

ON COMPOUND OPERATORS CONSTRUCTED WITH
BINOMIAL AND SHEFFER SEQUENCES*

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Abstract. In this note we consider a general compound approximation operator using binomial sequences and we give a representation for its corresponding remainder term. We also introduce a more general compound approximation operator using Sheffer sequences. We provide convergence theorems for both studied operators.

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1. INTRODUCTION

A sequence of polynomials $(p_m(x))_{m \geq 0}$ is called a sequence of binomial type if $\deg p_m = m$, $\forall m \in \mathbb{N}$, and satisfies the identities

$$p_m(x+y) = \sum_{k=0}^m \binom{m}{k} p_k(x) p_{m-k}(1-x).$$

We will denote by E^a the shift operator defined by $(E^a p)(x) = p(x+a)$, for every polynomial p and every real number x .

A linear operator T is said to be shift invariant if it commutes with the shift operator E^a , for every real number a .

Sequences of binomial type are connected with the notion of theta operators (J. F. Steffensen [27], [28]) which were called delta operators by F. B. Hildebrand [7] and G.-C. Rota and his collaborators [15].

A delta operator Q is a shift invariant operator for which $Qx = \text{const.} \neq 0$.

DEFINITION 1. *Let Q be a delta operator. A sequence $(p_m(x))_{m \geq 0}$ is a sequence of basic polynomials for Q (basic sequence, for short) if:*

- i) $p_0 = 1$,
- ii) $p_m(0) = 0$, if $m \geq 1$,
- iii) $Qp_m = mp_{m-1}$, if $m \geq 1$.

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For every delta operator there exists a unique basic sequence. A polynomial sequence is a sequence of binomial type if and only if it is the sequence of basic polynomials for a delta operator.

Sequences of binomial type were called poweroids by Steffensen [27], because the action of any delta operator on the binomial sequence, which is its basic sequence, is the same as the action of the derivative D on x^m .

DEFINITION 2. A sequence of polynomials $(s_m(x))_{m \geq 0}$ is called a Sheffer sequence for Q if:

- i) $s_0 = \text{const.} \neq 0$,
- ii) $Qs_m = ms_{m-1}$, if $m \geq 1$.

It is known [15] that if $(s_m(x))_{m \geq 0}$ is a Sheffer sequence for a delta operator Q with the basic sequence $(p_m(x))_{m \geq 0}$ then there exists a shift invariant and invertible operator S such that $s_m = S^{-1}p_m$, $\forall m \in \mathbb{N}$, so every pair (Q, S) gives us a unique Sheffer sequence.

A Sheffer sequence satisfies the relations

$$(1) \quad s_m(x+y) = \sum_{k=0}^m \binom{m}{k} p_k(x) s_{m-k}(1-x), \quad \forall m \in \mathbb{N}.$$

A Sheffer sequence for the usual derivative D is an Appell sequence.

The Umbral Calculus allows a unified and simple study of binomial, Appell and Sheffer sequences. More details about Umbral Calculus can be found in [15], [5] and [6].

T. Popoviciu proposed in [14] the use of binomial sequences in order to construct a class of approximation operators of the form

$$(2) \quad (T_m^Q f)(x) = \frac{1}{p_m(1)} \sum_{k=0}^m \binom{m}{k} p_k(x) p_{m-k}(1-x) f\left(\frac{k}{m}\right),$$

for every function $f \in C[0, 1]$.

This kind of operators and their generalizations were intensively studied. They interpolate the function f at 0 and 1 and preserve the polynomials of degree one. The expressions for $T_m^Q e_n$, $n \geq 2$, were computed by C. Manole (see [10] and [11]) using the umbral calculus and later by P. Sablonnière using the generating function for the binomial sequences (see [16]). We mention that Sablonnière called them Bernstein–Sheffer operators while D. D. Stancu called them binomial operators of Tiberiu Popoviciu type.

Different results regarding the operator T_m^Q were obtained by several authors: D. D. Stancu and M. R. Occorsio found representations for the remainder in the approximation formula $f(x) = (T_m^Q f)(x) + (R_m^Q f)(x)$ [24]; V. Miheşan proved that T_m^Q preserve the Lipschitz constant for a Lipschitz function [12]; D. D. Stancu and A. Vernescu studied bivariate operators of this

type [26]; O. Agratini considered a generalization of T_m^Q in the Kantorovich sense [1]; L. Lupaş and A. Lupaş introduced and studied a modified operator of binomial type replacing x by mx and 1 by m [9], [8]. More details about the role of the binomial polynomials in the Approximation Theory can be found in [2], [8] and [24].

2. COMPOUND POWEROID OPERATORS

Let Q be a delta operator with the basic sequence $(p_m(x))_{m \geq 0}$. If $p_m(1) \neq 0$, $\forall m \in \mathbb{N}$, for every function $f \in C[0, 1]$ we consider the general approximation operator defined by

$$(3) \quad (S_{m,r,s}^Q f)(x) = \sum_{k=0}^{m-sr} p_{m-sr,k}^Q(x) \sum_{j=0}^s p_{s,j}^Q(x) f\left(\frac{k+jr}{m}\right),$$

where $p_{n,k}^Q(x) = \binom{n}{k} \frac{p_k(x)p_{n-k}(1-x)}{p_n(1)}$, while s and r are two nonnegative integers satisfying the condition $2sr \leq m$.

If $p'_m(0) \geq 0$, $\forall m \in \mathbb{N}$, then $p_m(x) \geq 0$, $\forall x \in [0, 1]$, so the operator $S_{m,r,s}^Q f$ is a positive approximation operator.

Different instances of this compound poweroid operator were previously studied by D. D. Stancu and his collaborators as follows:

1. For $Q = D$, $p_k(x) = x^k$, $s = 1$ the corresponding compound operator was introduced and studied by D. D. Stancu (see [18]); if s is arbitrary, the operator $S_{m,r,s}^D$ is a special case of the operator $L_{m,r_1,\dots,r_s}^{\alpha,\beta}$, considered by D. D. Stancu in [19] (in fact $S_{m,r,s}^D$ is obtained from $L_{m,r_1,\dots,r_s}^{\alpha,\beta}$ when $\alpha = \beta = 0$ and $r_1 = r_2 = \dots = r_s = r$);
2. The case obtained for $Q = \frac{1}{\alpha} \nabla_\alpha = \frac{I-E^{-\alpha}}{\alpha}$, $p_k^\alpha(x) = x^{[k,-\alpha]}$ was studied by D. D. Stancu and J. W. Drane in [23];
3. D. D. Stancu and A. C. Simoncelli studied in [25] the compound poweroid operator for $Q = \frac{1}{\alpha} E^{-\beta} \nabla_\alpha = \frac{1}{\alpha} (E^{-\beta} - E^{-\alpha-\beta})$, $p_k^{\alpha,\beta}(x) = x(x + \alpha + k\beta)^{[k-1,-\alpha]}$. They proved that if $\alpha = \alpha(m) \rightarrow 0$, $m\beta(m) \rightarrow 0$, as $m \rightarrow \infty$, then $(S_{m,r,s}^{\alpha,\beta} f)$ converges uniformly to f on the interval $[0, 1]$. Using the Peano theorem, the authors also gave a representation of the remainder $R_{m,r,s}^{\alpha,\beta} f$ for the approximation formula

$$f(x) = (S_{m,r,s}^{\alpha,\beta} f)(x) + (R_{m,r,s}^{\alpha,\beta} f)(x)$$

D. D. Stancu considered also a class of linear positive compound operators $S_{m,r,s}^{\alpha,\beta,\gamma,\delta} f$ (see [22]) with modified knots defined by the following relation

$$(S_{m,r,s}^{\alpha,\beta,\gamma,\delta} f)(x) = \sum_{k=0}^{m-sr} p_{m-sr,k}^{\alpha,\beta}(x) \sum_{j=0}^s p_{s,j}^{\alpha,\beta}(x) f\left(\frac{k+jr+\gamma}{m+\delta}\right),$$

where $0 \leq \gamma \leq \delta$.

If $s = 0$ or $r = 0$ then $S_{m,r,0}^Q$ and $S_{m,0,s}^Q$ reduce to the binomial operator of T. Popoviciu T_m^Q , defined by (2).

From the definition of a basic sequence it results that

$$p_{n,k}^Q(0) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0 \end{cases} \quad \text{and} \quad p_{n,k}^Q(1) = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n. \end{cases}$$

Using these relations we obtain that the polynomial $S_{m,r,s}^Q f$ interpolates f at both sides of the interval $[0, 1]$, that is, $(S_{m,r,s}^Q f)(0) = f(0)$, $(S_{m,r,s}^Q f)(1) = f(1)$.

LEMMA 3. *The values of the operator $S_{m,r,s}^Q$ for the test functions are*

$$(4) \quad \begin{aligned} (S_{m,r,s}^Q e_0)(x) &= e_0(x), \\ (S_{m,r,s}^Q e_1)(x) &= e_1(x), \\ (S_{m,r,s}^Q e_2)(x) &= x^2 + x(1-x)A_{m,s,r}^Q, \end{aligned}$$

where $A_{m,s,r}^Q = \frac{(m-sr)^2 d_{m-sr}^Q + r^2 s^2 d_s^Q}{m^2}$ and $d_m^Q = 1 - \frac{m-1}{m} \frac{(Q')^{-2} p_{m-2}(1)}{p_m(1)}$.

Proof. From the definition of a sequence of binomial type we have that $\sum_{k=0}^m p_{m,k}^Q(x) = 1$, so it follows that

$$\begin{aligned} (S_{m,r,s}^Q e_0)(x) &= \sum_{k=0}^{m-sr} p_{m-sr,k}^Q(x) \sum_{j=0}^s p_{s,j}^Q(x) = 1 = e_0(x), \\ (S_{m,r,s}^Q e_1)(x) &= \frac{1}{m} \sum_{k=0}^{m-sr} p_{m-sr,k}^Q(x) \left[k \sum_{j=0}^s p_{s,j}^Q(x) + r \sum_{j=0}^s j p_{s,j}^Q(x) \right] \\ &= \frac{1}{m} \sum_{k=0}^{m-sr} p_{m-sr,k}^Q(x) \left[k (T_s^Q e_0)(x) + rs (T_s^Q e_1)(x) \right] \\ &= \frac{1}{m} \left[(m-sr)(T_{m-sr}^Q e_1)(x) + xrs(T_{m-sr}^Q e_0)(x) \right] \\ &= \frac{1}{m} \left[(m-sr)x + rsx \right] \\ &= x. \end{aligned}$$

Hence the operator $S_{m,r,s}^Q$ preserves the polynomials of degree one.

Analogously, we obtain that

$$\begin{aligned} (S_{m,r,s}^Q e_2)(x) &= \\ &= \frac{1}{m^2} \left[(m-sr)^2 (T_{m-sr}^Q e_2)(x) + 2(m-sr)srx^2 + r^2 s^2 (T_s^Q e_2)(x) \right]. \end{aligned}$$

Using the expression found by C. Manole in [11] for $T_m^Q e_2$,

$$(T_m^Q e_2)(x) = x^2 + x(1-x)d_m^Q,$$

with $d_m^Q = 1 - \frac{m-1}{m} \frac{(Q')^{-2} p_{m-2}(1)}{p_m(1)}$, we obtain

$$(S_{m,r,s}^Q e_2)(x) = x^2 + x(1-x)A_{m,s,r}^Q,$$

where $A_{m,s,r}^Q = \frac{(m-sr)^2 d_{m-sr}^Q + r^2 s^2 d_s^Q}{m^2}$. □

If $Q = D$ then $d_m^D = \frac{1}{m}$ and $A_{m,s,r}^D = \frac{m+sr(r-1)}{m^2}$.

For $Q = \frac{\nabla_\alpha}{\alpha}$ we have $d_m^{\frac{\nabla_\alpha}{\alpha}} = \frac{1+\alpha m}{(1+\alpha)m}$ so it follows that

$$A_{m,s,r}^{\frac{\nabla_\alpha}{\alpha}} = \frac{sr^2(1+\alpha s) + (m-sr)(1+\alpha(m-sr))}{(1+\alpha)m^2}.$$

From the relations (4) it results that $(S_{m,r,s}^Q e_2)(x)$ converges to $e_2(x)$ if $A_{m,s,r}^Q \rightarrow 0$, as $m \rightarrow \infty$. But, if $d_m^Q \rightarrow 0$, then $A_{m,s,r}^Q \rightarrow 0$, so using the Bohman–Korokvin criterion of convergence we have the following result.

THEOREM 4. *Let Q be a delta operator with the basic sequence $(p_m(x))_{m \geq 0}$, $p_m(1) \neq 0$ and $p'_m(0) \geq 0$, for every positive integer m . If $d_m^Q \rightarrow 0$ then the sequence of linear and positive operators $S_{m,r,s}^Q f$ converges to the function f , uniformly on the interval $[0, 1]$.*

Now we establish an estimate for the order of approximation of a function $f \in C[0, 1]$ by means of the operator $S_{m,r,s}^Q$ using the first modulus of continuity.

Taking into account an inequality proved by O. Shisha and B. Mond (see [17]), we can write

$$|f(x) - (S_{m,r,s}^Q f)(x)| \leq \left[1 + \frac{1}{\delta^2} S_{m,r,s}^Q((t-x)^2; x)\right] \omega_1(f; \delta).$$

Using the expressions obtained for $S_{m,r,s}^Q e_i$, for $i = 0, 1, 2$, we obtain that $S_{m,r,s}^Q((t-x)^2; x) = x(1-x)A_{m,s,r}^Q$. Taking into account that $x(1-x) \leq \frac{1}{4}$, $\forall x \in [0, 1]$ and replacing δ by $\sqrt{A_{m,s,r}^Q}$, we obtain that

$$|f(x) - (S_{m,r,s}^Q f)(x)| \leq \frac{5}{4} \omega_1\left(f; \sqrt{A_{m,s,r}^Q}\right).$$

EXAMPLE 1. If we consider the delta operator $T = \ln(I + D)$, its basic sequence is the sequence of exponential polynomials:

$$\varphi_m(x) = \sum_{k=1}^m S(m, k) x^k,$$

where $S(m, k) = [0, 1, \dots, k; e_m]$ are the Stirling numbers of second kind. In this case, Manole obtained

$$d_m^T = \frac{1}{m} + \frac{m-1}{m} \frac{\varphi_{m-1}(1)}{\varphi_m(1)},$$

and he proved that there exist two positive constants c_1 and c_2 such that

$$c_1 \frac{\ln m}{m} \leq \frac{\varphi_{m-1}(1)}{\varphi_m(1)} \leq c_2 \frac{\ln m}{m}.$$

Hence, $\frac{\varphi_{m-1}(1)}{\varphi_m(1)} \rightarrow 0$, as $m \rightarrow \infty$, which implies $d_m^T \rightarrow 0$. Consequently, $S_{m,r,s}^T f$ defined by

$$S_{m,r,s}^T = \sum_{k=0}^{m-sr} \frac{\varphi_k(x) \varphi_{m-sr-k}(1-x)}{\varphi_{m-sr}(1)} \sum_{j=0}^s \frac{\varphi_j(x) \varphi_{s-j}(1-x)}{\varphi_s(1)} f\left(\frac{k+jr}{m}\right)$$

converges to the function f , uniformly on the interval $[0, 1]$. \square

3. EVALUATION OF THE REMAINDER

Using Peano's theorem, the remainder of the approximation formula

$$(5) \quad f(x) = (S_{m,r,s}^Q f)(x) + (R_{m,r,s}^Q f)(x)$$

for $f \in C^2[0, 1]$ can be represented as

$$(6) \quad (R_{m,r,s}^Q f)(x) = \int_0^1 G_{m,r,s}^Q(t; x) f''(t) dt,$$

where $G_{m,r,s}^Q(t; x)$ is the Peano kernel defined by

$$G_{m,r,s}^Q(t; x) = (R_{m,r,s}^Q \varphi_x)(t), \quad \varphi_x(t) = (x-t)_+ = \frac{1}{2} [x-t + |x-t|].$$

Since the expression

$$G_{m,r,s}^Q(t; x) = (x-t)_+ - \sum_{k=0}^{m-sr} p_{m-sr,k}^Q(x) \sum_{j=0}^s p_{s,j}^Q(x) \left(\frac{k+jr}{m} - t\right)_+$$

is negative, one can apply the mean value theorem to the integral from (6) and we obtain that there exists $\xi \in [0, 1]$ such that

$$(R_{m,r,s}^Q f)(x) = f''(\xi) \int_0^1 G_{m,r,s}^Q(t; x) dt.$$

Taking $f(x) = x^2$ in the previous relation, we obtain that

$$\int_0^1 G_{m,r,s}^Q(t; x) dt = \frac{1}{2} (R_{m,r,s}^Q e_2)(x) = -\frac{1}{2} x(1-x) \frac{(m-sr)^2 d_{m-sr}^Q + r^2 s^2 d_s^Q}{m^2},$$

so it follows that, for every function $f \in C^2[0, 1]$, the remainder in formula (5) is of the following form:

$$(R_{m,r,s}^Q f)(x) = \frac{x(x-1)}{2m^2} \left[(m-sr)^2 d_{m-sr}^Q + r^2 s^2 d_s^Q \right] f''(\xi).$$

4. COMPOUND SHEFFER OPERATORS

Let Q be a delta operator with the basic sequence $(p_m(x))$, S a shift invariant and invertible operator and $s_m = S^{-1}p_m$ a Sheffer sequence. We can also generalize the operator defined in (3), by considering another compound approximation operator containing a Sheffer sequence (additionally to the basic sequence p_m):

$$(7) \quad (S_{m,r,s}^{Q,S} f)(x) = \sum_{k=0}^{m-sr} w_{m-sr,k}^{Q,S}(x) \sum_{j=0}^s w_{s,j}^{Q,S}(x) f\left(\frac{k+jr}{m}\right),$$

where $w_{n,k}^{Q,S}(x) = \binom{n}{k} \frac{p_k(x)s_{n-k}(1-x)}{s_n(1)}$.

When $S = I$ this operator reduces to the operator defined by (3).

For $s = 0$, $S_{m,r,0}^{Q,S}$ is in fact the operator that we studied in our paper [3], $(L_m^{Q,S} f)(x) = \sum_{k=0}^m w_{m,k}^{Q,S}(x) f\left(\frac{k}{m}\right)$. We remind that for this operator we have obtained the following expressions for the test functions

$$(8) \quad \begin{aligned} L_m^{Q,S} e_0 &= e_0 \\ (L_m^{Q,S} e_1)(x) &= a_m e_1(x) \\ (L_m^{Q,S} e_2)(x) &= b_m x^2 + x(a_m - b_m - c_m), \end{aligned}$$

where

$$a_m = \frac{(Q')^{-1}s_{m-1}(1)}{s_m(1)}, \quad b_m = \frac{m-1}{m} \frac{[(Q')^{-2}s_{m-2}](1)}{s_m(1)}, \quad c_m = \frac{m-1}{m} \frac{[(Q')^{-2}(S^{-1})'p_{m-2}]}{s_m(1)}.$$

If $p'_n(0) \geq 0$ and $s_n(0) \geq 0$, $\forall n \in \mathbb{N}$, then $w_{n,k}^{Q,S}(x) \geq 0$, $\forall x \in [0, 1]$ (see [3]) and so the operator $S_{m,r,s}^{Q,S} f$ is a positive approximation operator.

For the operator (7) we have $(S_{m,r,s}^{Q,S} f)(0) = f(0)$.

In the following we compute the values of the operator $S_{m,r,s}^{Q,S}$ for the test functions.

From the convolution-type relation (1) satisfied by a Sheffer sequence, it is obvious that

$$S_{m,r,s}^{Q,S} e_0 = e_0.$$

For e_1 we have

$$\begin{aligned} (S_{m,r,s}^{Q,S} e_1)(x) &= \sum_{k=0}^{m-sr} w_{m-sr,k}^{Q,S}(x) \left[\frac{k}{m} (L_s^{Q,S} e_0)(x) + \frac{rs}{m} (L_s^{Q,S} e_1)(x) \right] \\ &= \frac{m-sr}{m} (L_{m-sr}^{Q,S} e_1)(x) + \frac{a_s r s x}{m} (L_{m-sr}^{Q,S} e_0)(x), \end{aligned}$$

and using the relations (8) we obtain that

$$(S_{m,r,s}^{Q,S} e_1)(x) = x \frac{(m-sr)a_{m-sr} + r s a_s}{m}.$$

Finally, for e_2 we have

$$\begin{aligned} (S_{m,r,s}^{Q,S} e_2)(x) &= \frac{1}{m^2} \left[(m-sr)^2 (L_{m-sr}^{Q,S} e_2)(x) \right. \\ &\quad + 2rs(m-sr) (L_{m-sr}^{Q,S} e_1)(x) (L_s^{Q,S} e_1)(x) \\ &\quad \left. + r^2 s^2 (L_s^{Q,S} e_2)(x) \right]. \end{aligned}$$

Using again the relations (8) we can rewrite the last expression as

$$\begin{aligned} (S_{m,r,s}^{Q,S} e_2)(x) &= \\ &= \frac{1}{m^2} \left\{ x^2 [(m-sr)^2 b_{m-sr} + 2sr(m-sr)a_s a_{m-sr} + s^2 r^2 b_s] \right. \\ &\quad \left. + x [(m-sr)^2 (a_{m-sr} - b_{m-sr} - c_{m-sr}) + s^2 r^2 (a_s - b_s - c_s)] \right\} \end{aligned}$$

In [3] we proved that if $L_m^{Q,S}$ is a positive operator then

$$0 \leq c_m \leq \min \left\{ \frac{1-b_m}{2}, a_m - a_m^2 \right\}.$$

So, if $\lim_{m \rightarrow \infty} a_m = \lim_{m \rightarrow \infty} b_m = 1$, then $\lim_{m \rightarrow \infty} c_m = 0$. Taking into account the previous relations and applying the Bohman–Korokvin criterion convergence we can state the following result.

THEOREM 5. *If $f \in C[0, 1]$ and $\lim_{m \rightarrow \infty} a_m = \lim_{m \rightarrow \infty} b_m = 1$, then the sequence of compound operators constructed with Sheffer sequences defined by (7) converges uniformly to f on the interval $[0, 1]$.*

EXAMPLES. 1. If we consider the special case when $Q = D$, then in the expression of $S_{m,r,s}^{D,S} f$, instead of s_m , there appears an Appell sequence A_m . Because the Pincherle derivative of D is the identity operator I , we have

$$a_m = \frac{A_{m-1}(1)}{A_m(1)}, \quad b_m = \frac{m-1}{m} a_m a_{m-1}, \quad c_m = \frac{m-1}{m} a_m (1 - a_{m-1}),$$

so the condition for the convergence of the operator $S_{m,r,s}^{D,S}$ is $\lim_{m \rightarrow \infty} \frac{A_{m-1}(1)}{A_m(1)} = 1$.

2. If we take the Gould delta operator

$$G = \frac{1}{\alpha} E^{-\beta} \nabla_{\alpha} = \frac{1}{\alpha} (E^{-\beta} - E^{-\alpha-\beta})$$

and the invertible operator

$$S = E^{\alpha+\beta} G' = \frac{1}{\alpha} ((\alpha + \beta) I - \beta E^\alpha)$$

then the corresponding basic sequence and Sheffer sequence are

$$p_m(x) = x(x + \alpha + m\beta)^{[m-1, -\alpha]}, \quad \text{resp.} \quad s_m(x) = (x + m\beta)^{[m, -\alpha]},$$

so in this case

$$w_{m,k}^{G,S}(x) = \frac{1}{(1+m\beta)^{[m, -\alpha]}} \binom{m}{k} x(x + \alpha + k\beta)^{[k-1, -\alpha]} (1 - x + (m-k)\beta)^{[m-k, -\alpha]}.$$

We mention that the operator

$$(L_m^{G,S} f)(x) = \sum_{k=0}^m w_{m,k}^{G,S}(x) f\left(\frac{k}{m}\right)$$

was studied by G. Moldovan (see for example [13]).

If $\alpha \rightarrow 0$, $m\beta \rightarrow 0$, $\frac{m\beta}{\alpha} \rightarrow 0$, as $m \rightarrow \infty$, or $m\alpha \rightarrow 0$, $m\beta \rightarrow 0$, $\frac{m\beta}{\alpha} \rightarrow c$, as $m \rightarrow \infty$, then the sequence of operators $L_m^{G,S}$ converges uniformly to f on $[0, 1]$.

It can be easily proved that, in the same conditions, the operator $S_{m,r,s}^{G,S}$ converges also uniformly to f on $[0, 1]$. \square

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