.

Let P be the linear space of all polynomials with real coefficients.

A polynomial sequence is a sequence of polynomials (p_n) with deg $p_n = n$ for all $n \in \mathbb{N}$.

A sequence of binomial type (a binomial sequence) is a polynomial sequence which satisfies the binomial identity

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, p_{n-k}(y)$$

for all real x, y and $n = 0, 1, 2, \ldots$.

The shift operator $E^a : P \to P$ is defined by $E^a p(x) = p(x+a)$.

A linear operator T with $TE^a = E^a T$ for all real a is called a shift invariant operator.

We recall that if T_1 and T_2 are shift invariant operators then $T_1T_2 = T_2T_1$.

A delta operator is a shift invariant operator for which $Qx = const. \neq 0$.

A polynomial sequence (p_n) is called a basic sequence for a delta operator Q if $p_0(x) = 1$, $p_n(0) = 0$ and $Qp_n = np_{n-1}$, $n = 1, 2, \ldots$.

PROPOSITION 1. [9]. i) *Every delta operator has a unique basic sequence.*

ii) *A polynomial sequence is a binomial sequence if and only if it is a basic sequence for a delta operator Q.*

The Pincherle derivative of an operator T is defined by $T' = TX - XT$, where X is the multiplication operator, $Xp(x) = xp(x)$.

* "T. Popoviciu" Institute of Numerical Analysis, P.O. Box 68-1, 3400 Cluj-Napoca, Romania, e-mail: craciun@ictp.acad.ro.

The Pincherle derivative of a shift invariant operator is also a shift invariant operator and the Pincherle derivative of a delta operator is an invertible operator.

A polynomial sequence $(s_n)_{n\geq 0}$ is called a Sheffer sequence relative to a delta operator Q if $s_0(x) = const \neq 0$ and $Qs_n = ns_{n-1}$, $n = 1, 2, \ldots$.

An Appell sequence is a Sheffer sequence relative to the derivative D.

PROPOSITION 2. [9]. *Let Q be a delta operator with the basic sequence (p_n) and (s_n) a polynomial sequence. The following statements are equivalent:*

i) *s_n is a Sheffer set relative to Q.*

ii) *There exists an invertible shift invariant operator S such that $s_n(x) = S^{-1}p_n(x)$.*

iii) *For all x, $y \in \mathbb{R}$ and $n = 0, 1, 2, \ldots$ the following identity holds:*

$$s_n(x+y) = \sum_{k=n}^{n} \binom{n}{k} p_k(x)\, s_{n-k}(y).$$

2. AN APPROXIMATION OPERATOR OF KANTOROVICH TYPE

In our paper [3] we considered some linear approximation operators defined for all $f \in C[0,1]$ and $x \in [0,1]$ by

$$\left(L_n^{Q,S} f\right)(x) = \frac{1}{s_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, s_{n-k}(1-x)\, f\left(\tfrac{k}{n}\right), \tag{1}$$

where (p_n) is the basic sequence for a delta operator Q and (s_n) is a Sheffer sequence for the same delta operator, $s_n(1) \neq 0$, $\forall n \in \mathbb{N}$, $s_n = S^{-1}p_n$ with S an invertible shift invariant operator.

We remind that if $p_k'(0) \geq 0$ and $s_k(0) \geq 0$ for $n = 0, 1, 2, \ldots$ then the operator $L_n^{Q,S}$ defined by (1) is positive.

In this note we want to introduce an integral operator of Kantorovich type of the form

$$\left(K_n^{Q,S} f\right)(x) = \tfrac{(n+1)}{s_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, s_{n-k}(1-x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\, \mathrm{d}t, \tag{2}$$

where $f \in L_1([0,1])$, $x \in [0,1]$.

We mention that for $S = I$ (that means $s_n = p_n$) these operators were considered by O. Agratini in [1] and V. Miheşan in [6].

We recall that the expressions of the operator $L_n^{Q,S}$ on the test functions $e_k(x) = x^k$, $k = \overline{0,2}$ are (see [3]):

$$\begin{aligned}
\left(L_n^{Q,S} e_0\right)(x) &= e_0(x), \\
\left(L_n^{Q,S} e_1\right)(x) &= a_n e_1(x), \\
\left(L_n^{Q,S} e_2\right)(x) &= b_n x^2 + x\left(a_n - b_n - c_n\right),
\end{aligned}$$

where

$$a_n = \frac{\left((Q')^{-1}s_{n-1}\right)(1)}{s_n(1)}, \quad b_n = \frac{n-1}{n}\frac{\left((Q')^{-2}s_{n-2}\right)(1)}{s_n(1)}, \quad c_n = \frac{n-1}{n}\frac{\left((Q')^{-2}(S^{-1})'Ss_{n-2}\right)(1)}{s_n(1)}$$

and Q' is the Pincherle derivative of Q.

Lemma 3. *If $K_n^{Q,S}$ is the linear operator defined by (2) then:*

$$\left(K_n^{Q,S}e_0\right)(x) = e_0(x),$$
$$\left(K_n^{Q,S}e_1\right)(x) = \tfrac{n}{n+1}a_n e_1(x) + \tfrac{1}{2(n+1)},$$
$$\left(K_n^{Q,S}e_2\right)(x) = \tfrac{1}{(n+1)^2}\left\{x^2n^2b_n + x[n^2(a_n - b_n - c_n) + na_n] + \tfrac{1}{3}\right\}.$$

Proof. If we denote $s_{n,k}(x) = \frac{1}{s_n(1)}\binom{n}{k}p_k(x)\,s_{n-k}(1-x)$ we have

$$\left(K_n^{Q,S}e_0\right)(x) = (n+1)\sum_{k=0}^{n} s_{n,k}(x)\left(\tfrac{k+1}{n+1} - \tfrac{k}{n+1}\right) = 1 = e_0(x),$$

$$\begin{aligned}\left(K_n^{Q,S}e_1\right)(x) &= \tfrac{n}{n+1}\sum_{k=0}^{n} s_{n,k}(x)\tfrac{k}{n} + \tfrac{1}{2(n+1)}\sum_{k=0}^{n} s_{n,k}(x)\\ &= \tfrac{n}{n+1}\left(L_n^{Q,S}e_1\right)(x) + \tfrac{1}{2(n+1)}\left(L_n^{Q,S}e_0\right)(x)\\ &= \tfrac{n}{n+1}a_n e_1(x) + \tfrac{1}{2(n+1)},\end{aligned}$$

$$\begin{aligned}\left(K_n^{Q,S}e_2\right)(x) &= \tfrac{n^2}{(n+1)^2}\left(L_n^{Q,S}e_2\right)(x) + \tfrac{n}{(n+1)^2}\left(L_n^{Q,S}e_1\right)(x)\\ &\quad + \tfrac{1}{3(n+1)^2}\left(L_n^{Q,S}e_0\right)(x)\\ &= \tfrac{1}{(n+1)^2}\left\{x^2n^2b_n + x\left[n^2(a_n - b_n - c_n) + na_n\right] + \tfrac{1}{3}\right\}. \qquad \square\end{aligned}$$

From this Lemma, the central moments of $K_n^{Q,S}$ defined by $\Omega_{n,k}(x) = K_n^{Q,S}\left((e_1 - xe_0)^k, x\right)$, $k \in \mathbb{N}$ are

$$\Omega_{n,0}(x) = 1,$$
$$\Omega_{n,1}(x) = \tfrac{1}{n+1}\left[x(na_n - (n+1)) + \tfrac{1}{2}\right] \quad \text{and}$$

$$\begin{aligned}\Omega_{n,2}(x) = \tfrac{1}{(n+1)^2}\Big\{&x^2\left[n^2(b_n - 2a_n + 1) + 2n(1 - a_n)\right]\\ &+ x\left[n^2(a_n - b_n - c_n) + n(a_n - 1) - 1\right] + \tfrac{1}{3}\Big\}.\end{aligned}$$

Theorem 4. *Let $K_n^{Q,S}$ be the linear operator defined by (2) with $p_k'(0) \geq 0$ and $s_k(0) \geq 0$, $\forall k \in \mathbb{N}$.*

i) *If $f \in C[0,1]$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1$ then $K_n^{Q,S}$ converges uniformly to f.*

ii) *If $f \in L_p[0,1]$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1$ then $\left\|K_n^{Q,S}f - f\right\|_p = 0$.*

Proof. In [3] we proved that if $p_k'(0) \geq 0$ and $s_k(0) \geq 0$, $\forall k \in \mathbb{N}$ then $0 \leq c_n \leq \min\left\{(1-b_n)/2, a_n - a_n^2\right\}$, so from $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1$ we have $\lim_{n\to\infty} c_n = 0$. Using Lemma 3 it results that $\lim_{n\to\infty}\left(K_n^{Q,S} e_i\right)(x) = e_i(x)$ for $i = 0, 1, 2$, and applying the convergence criterion of Bohman-Korovkin we obtain the first affirmation.

The second assertion follows immediately because the Korovkin subspaces in $C[0,1]$ are also Korovkin subspaces in $L_p[0,1]$. □

3. LIPSCHITZ CONSTANTS FOR $L_n^{Q,S}$ AND $K_n^{Q,S}$

In this section we want to find the Lipschitz constants for $L_n^{Q,S}$ and $K_n^{Q,S}$ if $f \in Lip_M \alpha$.

In [2] B.M. Brown, D. Elliot and D.F. Paget proved that the Bernstein operator ($B_n = L_n^{D,I}$) preserves the Lipschitz constant of the function f for $\alpha \in (0,1]$. V. Miheşan showed in [6] that all positive binomial operators (which can be obtained by $L_n^{Q,S}$ when $S = I$) preserve the Lipschitz constant of the function f for $\alpha \in (0,1]$ and if $f \in Lip_M^*(\alpha, [0,1])$ then $L_n^{Q,I} f \in Lip_{2M}^*(\alpha, [0,1])$, where

$$Lip_M^*(\alpha, [0,1]) = \left\{f \in C[0,1], \omega_2(f,h) \leq Mh^\alpha,\ 0 < h \leq \tfrac{1}{2}\right\}.$$

Theorem 5. *If $f \in Lip_M \alpha$, $\alpha \in (0,1]$, then $L_n^{Q,S} f \in Lip_{Ma_n^\alpha} \alpha$.*

Proof. Let $x \leq y$ be any two points of $[0,1]$. Using the binomial identity for p_n we can write

$$\begin{aligned}\left(L_n^{Q,S} f\right)(y) &= \tfrac{1}{s_n(1)} \sum_{j=0}^{n} \tbinom{n}{j} p_j\big(x + (y-x)\big) s_{n-j}(1-y) f\big(\tfrac{j}{n}\big)\\ &= \tfrac{1}{s_n(1)} \sum_{j=0}^{n} \tbinom{n}{j} s_{n-j}(1-y) f\big(\tfrac{j}{n}\big) \sum_{k=0}^{j} \tbinom{j}{k} p_k(x) p_{j-k}(y-x).\end{aligned}$$

If we change the order of summation and note $j - k = l$ then we obtain

$$\left(L_n^{Q,S} f\right)(y) = \tag{3}$$

$$= \tfrac{1}{s_n(1)} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \tfrac{n!}{k!l!(n-k-l)!} p_k(x) p_l(y-x) s_{n-k-l}(1-y) f\left(\tfrac{k+l}{n}\right),$$

$$\begin{aligned}\left(L_n^{Q,S} f\right)(x) &= \tfrac{1}{s_n(1)} \sum_{k=0}^{n} \tbinom{n}{k} p_k(x) s_{n-k}\big((y-x) + (1-y)\big) f\left(\tfrac{k}{n}\right)\\ &= \tfrac{1}{s_n(1)} \sum_{k=0}^{n} \tbinom{n}{k} p_k(x) f\left(\tfrac{k}{n}\right) \sum_{l=0}^{n-k} \tbinom{n-k}{l} p_l(y-x) s_{n-k-l}(1-y) f\left(\tfrac{k}{n}\right),\end{aligned}$$

$$\begin{aligned}\left(L_n^{Q,S} f\right)(x) &= \\ &= \tfrac{1}{s_n(1)} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \tfrac{n!}{k!l!(n-k-l)!} p_k(x) p_l(y-x) s_{n-k-l}(1-y) f\left(\tfrac{k}{n}\right).\end{aligned} \tag{4}$$

From (3) and (4) we have

$$\left| \left(L_n^{Q,S} f\right)(y) - \left(L_n^{Q,S} f\right)(x) \right| =$$

$$= \tfrac{1}{s_n(1)} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \tfrac{n!}{k!l!(n-k-l)!} p_k(x)\, p_l(y-x)\, s_{n-k-l}(1-y) \left| f\left(\tfrac{k+l}{n}\right) - f\left(\tfrac{k}{n}\right) \right|.$$

Because $f \in Lip_M \alpha$ we have $\left|f\left(\frac{k+l}{n}\right) - f\left(\frac{k}{n}\right)\right| \leq M\left(\frac{l}{n}\right)^\alpha$ so we obtain

$$\begin{aligned}
&\left|\left(L_n^{Q,S} f\right)(y) - \left(L_n^{Q,S} f\right)(x)\right| \leq \\
&\leq \tfrac{M}{s_n(1)} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \tfrac{n!}{k!l!(n-k-l)!} p_k(x)\, p_l(y-x)\, s_{n-k-l}(1-y) \left(\tfrac{l}{n}\right)^\alpha \\
&= \tfrac{M}{s_n(1)} \sum_{l=0}^{n} \sum_{k=0}^{n-l} \tbinom{n-l}{k} p_k(x)\, s_{n-k-l}(1-y) \tbinom{n}{l} p_l(y-x) \left(\tfrac{l}{n}\right)^\alpha \\
&= \tfrac{M}{s_n(1)} \sum_{l=0}^{n} \tbinom{n}{l} p_l(y-x)\, s_{n-l}(x+1-y) \left(\tfrac{l}{n}\right)^\alpha \\
&= M L_n^{Q,S}(x^\alpha; y-x).
\end{aligned}$$

We remind that for a convex function f we have $f(a_n x) \leq \left(L_n^{Q,S} f\right)(x)$ (see [3]). Since the function $g(x) = -x^\alpha$, $\alpha \in (0,1]$, is convex on $[0,1]$ we obtain

$$\left(L_n^{Q,S} x^\alpha; y-x\right) \leq \left(a_n(y-x)\right)^\alpha$$

and we get

$$\left| \left(L_n^{Q,S} f\right)(y) - \left(L_n^{Q,S} f\right)(x) \right| \leq M a_n^\alpha (y-x)^\alpha.$$

Therefore $L_n^{Q,S} f \in Lip_{M a_n^\alpha} \alpha$. □

THEOREM 6. *If* $f \in Lip_M \alpha$, $\alpha \in (0,1]$, *then* $K_n^{Q,S} f \in Lip_{N_n} \alpha$, *where* $N_n = M\left(\frac{n a_n}{n+1}\right)^\alpha$.

Proof. We can write $K_n^{Q,S} f = L_n^{Q,S} h_n$, where

$$h_n(x) = \int_0^1 f\left(\tfrac{t+nx}{n+1}\right) \mathrm{d}t,$$

$$\begin{aligned}
|h_n(x) - h_n(y)| &= \left| \int_0^1 \left[f\left(\tfrac{t+nx}{n+1}\right) - f\left(\tfrac{t+ny}{n+1}\right) \right] \mathrm{d}t \right| \\
&\leq M \left| \tfrac{t+nx}{n+1} - \tfrac{t+ny}{n+1} \right|^\alpha \leq M\left(\tfrac{n}{n+1}\right)^\alpha |x-y|^\alpha.
\end{aligned}$$

So, $f \in Lip_M \alpha$ implies $h_n \in Lip_{M\left(\frac{n}{n+1}\right)^\alpha} \alpha$. From $K_n^{Q,S} f = L_n^{Q,S} h_n$ and the previous theorem we obtain the conclusion. □

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