JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY

J. Numer. Anal. Approx. Theory, vol. 47 (2018) no. 2, pp. 114–123

ictp.acad.ro/jnaat

BALL CONVERGENCE FOR AN AITKEN-NEWTON METHOD

IOANNIS K. ARGYROS*, MUNISH KANSAL† and VINAY KANWAR‡

Abstract. We present a local convergence analysis of an eighth-order Aitken-Newton method for approximating a locally unique solution of a nonlinear equation. Earlier studies have shown convergence of these methods under hypotheses up to the eighth derivative of the function although only the first derivative appears in the method. In this study, we expand the applicability of these methods using only hypotheses up to the first derivative of the function. This way the applicability of these methods is extended under weaker hypotheses. Moreover, the radius of convergence and computable error bounds on the distances involved are also given in this study. Numerical examples are also presented in this study.

MSC 2010. 65D10, 65D99.

Keywords. Nonlinear equations, Aitken-Newton method, local convergence, eighth order of convergence.

1. INTRODUCTION

Let X, Y be Banach spaces and D be a convex subset of X. Let also $L(X,Y)$ denote the set of bounded linear operators from X into Y. Many problems can be written in the form

$$F(x) = 0 \tag{1}$$

using Mathematical Modelling [1], [6], [7], [9], [23] where $F : D \subseteq X \to Y$ is a Fréchet-differentiable operator. Most solution methods for finding a solution x^* of equation (1) are iterative, since closed form solutions can be found only in special cases [1]-[23]. In this paper, we study the local convergence of

*Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA, e-mail: `iargyros@cameron.edu`.

† School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, Punjab-147004, India, e-mail: `mkmaths@gmail.com`.

‡ University Institute of Engineering and Technology, Panjab University, Chandigarh-160 014, India, e-mail: `vmithil@yahoo.co.in`.

Aitken-Newton method [21] defined for each $n = 0, 1, 2, \dots$ by

$$
(2) \quad \begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ z_n = y_n - F'(y_n)^{-1}F(y_n), \\ x_{n+1} = z_n - [z_n, y_n; F]^{-1}F(z_n) \\ \qquad - [y_n, z_n; F]^{-1}[z_n, y_n, y_n; F][y_n, z_n; F]^{-1}F(z_n)F'(y_n)^{-1}F(y_n), \end{cases}
$$

where $[\cdot, \cdot\, ; F]$, $[\cdot, \cdot, \cdot\, ; F]$ are divided differences of order one and two, respectively and

$$
(3) \qquad [x, y; F] = \int_0^1 F'(y + \theta(x - y))d\theta.
$$

The above method (2) attains eighth-order of convergence using five functional evaluations, viz. $F(x_n)$, $F'(x_n)$, $F(y_n)$, $F'(y_n)$ and $F(z_n)$, per iteration. Therefore, the efficiency index [23] of the proposed method is $E = \sqrt[5]{8} \approx 1.51$, when $X = Y = \mathbb{R}$. The convergence of method (2) was shown in [21] using Taylor expansions and hypotheses reaching up to the eighth derivative of the function F although only first derivative appears in the method. We will show that method (2) is well-defined and convergent using hypotheses only on the first derivative in the more general setting of a Banach space. Notice that the method (2) was not shown to be well defined in [21]. However, the eighth order of convergence was shown assuming that method (2) is well defined which may not be the case. These hypotheses limit the applicability of method (2).

As a motivational example, define function F on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$
F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}
$$

We have $x^* = 1$,

$$
(4) \qquad \begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x, \\ F'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}
$$

Clearly, function $F'''(x)$ is unbounded on D. Hence, the results in [21] cannot be applied to solve equation $F(x) = 0$, where F is given by (4). Moreover, the results in [21] do not provide computable convergence radii, error bounds on the distances $|x_n - x^*|$ and uniqueness of the solution results. We address all these problems using only hypotheses on the first derivative. We use *the computational order of convergence* (COC) to approximate the convergence order (which does not depend upon the solution x^*). Moreover, we present the results in a more general setting of a Banach space.

The rest of the paper is organized as follows: In Section 2, we present the local convergence of method (2). The numerical examples are presented in the concluding Section 3.

2. LOCAL CONVERGENCE ANALYSIS

We present the local convergence analysis of method (2) in this section using some scalar functions and parameters.

Let $L_0 > 0$, $L > 0$, $K > 0$, and $M \geq 1$ be given parameters. Define function g_1 on the interval $[0, \frac{1}{L_0})$ by

$$g_1(t) = \frac{Lt}{2(1-L_0t)},$$

and parameter r_1 by

$$r_1 = \frac{2}{2L_0+L}.$$

We have that $g_1(r_1) = 1$ and $0 \leq g_1(t) < 1$ for each $t \in [0, r_1)$. Define functions p_1 and h_{p_1} on the interval $[0, \frac{1}{L_0})$ by

$$p_1(t) = L_0 g_1(t) t$$

and $h_{p_1}(t) = p_1(t) - 1$.

We get that $h_{p_1}(0) = -1 < 0$ and $h_{p_1}(t) \to +\infty$ as $t \to \frac{1^-}{L_0}$. It follows from the intermediate value theorem that function h_{p_1} has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_{p_1} the smallest such zero. Moreover, define functions g_2 and h_2 on the interval $[0, r_{p_1})$ by

$$g_2(t) = \frac{L g_1^2(t) t^3}{2(1-p(t))}$$

and $h_2(t) = g_2(t) - 1$.

We get that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to r_{p_1}^-$. Denote by r_2 the smallest zero of function h_2 in the interval $(0, r_2)$. Furthermore, define functions p_2 and h_{p_2} on the interval $[0, r_{p_1})$ by

$$p_2(t) = \frac{L_0}{2}\Big(g_1(t) + g_2(t)\Big)t$$

and $h_{p_2}(t) = p_2(t) - 1$.

We have that $h_{p_2}(0) = -1 < 0$ and $h_{p_2}(t) \to +\infty$ as $t \to r_{p_1}^-$. Denote by r_{p_2} the smallest such zero of function h_{p_2} in the interval $(0, r_{p_1})$. Finally, define functions g_3 and h_3 on the interval $[0, r_{p_2})$ by

$$g_3(t) = \Big(1 + \frac{M}{1-p_2(t)} + \frac{KM^2 g_1(t)t}{(1-p_2(t))^2(1-p_1(t))}\Big)\, g_2(t)$$

and $h_3(t) = g_3(t) - 1$.

We obtain that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to r_{p_2}^-$. Denote by r_3 the smallest zero of function h_3 in the interval $(0, r_{p_2})$. Define the radius of convergence r by

$$r = \min\{r_i\},\ i = 1, 2, 3. \tag{5}$$

Then, we have that

$$0 < r \leq r_1 < \frac{1}{L_0} \tag{6}$$

and for each $t \in [0, r)$

$$0 \leq g_i(t) < 1, \ i = 1, 2, 3 \tag{7}$$

and

$$0 \leq p_j(t) < 1, \ j = 1, 2. \tag{8}$$

Let $U(z, \rho)$ and $\bar{U}(z, \rho)$ stand respectively for the open and closed balls in X with center at $z \in X$ and of radius $\rho > 0$. Next, the local convergence analysis of method (2) shall be presented using previous notations.

THEOREM 1. *Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator. Let $[\cdot, \cdot\ ; F]$, $[\cdot, \cdot, \cdot\ ; F]$ are divided differences of order one and two on D, respectively. Suppose there exist $x^* \in D$ and $L_0 > 0$, such that for each $x \in D$*

$$F(x^*) = 0, \ F'(x^*)^{-1} \in L(Y, X) \tag{9}$$

and

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0 \|x - x^*\|. \tag{10}$$

Moreover, suppose there exist $L > 0$, $K > 0$ and $M \geq 1$ such that for each $x, y \in D_0 := D \cap U\left(x^*, \frac{1}{L_0}\right)$, we have

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|, \quad , \tag{11}$$

$$\|F'(x^*)^{-1} F'(x)\| \leq M, \tag{12}$$

$$\|F'(x^*)^{-1}[x, y, y; F]\| \leq K, \tag{13}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{14}$$

where radius of convergence r is defined by (5). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r)$-$\{x^*\}$ by method (2) is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to the solution x^*. Moreover, the following error estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{15}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| \tag{16}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{17}$$

where the "$g_i, \ i = 1, 2, 3$" functions are defined previously. For $T \in [r, \frac{2}{L_0})$, the limit point x^* is the only solution of equation $F(x) = 0$ in $D_1 := D \cap \bar{U}(x^*, T)$.

Proof. We shall show using mathematical induction that sequence $\{x_n\}$ of iterates generated by (2) is well defined, remains in $U(x^*, r)$ and satisfies estimations (15)–(17). By hypothesis, $x_0 \in U(x^*, r) - x^*$, (6) and (10), we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0\|x_0 - x^*\| < L_0 r < 1. \tag{18}$$

It follows from estimate (18) and the Banach lemma on invertible operators [6, 9, 20, 22, 23] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \tfrac{1}{1-L_0\|x_0-x^*\|}. \tag{19}$$

Hence, y_0 is well defined. Using (6), (7), (11), (19) and the first substep of method (2) for $n = 0$, we get in turn that

(20)

$$\begin{aligned}
&\|y_0 - x^*\| \leq \\
&\leq \|x_0 - x^* - F'(x_0)^{-1}F'(x_0)\| \\
&\leq \|F'(x_0)^{-1}F'(x^*)\| \Big\| \int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0))](x_0 - x^*)d\theta \Big\| \\
&\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} = g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r,
\end{aligned}$$

which shows (15) for $n = 0$ and $y_0 \in U(x^*, r)$. Then, we have as in (18) that

(21)

$$\begin{aligned}
\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| \leq L_0\|y_0 - x^*\| &\leq L_0 g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&= p_1(\|x_0 - x^*\|) < p_1(r) < 1,
\end{aligned}$$

so $F'(y_0)^{-1} \in L(Y, X)$. Therefore, we have

$$\|F'(y_0)^{-1}F'(x^*)\| \leq \tfrac{1}{1-p_1\|x_0-x^*\|} \tag{22}$$

and as in (20)

$$\begin{aligned}
\|z_0 - x^*\| &\leq \tfrac{L\|y_0-x^*\|^2}{2(1-L_0\|y_0-x^*\|)} \\
&\leq g_2(\|x_0 - \xi\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r,
\end{aligned} \tag{23}$$

which implies that (16) holds for $n = 0$ and $z_0 \in U(x^*, r)$.

Next, we show that $[y_0, z_0; F]^{-1}$ exists. In view of the definition of divided difference $[\cdot, \cdot; F]$, (6), (7), (10), (20) and (22) that

$$\begin{aligned}
&\|F'(x^*)^{-1}\left([z_0, y_0; F] - F'(x^*)\right)\| \leq \\
&\leq \tfrac{L_0}{2}\left(\|z_0 - x^*\| + \|y_0 - x^*\|\right) \\
&\leq \tfrac{L_0}{2}\Big(g_1(\|x_0 - x^*\|) + g_2(\|x_0 - x^*\|)\Big)\|x_0 - x^*\| \\
&= p_2(\|x_0 - x^*\|) < p_2(r) < 1,
\end{aligned} \tag{24}$$

so

$$\|[z_0, y_0; F]^{-1} - F'(x^*)\| \leq \tfrac{1}{1-p_2\|x_0-x^*\|}. \tag{25}$$

Hence, x_1 is well defined. Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| \leq \theta\|x_0 - x^*\| < r$ for each $\theta \in [0,1]$, so $x^* + \theta(x_0 - x^*) \in U(x^*, r)$ for each $\theta \in [0,1]$. Then, by (9) and (12), we get that

$$\begin{aligned} \|F'(x^*)^{-1}F(x_0)\| &= \\ &= \|F'(x^*)^{-1}(F(x_0) - F(x^*)\| \\ &= \|\int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)\| \leq M\|x_0 - x^*\| \end{aligned} \tag{26}$$

and similarly

$$\|F'(x^*)^{-1}F(y_0)\| \leq M\|y_0 - x^*\| \leq Mg_1(\|x_0 - x^*\|)\|x_0 - x^*\|. \tag{27}$$

By the last substep of method (2) for $n = 0$, (5), (6), (20), (22), (23), (25)–(27), we obtain in turn that

$$\begin{aligned} \|x_1 - x^*\| &\leq \\ &\leq \|z_0 - x^*\| + \|[z_0, y_0; F]^{-1}F'(x^*)\|\|F'(x^*)^{-1}F'(z_0)\| + \\ &\quad + \|[z_0, y_0; F]^{-1}F'(x^*)\|\|[z_0, y_0; F]F'(x^*)^{-1}\|\|[z_0, y_0; F]^{-1}F'(x^*)\| \cdot \\ &\quad \cdot \|F'(x^*)^{-1}[z_0, y_0, y_0; F]^{-1}F'(x^*)\|\|F'(y_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(y_0)\| \\ &\leq \|z_0 - x^*\| + \tfrac{M\|z_0-x^*\|}{1-p_2(\|x_0-x^*\|)} + \tfrac{KM^2\|z_0-x^*\|\|y_0-x^*\|}{(1-p_2(\|x_0-x^*\|))^2(1-p_1(\|x_0-x^*\|))} \\ &= \Big(1 + \tfrac{M}{1-p_2(\|x_0-x^*\|)} + \tfrac{KM^2\|y_0-x^*\|}{(1-p_2(\|x_0-x^*\|))^2(1-p_1(\|x_0-x^*\|))}\Big)\|z_0 - x^*\| \\ &\leq g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{28}$$

which implies that (17) holds for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing y_0, z_0, x_1 by y_n, z_n, x_{n+1} in the preceding estimates, we complete the induction for estimates (15)–(17). Then, in view of the estimate $\|x_{n+1} - x^*\| \leq c\|x_n - x^*\| < r$, $c = g_3(\|x_0 - x^*\|) \in [0,1)$, we deduce that $\lim_{n\to\infty} x_n = x^*$ and $x_{n+1} \in U(x^*, r)$. The proof of the uniqueness follows using standard arguments [11]. □

REMARK 2. 1) It follows from (10) that condition (12) can be dropped, if we set

$$M(t) = 1 + L_0 t$$

or

$$M(t) = M = 2, \text{ since } t \in \Big[0, \tfrac{1}{L_0}\Big).$$

2) The results obtained here can also be used for operators F satisfying autonomous differential equations [6, 9] of the form:

$$F'(x) = P(F(x)),$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $f(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$.
3) The radius $\bar{r_1} = \frac{2}{2L_0+L_1}$ was shown by Argyros [6] to be the convergence radius of Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \text{ for each } n = 0, 1, 2, \ldots \tag{29}$$

under the conditions (9)–(11) on D, where L_1 is the Lipschitz constant on D. We have that $L \leq L_1$ and $L_0 \leq L_1$, so $\bar{r_1} \leq r_1$. It follows that the convergence radius r of the method (2) cannot be larger than the convergence radius r_1 of the second order Newton's method (29). As already noted in [6], $\bar{r_1}$ is at least as large as the convergence ball given by Rheinboldt [22]

$$r_R = \frac{2}{3L_1}.$$

In particular, for $L_0 < L_1$, we have that

$$r_R < \bar{r_1}$$

and

$$\frac{r_R}{\bar{r_1}} \to \frac{1}{3} \text{ as } \frac{L_0}{L_1} \to 0.$$

That is our convergence ball $\bar{r_1}$ is at most three times larger than Rheinboldt's. The same value of r_R was given by Traub [23]. 4) It is worth noticing that method (2) is not changing when we use the conditions of Theorem 1 instead of stronger conditions used in previous studies. Moreover, we can consider the semi-computational order of convergence defined by

$$\begin{aligned} Q_L(k) &= \frac{\ln|x_{n+1}-x^*|}{\ln|x_n-x^*|} \\ Q_\Lambda(k) &= \ln\left(\frac{|x_{n+1}-x^*|}{|x_n-x^*|}\right)/\ln\left(\frac{|x_n-x^*|}{|x_{n-1}-x^*|}\right), \end{aligned}$$

for which some interesting properties were obtained in [19] and [12]. We can even compute the (full) computational order of convergence defined by

$$\begin{aligned} Q'_L(k) &= \frac{\ln|x_n-x_{n-1}|}{\ln|x_{n-1}-x_{n-2}|} \\ Q'_\Lambda(k) &= \ln\left(\frac{|x_n-x_{n-1}|}{|x_{n-1}-x_{n-2}|}\right)/\ln\left(\frac{|x_{n-1}-x_{n-2}|}{|x_{n-2}-x_{n-3}|}\right), \end{aligned}$$

for which interesting properties were obtained in [12]. We also recommend to the motivated reader the excellent survey on these notions [13], containing full proofs and historical aspects.

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F. Notice also that the computation of ξ^* does not require knowledge of x^*. 5) Also, condition (10) can be replaced by

$$\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| \leq \frac{\bar{L_0}}{2}(\|x - x^*\| + \|y - x^*\|). \tag{30}$$

In this case, the choice of the divided difference given by (4) can be dropped. Moreover, $\bar{L_0}$ and (30) can replace L_0 and (10), respectively in the proof of Theorem 1. □

3. NUMERICAL EXAMPLES

We present numerical examples in this section.

EXAMPLE 3. Let $X = Y = \mathbb{R}^3$, $D = \bar{U}(0,1)$, $x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \tfrac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet derivative is given by

$$F'(w) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have that $L_0 = e - 1$, $L = e^{\frac{1}{L_0}}$, $M = e^{\frac{1}{L_0}}$, $K = \frac{e^{\frac{1}{L_0}}}{2}$ and $L_1 = e$. The parameters using method (2) are:

$$r_1 = 0.382692,\ r_2 = 0.45738,\ r_3 = 0.38726,$$
$$r = 0.38726,\ \bar{r_1} = 0.324947,\ r_R = 0.245253.$$

EXAMPLE 4. Let $X = Y = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $D = \bar{U}(0,1)$ and $B(x) = F''(x)$ for each $x \in D$. Define function F on D by

$$F(\phi)(x) = \phi(x) - 5\int_0^1 x\theta\phi(\theta)^3 d\theta. \tag{31}$$

We have that

$$F'(\phi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\phi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D. \tag{32}$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L_1 = 15$, $L = 15$, $K = 15$ and $M = 2$. The parameters using method (2) are:
$r_1 = 0.0666667$, $r_2 = 0.114372$, $r_3 = 0.0783549$, $r = 0.0783549$,
$\bar{r_1} = 0.0666667$, $r_R = 0.0444444$. □

EXAMPLE 5. Returning back to the motivational example at the introduction of this paper, we have that $L = L_0 = 146.6629073$, $M = 2$, $L_1 = L$ and $K = 48.3315$. The parameters using method (2) are:

$$r_1 = 0.00454557,\ r_2 = .006787,\ r_3 = 0.004991287,$$
$$\bar{r_1} = 0.00454557,\ r_R = 0.00454557,\ \xi^* = 7.8403.$$ □

Example 6. Let $X = Y = \mathbb{R}$ and define function F on $D = \mathbb{R}$ by

$$F(x) = \beta x - \gamma \sin(x) - \delta, \tag{33}$$

where β, γ, δ are given real numbers. Suppose that there exists a solution x^* of $F(x) = 0$ with $F'(x^*) \neq 0$. Then, we have

$$L_1 = L_0 = L = \frac{|\gamma|}{|\beta - \gamma \cos x^*|}, \quad M = \frac{|\gamma| + |\beta|}{|\beta - \gamma \cos x^*|} \text{ and } K = \frac{|\gamma|}{2|\beta - \gamma \cos x^*|}.$$

Then one can find the convergence radii for different values of β, γ and δ. As a specific example, let us consider Kepler's equation (33) with $\beta = 1$, $0 \leq \gamma < 1$ and $0 \leq \delta \leq \pi$. A numerical study was presented in [14] for different values of γ and δ.

Let us take $\gamma = 0.9$ and $\delta = 0.1$. Then the solution is given by $x^* = 0.6308435$. Hence, for method (2) the parameters are:

$$\begin{aligned} &r_1 = 0.202387, \ r_2 = 0.261858, \ r_3 = 0.196578, \\ &r = 0.196578, \bar{r_1} = 0.202387, r_R = 0.202387, \ \xi^* = 8.0353. \end{aligned} \tag{34}$$

□

REFERENCES

[1] G. Adomian, *Solving Frontier problem of physics: The decomposition method*, Kluwer Academic Publishers, Doordrecht, 1994.

[2] S. Amat, S. Busquier, J.M. Guttiérrez, *Geometric constructions of iterative functions to solve nonlinear equations*, J. Comput. Appl. Math., **157** (2003), 197–205.

[3] S. Amat, S. Busquier, S. Plaza, *Dynamics of the King's and Jarratt iterations*, Aeq. Math., **69** (2005), 212–213.

[4] S. Amat, M.A. Hernández, N. Romero, *A modified Chebyshev's iterative method with at least sixth order of convergence*, Appl. Math. Comput. **206 (1)** (2008), 164–174.

[5] I.K. Argyros, *Quadratic equations and applications to Chandershekhar's and related equations*, Bull Austral. Math. Soc. **32** (1985), 275–292.

[6] I.K. Argyros, D. Chen, *Results on the Chebyshev method in Banach spaces*, Proyecciones **12 (2)** (1993), 119–128.

[7] I.K. Argyros, *A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space*, J. Math. Anal. Appl., **298** (2004), 374–397

[8] I.K. Argyros, *Computational Theory of Iterative Methods*, Series: Studies Comput. Math. 15, Editors: C.K. Chui and L. Wuytack, Elsevier, New York, USA, (2007)

[9] I.K. Argyros, S. Hilout, *Weaker conditions for the convergence of Newton's method*, J. Complexity. **28** (2012), 364–387.

[10] I.K. Argyros, *Numerical Methods in Nonlinear Analysis*, World Scientific Publ. Comp. New Jersey, 2013.

[11] I.K. Argyros, S. George, *Local convergence for an efficient eighth order iterative method with a parameter for solving equations under weaker conditions*, Int. J. Appl. Comput. Math.

[12] W.A. Beyer, B.R. Ebanks, C.R. Qualls, *Convergence rates and convergence-order profiles for sequences*, Acta Appl. Math., **20** (1990), 267–284.

[13] E. Catinas, *A survey on the high convergence orders and computational convergence orders of sequences*, Appl. Math. Comput., **343** (2019), 1–20.

[14] J.M.A. Danby, T.M. Burkardt, *The solution of Kepler's equation I.*, Celest. Mech. **31** (1983), 95–107.

[15] J.A. Ezquerro, M.A. Hernández, *New iterations of R-order four with reduced computational cost*, BIT Numer Math. **49** (2009), 325–342.
[16] J.M. Guttiérrez, M.A. Hernández, *Recurrence relations for the super-Halley method*, Comput. Math. Appl., **36** (1998), 1–8.
[17] L.V. Kantorovich, G.P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, 1982.
[18] A.A. Magreñán, *Estudio de la dinámica del método de Newton amortiguado*, (PhD Thesis), Servicio de Publicaciones Universidad de La Rioja, 2013.
[19] F.A. Potra, *On Q-order and R-order of convergence*, J. Optim. Theory Appl., **63** (1989) no. 3, 415–431.
[20] F.A. Potra, V. Pták, *Nondiscrete introduction and iterative processes*, Research Notes in Mathematics. **103**, Pitman, Boston, MA, 1984.
[21] I. Pavaloiu, E. Catinas, *On a robust Aitken-Newton method based on the Hermite polynomial*, Appl. Math. Comput., **287-288** (2016), 224–231.
[22] W.C. Rheinboldt, *An adaptive continuation process for solving systems of nonlinear equations. Mathematical models and numerical methods* (A.N.Tikhonov et al. eds.) pub.3, (19), 129–142 Banach Center, Warsaw, Poland.
[23] J.F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

Received by the editors: March 18, 2016. Accepted: February 18, 2017. Published online: February 17, 2019.