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NEWTON TYPE ITERATIVE METHODS WITH HIGHER ORDER OF CONVERGENCE

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Abstract. Newton type iterative methods with higher order of convergence are obtained. The order of convergence is further increased by amalgamating these methods with the standard secant method. The methods are compared to the similar recent methods.

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1. INTRODUCTION

Quite often, we come across numerous nonlinear equations which need to be solved. If the equation is not a polynomial equation, then it is not always easy to deal with such equations. To this end, one or the other numerical iterative method is employed. One such classical standard method is the Newton method

$$x_{n+1} = x_n - \tfrac{f(x_n)}{f'(x_n)}$$

which is quadratically convergent. Over the years, a lot of methods have appeared, each one claims to be better than the other in some or the other aspect. We mention here the method given by Weerakoon and Fernando [8] which is based on the Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(\lambda)\, d\lambda$$

and the integral involved is approximated by the trapezoidal rule, *i.e.*,

$$\int_{x_n}^{x} f'(\lambda)\, d\lambda = \tfrac{(x-x_n)}{2}\big(f'(x) + f'(x_n)\big).$$

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As a result, Weerakoon and Fernando obtained the following iterative method for solving the nonlinear equation $f(x) = 0$:

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(z_{n+1})}, \tag{1}$$

where $z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

The method so obtained is of third order. In the present paper, the aim is to modify method (1). In fact, in (1), f' is a function of the previously calculated iterate. In our modification, f' would be a function of some other convenient point. It is proved that the corresponding method has order of convergence 5.1925. We follow the technique of McDougall and Wotherspoon [7] who modified Newton's method in a similar way yielding the order of convergence of their method as $1 + \sqrt{2}$.

Further, in [3], it was proved that if any method for solving nonlinear equation is used in conjunction with the standard secant method then the order of the resulting method is increased by 1. We shall show, in this paper (see Theorem 3.2), that this order can be increased by more than 1. In fact, we prove that if our own method (which is of order 5.1925) is combined with the secant method than the new method is of order 7.275.

2. THE METHOD AND THE CONVERGENCE

We propose the following method:

If x_0 is the initial approximation, then

$$\left.\begin{aligned} x_0^* &= x_0 \\ x_1 &= x_0^* - \frac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)}, \\ \text{where} \quad z_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_1^* &= x_1 - \frac{2f(x_1)}{f'(x_1)+f'(z_1^*)}, \\ \text{with} \quad z_1^* &= x_1 - \frac{f(x_1)}{f'[\frac{1}{2}(x_0+x_0^*)]} = x_1 - \frac{f(x_1)}{f'(x_0)}. \end{aligned}\right\} \tag{2}$$

Subsequently, for $n \geq 1$, the iterations can be obtained as follows:

$$\left.\begin{aligned} x_n^* &= x_n - \frac{2f(x_n)}{f'(x_n)+f'(z_n^*)}, \\ \text{where} \quad z_n^* &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_{n-1}+x_{n-1}^*)]} \\ x_{n+1} &= x_n^* - \frac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})}, \\ \text{with} \quad z_{n+1} &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n+x_n^*)]}. \end{aligned}\right\} \tag{3}$$

Below, we prove the convergence result for the method (2)–(3).

THEOREM 1. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then the method* (2)–(3) *is convergent and has the order of convergence* 5.1925.

Proof. Let e_n and e_n^* denote respectively the errors in the terms x_n and x_n^*. Also, we denote $c_j = \frac{f^j(\alpha)}{j!f'(\alpha)}$, $j = 2, 3, 4...$, which are constants. The error equation for the method (1) as obtained by Weerakoon and Fernando [8] is given by

$$e_{n+1} = ae_n^3,$$

where $a = c_2^2 + \frac{1}{2}c_3$ and we have neglected higher power terms of e_n. In particular, the error e_1 in x_1 in the equations (2) is given by

$$e_1 = ae_0^3. \tag{4}$$

We now proceed to calculate the error e_1^* in x_1^*. By using Taylor series expansion and binomial expansion, we get

$$\begin{aligned}\frac{f(x_1)}{f'(x_0)} &= \frac{f(\alpha+e_1)}{f'(\alpha+e_0)}\\ &= \big(e_1 + c_2e_1^2 + c_3e_1^3 + \mathcal{O}(e_1^4)\big)\big(1 + 2c_2e_0 + 3c_3e_0^2 + \mathcal{O}(e_0^3)\big)^{-1}\\ &= e_1 - 2c_2e_0e_1 + \mathcal{O}(e_0^5)\end{aligned}$$

so that

$$x_1 - \frac{f(x_1)}{f'(x_0)} = \alpha + 2c_2e_0e_1 + \mathcal{O}(e_0^5).$$

Consequently, by Taylor series expansion, it can be calculated that

$$f'(z_1^*) = f'(\alpha)\big(1 + 4c_2^2e_0e_1 + \mathcal{O}(e_0^5)\big).$$

Also

$$f'(x_1) = f'(\alpha)\big(1 + 2c_2^2e_1 + 3c_3e_1^2 + \mathcal{O}(e_1^3)\big)$$

so that

$$f'(x_1) + f'(z_1^*) = 2f'(\alpha)\big(1 + c_2e_1 + 2c_2^2e_0e_1 + \mathcal{O}(e_0^5)\big). \tag{5}$$

Now, using (4) and (5), the error e_1^* in x_1^* in the equation (2) can be calculated as

$$\begin{aligned}e_1^* &= e_1 - \big(e_1 + c_2e_1^2 + \mathcal{O}(e_1^3)\big)\big(1 + c_2e_1 + 2c_2^2e_0e_1 + \mathcal{O}(e_0^5)\big)^{-1}\\ &= 2c_2^2e_0e_1^2\\ &= ba^2e_0^7,\end{aligned}$$

where $b = 2c_2^2$. Using e_1^*, we now compute the error e_2 in the term

$$x_2 = x_1^* - \frac{2f(x_1^*)}{f'(x_1^*)+f'(z_2)},$$

where

$$z_2 = x_1 - \frac{f(x_1)}{f'\big(\frac{x_1+x_1^*}{2}\big)}.$$

Now

$$f'\left(\tfrac{x_1+x_1^*}{2}\right) = f'\left(\alpha + \tfrac{e_1+e_1^*}{2}\right) \\ = f'(\alpha)\left(1 + c_2e_1 + c_2e_1^* + \tfrac{3}{4}c_3e_1^2 + \mathcal{O}(e_0^9)\right)$$

so that

$$\frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)} = \left(e_1 + c_2e_1^2 + \mathcal{O}(e_1^3)\right)\left(1 + c_2e_1 + c_2e_1^* + \tfrac{3}{4}c_3e_1^2 + \mathcal{O}(e_0^9)\right)^{-1} \\ = e_1 + \tfrac{1}{4}c_3e_1^3 - c_2e_1e_1^*$$

and therefore

$$z_2 = \alpha - \tfrac{1}{4}c_3e_1^3 + c_2e_1e_1^*,$$

where the higher power terms are neglected. Thus

$$f'(z_2) = f'(\alpha)\left(1 - \tfrac{1}{2}c_2c_3e_1^3 + 2c_2^2e_1e_1^*\right)$$

and

$$f'(x_1^*) = f'(\alpha)\left(1 + 2c_2e_1^* + 3c_3e_1^{*2}\right).$$

Using the above considerations, the error e_2 in x_2 is given by

$$e_2 = e_1^* - \left(e_1^* + c_2e_1^{*2} + c_3e_1^{*3}\right)\left(1 + c_2e_1^* - \tfrac{1}{4}c_2c_3e_1^3\right)^{-1} \\ = -\tfrac{1}{4}c_2c_3e_1^3e_1^* \\ = ce_1^3e_1^*,$$

where $c = -\frac{1}{4}c_2c_3$. In fact, it can be worked out that for $n \geq 1$, the following relation holds:

$$e_{n+1} = ce_n^3e_n^*. \tag{6}$$

In order to compute e_{n+1} explicitly, we need to compute e_n^*. We already know e_1^*. We now compute e_2^*. We have

$$x_2^* = x_2 - \tfrac{2f(x_2)}{f'(x_2)+f'(z_2^*)},$$

where

$$z_2^* = x_2 - \frac{f(x_2)}{f'\left(\frac{x_1+x_1^*}{2}\right)}.$$

Like above, it can be calculated that the error e_2^* is given by

$$e_2^* = de_1e_2^2,$$

where $d = c_2^2$ and, again, it can be checked that in general, for $n \geq 2$, the following relation holds:

$$e_n^* = de_{n-1}e_n^2. \tag{7}$$

In the view of (6) and (7), the error at each stage in x_n^* and x_{n+1} are calculated which are tabulated below:

n	e_n	e_n^*
0	e_0	e_0
1	ae_0^3	$a^2be_0^7$
2	$a^5bce_0^{16}$	$a^{11}b^2c^2de_0^{35}$
3	$a^{26}b^5c^6de_0^{83}$	$a^{57}b^{11}c^{13}d^3e_0^{182}$
4	$a^{135}b^{26}c^{32}d^6e_0^{431}$	$a^{296}b^{57}c^{70}d^{14}e_0^{945}$
5	$a^{701}b^{135}c^{167}d^{32}e_0^{2238}$	
$\vdots$	$\vdots$	$\vdots$

Table 1. Successive errors.

It is observed that the powers of e_0 in the errors at each iterate form a sequence

$$3,\ 16,\ 83,\ 431,\ 2238,\ ... \tag{8}$$

and the sequence of their successive ratios is

$$\tfrac{16}{3},\ \tfrac{83}{16},\ \tfrac{431}{83},\ \tfrac{2238}{431},\ ...$$

or,

$$5.3334,\ 5.1875,\ 5.1927,\ 5.1925,\ ...$$

This sequence seems to converge to the number 5.1925 approximately. Indeed, if the terms of the sequence (8) are denoted by $\{\,\alpha_i\,\}$, then it can be seen that

$$\alpha_i = 5\alpha_{i-1} + \alpha_{i-2}, \qquad i = 2, 3, 4... \tag{9}$$

If we set the limit

$$\tfrac{\alpha_i}{\alpha_{i-1}} = \tfrac{\alpha_{i-1}}{\alpha_{i-2}} = R,$$

Then dividing (9) by α_{i-1}, we obtain

$$R^2 - 5R - 1 = 0$$

which has its positive root as $R = \frac{5+\sqrt{29}}{2} \approx 5.1925$. Hence the order of convergence of the method is at least 5.1925. □

Next, we give two variants of the method (2)–(3). Note that, in (2)–(3), the arithmetic average of the points $x_n, x_n^*,\ n = 0, 1, 2...$ has been used. We propose methods in which the arithmetic average is replaced by harmonic as well as geometric averages. With harmonic average, we propose the following method: If x_0 is the initial approximation, then

$$
(10)\qquad \left.\begin{aligned}
&& x_0^* &= x_0\\
&& x_1 &= x_0^* - \tfrac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)},\\
&\text{where} & z_1 &= x_0 - \frac{f(x_0)}{f'\left(\frac{2x_0x_0^*}{x_0+x_0^*}\right)} = x_0 - \frac{f(x_0)}{f'(x_0)}\\
&& x_1^* &= x_1 - \tfrac{2f(x_1)}{f'(x_1)+f'(z_1^*)},\\
&\text{with} & z_1^* &= x_1 - \frac{f(x_1)}{f'\left(\frac{2x_0x_0^*}{x_0+x_0^*}\right)} = x_1 - \frac{f(x_1)}{f'(x_0)}.
\end{aligned}\right\}
$$

Subsequently, for $n \geq 1$, the iterations can be obtained as follows:

$$
(11)\qquad \left.\begin{aligned}
&& x_n^* &= x_n - \tfrac{2f(x_n)}{f'(x_n)+f'(z_n^*)},\\
&\text{where} & z_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{2x_{n-1}x_{n-1}^*}{x_{n-1}+x_{n-1}^*}\right)}\\
&& x_{n+1} &= x_n^* - \tfrac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})},\\
&\text{with} & z_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{2x_nx_n^*}{x_n+x_n^*}\right)}.
\end{aligned}\right\}
$$

For the geometric average of the points $x_n, x_n^*,\ n = 0, 1, 2...$, the following method is proposed:

$$
(12)\qquad \left.\begin{aligned}
&& x_0^* &= x_0\\
&& x_1 &= x_0^* - \tfrac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)},\\
&\text{where} & z_1 &= x_0 - \frac{f(x_0)}{f'\left(\sqrt{x_0x_0^*}\right)} = x_0 - \frac{f(x_0)}{f'(x_0)}\\
&& x_1^* &= x_1 - \tfrac{2f(x_1)}{f'(x_1)+f'(z_1^*)},\\
&\text{with} & z_1^* &= x_1 - \frac{f(x_1)}{f'\left(\sqrt{x_0x_0^*}\right)} = x_1 - \frac{f(x_1)}{f'(x_0)}.
\end{aligned}\right\}
$$

Subsequently, for $n \geq 1$, the iteration can be obtained as follows:

$$
(13)\qquad \left.\begin{aligned}
&& x_n^* &= x_n - \tfrac{2f(x_n)}{f'(x_n)+f'(z_n^*)},\\
&\text{where} & z_n^* &= x_n - \frac{f(x_n)}{f'\left(\sqrt{x_{n-1}x_{n-1}^*}\right)}\\
&& x_{n+1} &= x_n^* - \tfrac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})},\\
&\text{with} & z_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\sqrt{x_nx_n^*}\right)}.
\end{aligned}\right\}
$$

The convergence of the methods (10)–(11) and (12)–(13) can be proved on the similar lines as those in Theorem 1. We only state the results below:

Theorem 2. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then for solving nonlinear equation $f(x) = 0$, the method* (10)–(11) *is convergent with order of convergence* 5.1925.

Theorem 3. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then for solving nonlinear equation $f(x) = 0$, the method* (12)–(13) *is convergent with order of convergence* 5.1925.

3. METHODS WITH HIGHER ORDER CONVERGENCE

In this section, we obtain a new iterative method by combining the iterations of method (2)–(3) with secant method and prove that the order of convergence is more than 5.1925. Precisely, we propose the following method: If x_0 is the initial approximation, then

$$\left.\begin{aligned} x_0^* &= x_0 \\ x_0^{**} &= x_0^* - \tfrac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)}, \\ \text{where} \quad z_1 &= x_0 - \tfrac{f(x_0)}{f'(x_0)} \\ x_1 &= x_0^{**} - \tfrac{x_0^{**}-x_0^*}{f(x_0^{**})-f(x_0^*)} f(x_0^{**}). \end{aligned}\right\} \tag{14}$$

Subsequently, for $n \geq 1$, the iterations can be obtained as follows:

$$\left.\begin{aligned} x_n^* &= x_n - \tfrac{2f(x_n)}{f'(x_n)+f'(z_n^*)}, \\ \text{where} \quad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)} \\ x_n^{**} &= x_n^* - \tfrac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})}, \\ \text{where} \quad z_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)} \\ x_{n+1} &= x_1^{**} - \tfrac{x_n^{**}-x_n^*}{f(x_n^{**})-f(x_n^*)} f(x_n^{**}). \end{aligned}\right\} \tag{15}$$

Remark 4. In [3], it was proved that if the iterations of any method of order p for solving nonlinear equations are used alternatively with secant method, then the new method will be of order $p + 1$. Thus, in view of that result, the method (14)–(15) is certainly of order at least 6.1925. However, we prove below that the order is more.

Theorem 5. *Let f be a function f having sufficient number of smooth derivatives in a neighborhood of α which is a simple root of the equation $f(x) = 0$. Then method* (14)–(15) *to approximate the root α is convergent with order of convergence* 7.275.

Proof. We argue on the lines of that of Theorem 1 and the error equation of the standard secant method. In particular, the errors e_0^*, e_0^{**} and e_1, respectively, in x_0^*, x_0^{**} and x_1 in equations (14) are given by

$$\begin{aligned} e_0^* &= e_0 \\ e_0^{**} &= ae_0^3, \qquad \text{where } a = c_2^2 + \tfrac{1}{2}c_3 \\ e_1 &= \lambda a e_0^4, \qquad \text{where } \lambda = c_2. \end{aligned}$$

Also, the errors e_1^* in x_1^* in equation (15) is given by

$$\begin{aligned} e_1^* &= 2c_2^2 e_0 e_1^2 \\ &= \lambda^2 a^2 b e_0^9, \qquad \text{where } b = 2c_2^2 \end{aligned}$$

and the error e_1^{**} in x_1^{**} in equation (15) is given by

$$\begin{aligned} e_1^{**} &= -\tfrac{1}{4}c_2c_3e_1^3e_1^* \\ &= ce_1^3e_1^*, \end{aligned}$$

where $c = -\frac{1}{4}c_2c_3$. In fact, it can be worked out that for $n \geq 1$, the following relation holds:

$$e_n^{**} = ce_n^3e_n^*. \tag{16}$$

In order to compute e_n^{**} explicitly, we need to compute e_n and e_n^*. We have already computed e_1 and e_1^*. From the proof of Theorem 1

$$e_2^* = de_1e_2^2,$$

where $d = c_2^2$ and, again, it can be checked that the following relation holds:

$$e_n^* = de_{n-1}e_n^2. \tag{17}$$

Also from (15), it can be shown that

$$e_2 = \lambda e_1^* e_2^{**}.$$

Thus, for $n \geq 1$, it can be shown that error e_{n+1} in x_{n+1} in the method (14)–(15) satisfies the following recursion formula

$$e_{n+1} = \lambda e_n^* e_n^{**} \tag{18}$$

Using the above information, the errors at each stage in x_n^*, x_n^{**} and x_n are obtained and tabulated as follows:

We do the analysis of Table 2 as done in the proof of Theorem 1 for Table 1. Note that the powers of e_0 in the error at each iterate from the sequence

$$4,\ 30,\ 218,\ 1586,\ 11538,\ \ldots. \tag{19}$$

and the sequence of their successive ratios is

$$\tfrac{30}{4},\ \tfrac{218}{30},\ \tfrac{1586}{218},\ \tfrac{11538}{1586},\ \cdots$$

or

$$7.5,\ 7.2667,\ 7.2752,\ 7.2749,\ \ldots.$$

n	e_n	e_n^*	e_n^{**}
0	e_0	e_0	ae_0^3
1	$\lambda a e_0^4$	$\lambda^2 a^2 b e_0^9$	$\lambda^5 a^5 bce_0^{21}$
2	$\lambda^8 a^7 b^2 c e_0^{30}$	$\lambda^{17} a^{15} b^5 c^2 e_0^{64}$	$\lambda^{42} a^{36} b^{11} c^6 e_0^{154}$
3	$\lambda^{60} a^{51} b^{13} c^8 e_0^{218}$	$\lambda^{128} a^{109} b^{29} c^{17} e_0^{466}$	$\lambda^{308} a^{260} b^{68} c^{42} e_0^{1120}$
4	$\lambda^{437} a^{369} b^{97} c^{59} e_0^{1586}$	$\lambda^{934} a^{789} b^{208} c^{126} e_0^{3390}$	$\lambda^{2245} a^{1896} b^{499} c^{304} e_0^{8148}$
5	$\lambda^{3180} a^{2685} b^{707} c^{430} e_0^{11538}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2. Successive errors.

If the terms of the sequence (19) are denoted by $\{N_i\}$, then it can be seen that

$$N_i = 7N_{i-1} + 2N_{i-2}, \quad i = 2, 3, 4, ...$$

Thus, as in Theorem 1, the rate of convergence of method (14)–(15) is at least 7.275. □

It is natural to consider the variants of the method (14)–(15), where in the expression of z_n and z_n^*, the arithmetic mean is replaced by harmonic mean as well as geometric mean as done in methods (10)–(11) and (12)–(13), respectively. Precisely, with harmonic mean, we propose the following method:

$$\left.\begin{aligned} x_0^* &= x_0 \\ x_0^{**} &= x_0^* - \tfrac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)}, \\ \text{where} \qquad z_1 &= x_0 - \frac{f(x_0)}{f'\left(\frac{2x_0x_0^*}{x_0+x_0^*}\right)} = x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_1 &= x_0^{**} - \tfrac{x_0^{**}-x_0^*}{f(x_0^{**})-f(x_0^*)} f(x_0^{**}) \end{aligned}\right\} \tag{20}$$

followed by (for $n \geq 1$)

$$\left.\begin{aligned} x_n^* &= x_n - \tfrac{2f(x_n)}{f'(x_n)+f'(z_n^*)}, \\ \text{where} \qquad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{2x_{n-1}x_{n-1}^*}{x_{n-1}+x_{n-1}^*}\right)} \\ x_n^{**} &= x_n^* - \tfrac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})}, \\ \text{where} \qquad z_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{2x_nx_n^*}{x_n+x_n^*}\right)} \\ x_{n+1} &= x_1^{**} - \tfrac{x_n^{**}-x_n^*}{f(x_n^{**})-f(x_n^*)} f(x_n^{**}) \end{aligned}\right\} \tag{21}$$

and with the geometric mean, we propose the following :

$$(22)\qquad \left.\begin{aligned} x_0^* &= x_0\\ x_0^{**} &= x_0^* - \tfrac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)},\\ \text{where}\qquad z_1 &= x_0 - \tfrac{f(x_0)}{f'\left(\sqrt{x_0x_0^*}\right)} = x_0 - \tfrac{f(x_0)}{f'(x_0)}\\ x_1 &= x_0^{**} - \tfrac{x_0^{**}-x_0^*}{f(x_0^{**})-f(x_0^*)}f(x_0^{**}) \end{aligned}\right\}$$

followed by (for $n \geq 1$)

$$(23)\qquad \left.\begin{aligned} x_n^* &= x_n - \tfrac{2f(x_n)}{f'(x_n)+f'(z_n^*)},\\ \text{where}\qquad z_n^* &= x_n - \tfrac{f(x_n)}{f'\left(\sqrt{x_{n-1}x_{n-1}^*}\right)}\\ x_n^{**} &= x_n^* - \tfrac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})},\\ \text{where}\qquad z_{n+1} &= x_n - \tfrac{f(x_n)}{f'(\sqrt{x_nx_n^*})}\\ x_{n+1} &= x_1^{**} - \tfrac{x_n^{**}-x_n^*}{f(x_n^{**})-f(x_n^*)}f(x_n^{**}). \end{aligned}\right\}$$

The convergence of the methods (20)–(21) and (22)–(23) can be proved by using the arguments as used in the proof of Theorem 5. We skip the details for conciseness.

4. ALGORITHMS AND NUMERICAL EXAMPLES

We give below an algorithm to implement the method (2)–(3):

ALGORITHM 6. *Step 1 : For the given tolerance* $\varepsilon > 0$ *and iteration* N*, choose the initial approximation* x_0 *and set* $n = 0$*.*

Step 2 : Follow the sequence of expressions:

$$\begin{aligned} x_0^* &= x_0\\ x_1 &= x_0^* - \tfrac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)},\\ \textit{where}\qquad z_1 &= x_0 - \tfrac{f(x_0)}{f'(x_0)}\\ x_1^* &= x_1 - \tfrac{2f(x_1)}{f'(x_1)+f'(z_1^*)},\\ \textit{where}\qquad z_1^* &= x_1 - \tfrac{f(x_1)}{f'\left(\frac{x_0+x_0^*}{2}\right)} = x_1 - \tfrac{f(x_1)}{f'(x_0)} \end{aligned}$$

Step 3 : For $n = 1, 2, 3, \ldots,$ calculate $x_2, x_3, x_4, \ldots$ by the following sequence of expressions:

$$x_n^* = x_n - \frac{2f(x_n)}{f'(x_n)+f'(z_n^*)},$$

$$\textit{where} \qquad z_n^* = x_n - \frac{f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)}$$

$$x_{n+1} = x_n^* - \frac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})},$$

$$\textit{where} \qquad z_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)}$$

Step 4 : Stop if either $|x_{n+1} - x_n| < \varepsilon$ or $n > N$.
Step 5 : Set $n = n + 1$ and repeat Step 3.

Example 7. We apply method (2)–(3) on the nonlinear equation

$$\cos x - xe^x + x^2 = 0. \tag{24}$$

This equation has a simple root in the interval $(0, 1)$. Taking initial approximation as $x_0 = 1$, Table 3 shows the iterations of McDougall-Wotherspoon method, a third order method (1) and our method (2)–(3).

n	W-F Method (1)	M-W method	(2)–(3) method
1.	1.1754860092539474	0.89033621746836966	0.64406452481689269
2.	0.7117526001461193	0.66469560530044569	0.63915407608296659
3.	0.63945030188514695	0.63928150457301036	0.63915411559451774
4.	0.63915408656045591	0.63915408990276223	0.6391540955014231
5.	0.63915410631623149	0.63915410965853769	0.63915407540832936
6.	0.63915412607200606	0.6391540698096656	0.6391541149198805
7.	0.63915408622313585	0.63915408956544117	0.63915409482678587
8.	0.63915410597891142	0.63915410932121663	0.63915407473369212
9.	0.639154125734686	0.63915406947234454	0.63915411424524327
10.	0.63915408588581579	0.63915408922812	0.63915409415214863
11.	0.63915410564159136	0.63915410898389557	0.63915407405905489
12.	0.63915412539736594	0.63915406913502348	0.63915411357060603
13.	0.63915408554849573	0.63915408889079894	0.6391540934775114
14.	0.63915410530427119	0.63915410864657451	0.63915407338441765
15.	0.63915412506004576	0.63915406879770231	0.6391541128959688
16.	0.63915408521117556	0.63915408855347788	0.63915409280287416
17.	0.63915410496695113	0.63915410830925345	0.63915407270978042
18.	0.6391541247227257	0.63915406846038125	0.63915411222133156
19.	0.6391540848738555	0.63915408821615682	0.63915409212823693
20.	0.63915410462963107	0.63915410797193239	0.63915407203514318

Table 3. Numerical results for different methods.

Example 8. We consider the same equation (24) but now implement method (14)–(15) and compare with other methods. Table 4, shows the corresponding iterates. One can also compare the last columns of Table 3 and Table 4 which correspond to methods (2)–(3) and (14)–(15), respectively. This clearly indicates the fast convergence of (14)–(15).

n	W-F Method (1)	M-W method	(14)–(15) method
1.	1.1754860092539474	0.89033621746836966	0.63919747126530391
2.	0.7117526001461193	0.66469560530044569	0.63915410580338361
3.	0.63945030188514695	0.63928150457301036	0.63915409891807362
4.	0.63915408656045591	0.63915408990276223	0.63915409203276374
5.	0.63915410631623149	0.63915410965853769	0.63915408514745375
6.	0.63915412607200606	0.6391540698096656	0.63915411145121981
7.	0.63915408622313585	0.63915408956544117	division by zero
8.	0.63915410597891142	0.63915410932121663	
9.	0.639154125734686	0.63915406947234454	
10.	0.63915408588581579	0.63915408922812	
11.	0.63915410564159136	0.63915410898389557	
12.	0.63915412539736594	0.63915406913502348	
13.	0.63915408554849573	0.63915408889079894	
14.	0.63915410530427119	0.63915410864657451	
15.	0.63915412506004576	0.63915406879770231	
16.	0.63915408521117556	0.63915408855347788	
17.	0.63915410496695113	0.63915410830925345	
18.	0.6391541247227257	0.63915406846038125	
19.	0.6391540848738555	0.63915408821615682	
20.	0.63915410462963107	0.63915410797193239	

Table 4. Numerical results for different methods.

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