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A SUMMATION-INTEGRAL TYPE MODIFICATION OF SZÁSZ-MIRAKJAN-STANCU OPERATORS

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Abstract. In this paper, we introduce a summation-integral type modification of Szasz-Mirakjan-Stancu operators. Calculation of moments, density theorem, a direct result and a Voronovskaja-type result are obtained for the operators.

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1. INTRODUCTION

In 1941, G.M. Mirakjan [17] defined the operators $SM_n : C_2[0,\infty) \to C[0,\infty)$ for any $x \in [0,\infty)$ and for any $n \in \mathbb{N}$ given by,

$$SM_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\tfrac{k}{n}\right), \tag{1}$$

where

$$s_{n,k}(x) = e^{-nx}\tfrac{(nx)^k}{k!}, \qquad 0 \le x < \infty. \tag{2}$$

and

$$C_2[0,\infty) = \left\{ f \in C[0,\infty) : \lim_{x\to\infty} \tfrac{f(x)}{1+x^2} \text{ exists and is finite} \right\}.$$

The operators $(SM_n)_{n\in\mathbb{N}}$ are named Szász-Mirakjan operators, where $s_{n,k}$'s are Szász basis functions. They were extensively studied in 1950 by O. Szász [19]. A modification of operators (1) was introduced by P.L. Butzer [7], in order to obtained an approximation process for spaces of integrable functions, on unbounded intervals, which are now known as Szász-Mirakjan-Kantorovich operators.

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Durrmeyer [9] defined the summation-integral type approximation process, using the Bernstein polynomials, as

$$D_n(f;x) = (n+1)\sum_{k=0}^{n} b_{n,k}(x)\left(\int_0^1 b_{n,k}(t)f(t)\,dt\right), \tag{3}$$

where Bernstein polynomial are given by

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\tfrac{k}{n}\right),$$

and

$$b_{n,k}(x) = \tbinom{n}{k} x^k (1-x)^{n-k},$$

$0 \le x \le 1$, $k = 0,1,\dots,n$ and $n \in \mathbb{N}$.

Derriennic [8] studied the operators given by (3) extensively. Motivated by Derriennic, Sahai and Prasad [18] studied many properties of the modified Lupaş operators of the type

$$M_n(f;x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\left(\int_0^{\infty} p_{n,k}(t)f(t)\,dt\right), \tag{4}$$

where

$$p_{n,k}(x) = \tbinom{n+k-1}{k}\tfrac{x^k}{(1+x)^{n+k}},$$

$0 \le x < \infty$, $k = 0,1,2,\dots$ and $n \in \mathbb{N}$.

Mazhar and Totik [16] introduced two Durrmeyer type modifications of Szász-Mirakjan operators (1) as

$$\bar{S}_n(f;x) = f(0)s_{n,0}(x) + n\sum_{k=1}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k-1}(t)f(t)\,dt$$

and

$$S_n(f;x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k}(t)f(t)\,dt, \tag{5}$$

where $s_{n,k}$'s are as given by (2).

Various properties, like global approximation in weight spaces, uniform approximation, simultaneous approximation, weighted approximations, of these operators, their generalizations and modifications are studied over the years. We can mention some important studies of this type (see [1]–[3], [5]–[15], [21]–[23], [25]–[32]).

In 2015, Mishra *et al.* [24] introduced Szász-Mirakjan-Durrmeyer-type generalization of (5) given by

$$S_n^*(f;x) = b_n\sum_{k=0}^{\infty} s_{b_n,k}(x)\int_0^{\infty} s_{b_n,k}(t)f(t)\,dt, \tag{6}$$

where

$$s_{b_n,k}(x) = e^{-b_n x}\tfrac{(b_n x)^k}{k!}, \quad k = 0,1,2,\dots; n \in \mathbb{N}, \tag{7}$$

(b_n) is an increasing sequence of positive real numbers, $b_n \to \infty$ as $n \to \infty$, $b_1 \geq 1$ and studied the simultaneous approximation properties of the operators (6).

In this article, we introduce Stancu [20] type summation integral type of modification for $0 \leq \alpha \leq \beta$, for any $x \in [0, \infty)$ and for any $n \in \mathbb{N}$ of (1) given by

$$S_n^{*,\alpha,\beta}(f;x) = b_n \sum_{k=0}^{\infty} s_{b_n,k}(x) \int_0^{\infty} s_{b_n,k}(t) f\left(\tfrac{b_n t+\alpha}{b_n+\beta}\right) dt, \tag{8}$$

where $s_{b_n,k}$'s are as given in (7), (b_n) is an increasing sequence of positive real numbers with $b_n \to \infty$ as $n \to \infty$ and $b_1 + \beta \geq 1$. The operators $\left(S_n^{*,\alpha,\beta}\right)_{n \in \mathbb{N}}$, given by (8), are positive linear operators. Here we studied the direct result related properties of the operators (8).

2. ESTIMATION OF MOMENTS

Let us denote by $\mu_{n,m}^{*,\alpha,\beta}$, the m^{th} moments of the operators given in (8), defined as

$$\mu_{n,m}^{*,\alpha,\beta}(x) = S_n^{*,\alpha,\beta}((t-x)^m;x), \qquad m = 0,1,2,\ldots. \tag{9}$$

LEMMA 1. *For $m = 1,2,\ldots$, the following relation holds:*

$$S_n^{*,\alpha,\beta}(t^m;x) = \sum_{j=0}^{m} \binom{m}{j} \tfrac{b_n^j \alpha^{m-j}}{(b_n+\beta)^m} S_n^*(t^j;x), \tag{10}$$

where $S_n^*(f;x)$ and $S_n^{*,\alpha,\beta}(f;x)$ are as given by (6) *and* (8), *respectively.*

Proof. Using (6) and (8),

$$\begin{aligned}
S_n^{*,\alpha,\beta}(t^m;x) &= b_n \sum_{k=0}^{\infty} s_{b_n,k}(x) \int_0^{\infty} s_{b_n,k}(t) \left(\tfrac{b_n t+\alpha}{b_n+\beta}\right)^m dt \\
&= b_n \sum_{k=0}^{\infty} s_{b_n,k}(x) \int_0^{\infty} s_{b_n,k}(t) \left(\sum_{j=0}^{m} \binom{m}{j} \tfrac{b_n^j \alpha^{m-j}}{(b_n+\beta)^m} t^j\right) dt \\
&= \sum_{j=0}^{m} \binom{m}{j} \tfrac{b_n^j \alpha^{m-j}}{(b_n+\beta)^m} \left(b_n \sum_{k=0}^{\infty} s_{b_n,k}(x) \int_0^{\infty} s_{b_n,k}(t) t^j\, dt\right) \\
&= \sum_{j=0}^{m} \binom{m}{j} \tfrac{b_n^j \alpha^{m-j}}{(b_n+\beta)^m} S_n^*(t^j;x).
\end{aligned}$$

□

LEMMA 2. *For $m = 1,2,\ldots$, the following holds:*

$$S_n^{*,\alpha,\beta}((t-x)^m;x) = \sum_{j=0}^{m} \binom{m}{j} (-x)^{m-j} S_n^{*,\alpha,\beta}(t^j;x),$$

where $S_n^{*,\alpha,\beta}(f;x)$ is as given in (8).

Proof. Using (8),

$$S_n^{*,\alpha,\beta}((t-x)^m;x) =$$
$$= b_n \sum_{k=0}^{\infty} s_{b_n,k}(x) \int_0^{\infty} s_{b_n,k}(t) \left(\tfrac{b_n t+\alpha}{b_n+\beta} - x\right)^m dt$$
$$= b_n \sum_{k=0}^{\infty} s_{b_n,k}(x) \int_0^{\infty} s_{b_n,k}(t) \left\{ \sum_{j=0}^{m} \tbinom{m}{j} (-x)^{m-j} \left(\tfrac{b_n t+\alpha}{b_n+\beta}\right)^j \right\} dt$$
$$= \sum_{j=0}^{m} \tbinom{m}{j} (-x)^{m-j} \left\{ b_n \sum_{k=0}^{\infty} s_{b_n,k}(x) \int_0^{\infty} s_{b_n,k}(t) \left(\tfrac{b_n t+\alpha}{b_n+\beta}\right)^j dt \right\}$$
$$= \sum_{j=0}^{m} \tbinom{m}{j} (-x)^{m-j} S_n^{*,\alpha,\beta}(t^j;x).$$

□

LEMMA 3. *For* $e_i(t) = t^i$, $\quad i = 0,1,2,3,4$, *the following holds:*

(a) $S_n^*(e_0;x) = 1$,
(b) $S_n^*(e_1;x) = \frac{1}{b_n}(b_n x + 1)$,
(c) $S_n^*(e_2;x) = \frac{1}{b_n^2}(b_n^2 x^2 + 4b_n x + 2)$,
(d) $S_n^*(e_3;x) = \frac{1}{b_n^3}(b_n^3 x^3 + 9b_n^2 x^2 + 18b_n x + 6)$,
(e) $S_n^*(e_4;x) = \frac{1}{b_n^4}(b_n^4 x^4 + 16b_n^3 x^3 + 72b_n^2 x^2 + 96b_n x + 24)$.

Proof. By (6),

$$\begin{aligned} S_n^*(e_2;x) &= \sum_{k=0}^{\infty} s_{b_n,k}(x) b_n \int_0^{\infty} s_{b_n,k}(t) t^2\, dt \\ &= \sum_{k=0}^{\infty} s_{b_n,k}(x) b_n \int_0^{\infty} e^{-b_n t} \tfrac{(b_n t)^k}{k!} t^2\, dt \\ &= \sum_{k=0}^{\infty} s_{b_n,k}(x) \tfrac{(k+1)(k+2)}{b_n^2} \\ &= \tfrac{1}{b_n^2} \sum_{k=0}^{\infty} e^{-b_n x} \tfrac{(b_n x)^k}{k!} [k(k-1) + 4k + 2] \\ &= \tfrac{1}{b_n^2}(b_n^2 x^2 + 4b_n x + 2). \end{aligned}$$

This proves (c).

Other relations follow on the same line. □

LEMMA 4. *For* $e_i(t) = t^i$, $\quad i = 0,1,2,3,4$, *the following holds:*

(a) $S_n^{*,\alpha,\beta}(e_0;x) = 1$,
(b) $S_n^{*,\alpha,\beta}(e_1;x) = \frac{1}{b_n+\beta}(b_n x + \alpha + 1)$,
(c) $S_n^{*,\alpha,\beta}(e_2;x) = \frac{1}{(b_n+\beta)^2}[b_n^2 x^2 + 2b_n x(\alpha+2) + (\alpha^2 + 2\alpha + 2)]$,

(d) $S_n^{*,\alpha,\beta}(e_3;x) = \frac{1}{(b_n+\beta)^3}\Big[b_n^3x^3 + 3b_n^2x^2(\alpha+3) + 3b_nx(\alpha^2+4\alpha+6) + (\alpha^3 + 3\alpha^2 + 6\alpha + 6)\Big]$,

(e) $S_n^{*,\alpha,\beta}(e_4;x) = \frac{1}{(b_n+\beta)^4}\Big[b_n^4x^4 + 4b_n^3x^3(\alpha+4) + 6b_n^2x^2(\alpha^2+6\alpha+12) + 4b_nx(\alpha^3+6\alpha^2+18\alpha+24) + (\alpha^4+4\alpha^3+12\alpha^2+24\alpha+24)\Big]$.

Proof. Using (10),

$$\begin{aligned} S_n^{*,\alpha,\beta}(e_2;x) &= \sum_{j=0}^{2}\tbinom{2}{j}\tfrac{b_n^j\alpha^{2-j}}{(b_n+\beta)^2}S_n^*(t^j;x) \\ &= \tfrac{1}{(b_n+\beta)^2}[\alpha^2S_n^*(1;x) + 2b_n\alpha S_n^*(t;x) + b_n^2S_n^*(t^2;x)] \\ &= \tfrac{1}{(b_n+\beta)^2}[b_n^2x^2 + 2b_nx(\alpha+2) + (\alpha^2+2\alpha+2)]. \end{aligned}$$

This proves (c).

Other relations follow on the same line. □

Consider the Banach lattice

$$C_\gamma[0,\infty) = \Big\{f \in C[0,\infty) : |f(t)| \le M(1+t)^\gamma\Big\}$$

for some $M > 0, \gamma > 0$.

Theorem 5. $\lim_{n\to\infty} S_n^{*,\alpha,\beta}(f;x) = f(x)$ *uniformly for* $x \in [0,a]$*, provided* $f \in C_\gamma[0,\infty)$*,* $\gamma \ge 2$ *and* $a > 0$*.*

Proof. For fix $a > 0$, consider the lattice homomorphism $T_a : C[0,\infty) \to C[0,a]$ defined by $T_a(f) := f|_{[0,a]}$ for every $f \in C[0,\infty)$, where $f|_{[0,a]}$ denotes the restriction of the domain of f to the interval $[0,a]$. In this case, we see that, for each $i = 0,1,2$ and by (a)–(c) of Lemma 4,

$$\lim_{n\to\infty} T_a\left(S_n^{\alpha,\beta}(e_i;x)\right) = T_a(e_i(x)), \text{ uniformly on } [0,a]. \tag{11}$$

Thus, by using (11) and with the universal Korovkin-type property with respect to positive linear operators (see Theorem 4.1.4 (vi) of [4], p.199) we have the result. □

Lemma 6. *For the moments defined in (9), the following holds:*

(a) $\mu_{n,1}^{*,\alpha,\beta}(x) = S_n^{*,\alpha,\beta}((t-x);x) = \frac{1}{b_n+\beta}(\alpha+1-\beta x)$,

(b) $\mu_{n,2}^{*,\alpha,\beta}(x) = S_n^{*,\alpha,\beta}((t-x)^2;x) = \frac{1}{(b_n+\beta)^2}\Big(\beta^2x^2 + 2x(b_n - \alpha\beta - \beta) + (\alpha^2 + 2\alpha + 2)\Big)$,

(c) $\mu_{n,3}^{*,\alpha,\beta}(x) = S_n^{*,\alpha,\beta}((t-x)^3;x) = \frac{1}{(b_n+\beta)^3}\Big[-\beta^3x^3 + 3\beta x^2\{(2b_n+\beta)(\alpha+1) - 2(\alpha+2)\} + 3x\{2b_n(\alpha+2) - \beta(\alpha^2+2\alpha+2)\} + (\alpha^3+3\alpha^2+6\alpha+6)\Big]$,

(d) $\mu_{n,4}^{*,\alpha,\beta}(x) = S_n^{*,\alpha,\beta}((t-x)^4;x) = \frac{1}{(b_n+\beta)^4}\Big[\beta^4x^4 + 4\beta^2x^3\{3b_n - \beta(\alpha + 1)\} + 6x^2\{2b_n^2 - 4b_n\beta(\alpha+3) + \beta^2(\alpha^2+2\alpha+2)\} + 4x\{3b_n(\alpha^2+4\alpha+6) - \beta(\alpha^3+3\alpha^2+6\alpha+6)\} + (\alpha^4+4\alpha^3+12\alpha^2+24\alpha+24)\Big]$.

Proof. The results follow from linearity of the operators $S_n^{*,\alpha,\beta}$ and lemma (4). □

3. DIRECT RESULT

Let us consider the space $C_B[0,\infty)$ of all continuous and bounded functions on $[0,\infty)$ and for $f \in C_B[0,\infty)$, consider the supremum norm $\|f\| = \sup\{|f(x)| : x \in [0,\infty)\}$. Also, consider the K-functional

$$K_2(f;\delta) = \inf_{g\in W^2}\left\{\|f-g\| + \delta\|g''\|\right\}, \tag{12}$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. For a constant $C > 0$, the following relationship exists:

$$K_2(f;\delta) \le C\omega_2(f,\sqrt{\delta}), \tag{13}$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h<\sqrt{\delta}}\ \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)| \tag{14}$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$; and for $f \in C_B[0,\infty)$, let the modulus of continuity be given by

$$\omega_1(f,\sqrt{\delta}) = \sup_{0<h<\sqrt{\delta}}\ \sup_{x\in[0,\infty)} |f(x+h) - f(x)|. \tag{15}$$

Theorem 7. *For $f \in C_B[0,\infty)$, we have*

$$|S_n^{*,\alpha,\beta}(f;x) - f(x)| \le \omega_1\left(f, \tfrac{|\alpha+1-\beta x|}{b_n+\beta}\right) + C\omega_2\left(f, \sqrt{\mu_{n,2}^{*,\alpha,\beta}(x) + \left(\tfrac{\alpha+1-\beta x}{b_n+\beta}\right)^2}\right),$$

where C is a positive constant.

Proof. Let the auxiliary operator denoted by $\bar{S}_n^{*,\alpha,\beta}$ be defined as

$$\bar{S}_n^{*,\alpha,\beta}(f;x) = S_n^{*,\alpha,\beta}(f;x) - f\left(\tfrac{b_nx+\alpha+1}{b_n+\beta}\right) + f(x)$$

for every $x \in [0,\infty)$. It is a linear operator which preserves the linear functions as $\bar{S}_n^{*,\alpha,\beta}(1;x) = 1$ and $\bar{S}_n^{*,\alpha,\beta}(t;x) = x$. This gives us $\bar{S}_n^{*,\alpha,\beta}(t-x;x) = 0$.

For $g \in W^2$, $x \in [0,\infty)$ and by Taylor's expansion, we have

$$g\left(\tfrac{b_nt+\alpha}{b_n+\beta}\right) = g(x) + \left(\tfrac{b_nt+\alpha}{b_n+\beta} - x\right)g'(x) + \int_x^{\frac{b_nt+\alpha}{b_n+\beta}} \left(\tfrac{b_nt+\alpha}{b_n+\beta} - u\right)g''(u)du.$$

Operating $\bar{S}_n^{*,\alpha,\beta}$ on both the sides,

$$\begin{aligned}\left|\bar{S}_n^{*,\alpha,\beta}(g;x)-g(x)\right| &= \left|\bar{S}_n^{*,\alpha,\beta}\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\ &\leq \left|S_n^{*,\alpha,\beta}\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\ &\quad + \left|\int_x^{\frac{b_nx+\alpha+1}{b_n+\beta}} \left(\tfrac{b_nx+\alpha+1}{b_n+\beta}-u\right) g''(u)du\right| \\ &\leq \|g''\| S_n^{*,\alpha,\beta}((t-x)^2;x) + \|g''\| \left(\tfrac{b_nx+\alpha+1}{b_n+\beta}-x\right)^2 \\ &= \|g''\| \left[\mu_{n,2}^{*,\alpha,\beta}(x) + \left(\tfrac{-\beta x+\alpha+1}{b_n+\beta}\right)^2\right].\end{aligned}$$

Also, we have $\left|S_n^{*,\alpha,\beta}(f;x)\right| \leq \|f\|$. Using these, we get

$$\begin{aligned}\left|S_n^{*,\alpha,\beta}(f;x)-f(x)\right| &\leq \left|\bar{S}_n^{*,\alpha,\beta}(f-g;x)-(f-g)(x)\right| \\ &\quad + \left|\bar{S}_n^{*,\alpha,\beta}(g;x)-g(x)\right| + \left|f\left(\tfrac{b_nx+\alpha+1}{b_n+\beta}\right)-f(x)\right| \\ &\leq 2\|f-g\| + \|g''\| \left[\mu_{n,2}^{*,\alpha,\beta}(x) + \left(\tfrac{-\beta x+\alpha+1}{b_n+\beta}\right)^2\right] \\ &\quad + \omega_1\left(f, \tfrac{|-\beta x+\alpha+1|}{b_n+\beta}\right).\end{aligned}$$

Taking infimum on the right hand side for all $g \in W^2$, we get

$$\begin{aligned}\left|S_n^{*,\alpha,\beta}(f;x)-f(x)\right| &\leq 2K_2\left(f, \tfrac{1}{2}\left[\mu_{n,2}^{*,\alpha,\beta}(x) + \left(\tfrac{-\beta x+\alpha+1}{b_n+\beta}\right)^2\right]\right) \\ &\quad + \omega_1\left(f, \tfrac{|-\beta x+\alpha+1|}{b_n+\beta}\right).\end{aligned}$$

Using (13) and $\omega_2(f,\lambda\delta) \leq (\lambda+1)^2\omega_2(f,\delta)$ for $\lambda > 0$, we get

$$\begin{aligned}\left|S_n^{*,\alpha,\beta}(f;x)-f(x)\right| &\leq C\omega_2\left(f, \sqrt{\mu_{n,2}^{*,\alpha,\beta}(x) + \left(\tfrac{-\beta x+\alpha+1}{b_n+\beta}\right)^2}\right) \\ &\quad + \omega_1\left(f, \tfrac{|-\beta x+\alpha+1|}{b_n+\beta}\right).\end{aligned}$$

□

4. A VORONOVSKAJA-TYPE RESULT

In this section we prove a Voronovskaja-type theorem for the operators $S_n^{*,\alpha,\beta}$ given in (8).

LEMMA 8. $\lim_{n\to\infty}(b_n+\beta)^2\mu_{n,4}^{*,\alpha,\beta}(x) = 12x^2$ *uniformly with respect to* $x \in [0,a]$, $a > 0$.

Proof. By lemma (6)(d), we may write that

$$\begin{aligned}(b_n+\beta)^2\mu_{n,4}^{*,\alpha,\beta}(x) &= \tfrac{1}{(b_n+\beta)^2}\Big[\beta^4x^4+4\beta^2x^3\{3b_n-\beta(\alpha+1)\}\\ &\quad +6x^2\{2b_n^2-4b_n\beta(\alpha+3)+\beta^2(\alpha^2+2\alpha+2)\}\\ &\quad +4x\{3b_n(\alpha^2+4\alpha+6)-\beta(\alpha^3+3\alpha^2+6\alpha+6)\}\\ &\quad +(\alpha^4+4\alpha^3+12\alpha^2+24\alpha+24)\Big]\end{aligned}$$

Taking limits on both sides, as $n\to\infty$, the Lemma is proved. □

THEOREM 9. *For every $f\in C_\gamma[0,\infty)$ such that $f',f''\in C_\gamma[0,\infty),\gamma\geq 4$, we have*

$$\lim_{n\to\infty}(b_n+\beta)\left[S_n^{*,\alpha,\beta}(f;x)-f(x)-\tfrac{-\beta x+\alpha+1}{b_n}f'(x)\right]=xf''(x)$$

uniformly with respect to $x\in\left[\frac{\alpha}{b_n+\beta},a\right]\ \left(a>\frac{\alpha}{b_n+\beta}\right)$.

Proof. Let $f,f',f''\in C_\gamma[0,\infty)$ and $x\geq\frac{\alpha}{b_n+\beta}$. Define, for $t\in[0,\infty)$ with $\frac{b_nt+\alpha}{b_n+\beta}\neq x$,

$$\Psi(t,x)=$$
$$=\frac{1}{\left(\frac{b_nt+\alpha}{b_n+\beta}-x\right)^2}\left[f\left(\tfrac{b_nt+\alpha}{b_n+\beta}\right)-f(x)-\left(\tfrac{b_nt+\alpha}{b_n+\beta}-x\right)f'(x)-\tfrac12\left(\tfrac{b_nt+\alpha}{b_n+\beta}-x\right)^2f''(x)\right],$$

and $\Psi\Big(\frac{(b_n+\beta)x-\alpha}{b_n},x\Big)=0$. The function $\Psi(\cdot,x)\in C_\gamma[0,\infty)$. Also, for $n\to\infty$, $\Psi\Big(\frac{(b_n+\beta)x-\alpha}{b_n},x\Big)=\Psi(x,x)$, so $\Psi(x,x)=0$, as $n\to\infty$. By Taylor's theorem we get

$$f\left(\tfrac{b_nt+\alpha}{b_n+\beta}\right)=$$
$$=f(x)+\left(\tfrac{b_nt+\alpha}{b_n+\beta}-x\right)f'(x)+\tfrac12\left(\tfrac{b_nt+\alpha}{b_n+\beta}-x\right)^2f''(x)+\left(\tfrac{b_nt+\alpha}{b_n+\beta}-x\right)^2\Psi(t,x).$$

Now from Lemma 6(a)–(b)

(16)
$$\begin{aligned}(b_n+\beta)\left[S_n^{*,\alpha,\beta}(f;x)-f(x)\right]&=(b_n+\beta)f'(x)\mu_{n,1}^{*,\alpha,\beta}(x)+\tfrac12(b_n+\beta)f''(x)\mu_{n,2}^{*,\alpha,\beta}(x)\\ &\quad+(b_n+\beta)S_n^{*,\alpha,\beta}((t-x)^2\Psi(t,x)).\end{aligned}$$

If we apply the Cauchy-Schwarz inequality to the third term on the right hand side of (16), then

$$(b_n+\beta)S_n^{*,\alpha,\beta}((t-x)^2\Psi(t,x);x)\leq\left((b_n+\beta)^2\mu_{n,4}^{*,\alpha,\beta}(x)\right)^{\frac12}(S_n^{*,\alpha,\beta}(\Psi^2(t,x);x))^{\frac12}$$

Now $\Psi^2(\cdot,x)\in C_{2\gamma}[0,\infty)$, using Theorem 5, we have $S_n^{*,\alpha,\beta}(\Psi^2(t,x);x)\to\Psi^2(x,x)=0$, as $n\to\infty$ and using Lemma 8, this third term on the right tends to zero for $x \in \left[\frac{\alpha}{b_n+\beta}, a\right]$ and we get

$$\lim_{n\to\infty}(b_n+\beta)\left[S_n^{*,\alpha,\beta}(f;x)-f(x)\right]=(1+\alpha-\beta x)f'(x)+xf''(x).$$

for $x \in \left[\frac{\alpha}{b_n+\beta}, a\right]$ $\left(a > \frac{\alpha}{b_n+\beta}\right)$. □

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REFERENCES

[1] T. Acar, L.N. Mishra and V.N. Mishra, *Simultaneous approximation for generalized Srivastava-Gupta operator*, J. Function Spaces, Article ID 936308, 12 pages, 2015.

[2] O. Agratini, *On approximation process of integral type*, Appl. Math. Comput., **236** (2014) pp. 195–201.

[3] P.N. Agrawal and H.S. Kasana, *On simultaneous approximation by Szász-Mirakjan operators.* Bull. Inst. Math. Acad. Sinica **22** (1994), pp. 181–188.

[4] P. Altomare and M. Campiti, *Korovkin-type Approximation Theory and its Applications*, Walter de Gruyter, Berlin - New York, 1994.

[5] M. Becker, *Global approximation theorems for Szász-Mirakjan and Baskakov operators in polynomial weight spaces*, Indiana Univ. Math. J., **27** (1978), pp. 127–142.

[6] M. Becker, D. Kucharski and R.J. Nessel, *Global approximation theorems for Szász-Mirakjan and Baskakov operators in exponential weight spaces*, In: Linear Spaces and Approximation (Proc. Conf. Oberwolfach, 1977), Birkhãuser Verlag, Basel.

[7] P.L. Butzer, *On the extension of Bernstein polynomials to the infinite interval.* Proc. Amer. Math. Soc., **5** (1954), pp. 547–553.

[8] M.M. Derriennic, *Sur l'approximation des fonctions d'une ou plusieurs variables par des polynomes de Bernstein modifiés et application au probléme des moments*, Thése de 3e cycle, Université de Rennes, 1978.

[9] J.L. Durrmeyer, *Une formule d'inversion de la transformée de Laplace: Applications á la théorie des moments*, Thése de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.

[10] A.R. Gairola, Deepmala, L.N. Mishra, *Rate of approximation by finite iterates of q-Durrmeyer operators*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, **86** (2016) no. 2, pp. 229–234.

[11] I. Gürhan and R.N. Mohapatra, *Approximation properties by q-Durrmeyer-Stancu operators*, Anal. Theory Appl. **29**(4) (2013), pp. 373–383.

[12] T. Hermann, *On the Szász-Mirakjan operator*, Acta Math. Sci. Acad. Hungar **32**(1-2) (1978), pp. 163–173.

[13] H.M. Srivastava, Z. Finta and V. Gupta, *Direct results for a certain family of summation-integral type operators*, Appl. Math. Comput. **190** (2007), pp. 449–457.

[14] H.M. Srivastava and V. Gupta, *A certain family of summation-integral type operators*, Math. Comput. Modelling **37**(2003), pp. 1307–1315.

[15] B. Ibrahim and E. Ibikli, *Inverse theorems for Bernstein-Chlodowsky type polynomials*, J. Math. Kyoto Univ., **46** (2006) 1, pp. 21–29.

[16] S.M. Mazhar and V. Totik, *Approximation by modified Szász operators*, Acta Sci. Math., **49** (1985), pp. 257–269.

[17] G.M. Mirakjan, *Approximation of continuous functions with the aid of polynomials* $e^{-nx}\sum_{k=0}^{m} C_{k,n}\chi^k$, Dokl. Akad. Nauk SSSR, **31** (1941), pp. 201–205.

[18] A. Sahai and G. Prasad, *On simultaneous approximation by modified Lupaş operators*, J. Approx. Theory, **45** (1985), 122–128.
[19] O. Szász, *Generalization of S. Bernstein's polynomials to the infinite interval*, J. Res. Nat. Bur. Standards Sect. B. **45** (1950), pp. 239–245.
[20] D.D. Stancu, *Asupra unei generalizări a polinoamelor lui Bernstein*, Studia Universitatis Babes-Bolyai, **14** (1969) 2, pp. 31–45 (in Romanian).
[21] V. Totik, V., *Uniform approximation by Szász-Mirakjan type operators*, Acta Math. Hungar. **41** (1983), pp. 291–307.
[22] B. Della Vecchia, G. Mastroianni and J. Szabados, *Weighted approximation of functions by Szász-Mirakyan-type operators*, Acta Math. Hungar., **111** (2006), pp. 325–345.
[23] B. Della Vecchia, G. Mastroianni and J. Szabados, *A weighted generalization of Szász-Mirakyan and Butzer operators*, Mediterr. J. Math., **12** (2015) no. 2, pp. 433–454.
[24] V.N. Mishra, R.B. Gandhi and F. Nasaireh, *Simultaneous approximation by Szász-Mirakjan-Durrmeyer-type operators*, Boll. Unione Mat. Ital., **8** (2016) 4, pp 297–305.
[25] V.N. Mishra and R.B. Gandhi, *Simultaneous approximation by Szász-Mirakjan-Stancu-Durrmeyer type operators*, Periodica Mathematica Hungarica, 2016.
[26] V.N. Mishra, H.H. Khan, K. Khatri and L.N. Mishra, *Hypergeometric representation for Baskakov-Durrmeyer-Stancu type operators*, Bulletin of Mathematical Analysis and Applications, **5** (2013) 3, pp. 18–26.
[27] V.N. Mishra, K. Khatri, L.N. Mishra and Deepmala, *Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators*, Journal of Inequalities and Applications 2013, 2013:586.
[28] V.N. Mishra, K. Khatri and L.N. Mishra, *Some approximation properties of q-Baskakov-Beta-Stancu type operators*, Journal of Calculus of Variations, Volume 2013, Article ID 814824, 8 pp.
[29] V.N. Mishra, K. Khatri and L.N. Mishra, *Statistical approximation by Kantorovich type discrete $q-$beta operators*, Advances in Difference Equations 2013, 2013:345.
[30] V.N. Mishra, P. Sharma and L.N. Mishra, *On statistical approximation properties of q-Baskakov-Szász-Stancu operators*, Journal of Egyptian Mathematical Society, **24** (2016) 3, pp. 396–401.
[31] A. Wafi, N. Rao and Deepmala Rai, *Approximation properties by generalized-Baskakov-Kantorovich-Stancu type operators*, Appl. Math. Inf. Sci. Lett., **4** (2016) 3, pp. 111–118.
[32] D.X. Zhou, *Weighted approximation by Szász-Mirakjan operators*, J. Approx. Theory **76** (1994), pp. 393–402.

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