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INEQUALITIES OF JENSEN TYPE FOR *AH*-CONVEX FUNCTIONS*

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Abstract. Some integral inequalities of Jensen type for AH-convex functions defined on intervals of real line are given. Applications for power and logarithm functions are provided as well. Some inequalities for functions of selfadjoint operators in Hilbert spaces are also established.

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1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \left\{ f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x) |f(x)| \, d\mu(x) < \infty \right\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x) \, d\mu(x)$.

If $f,\ g : \Omega \to \mathbb{R}$ are μ-measurable functions and $f,\ g,\ fg \in L_w(\Omega, \mu)$, then we may consider the *Čebyšev functional*

$$T_w(f, g) := \int_\Omega wfg d\mu - \int_\Omega wf d\mu \int_\Omega wg d\mu. \tag{1.1}$$

The following result is known in the literature as the *Grüss inequality*

$$|T_w(f, g)| \leq \tfrac{1}{4} (\Gamma - \gamma)(\Delta - \delta), \tag{1.2}$$

provided

$$-\infty < \gamma \leq f(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty \tag{1.3}$$

for μ-a.e. (almost every) $x \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

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If we assume that $-\infty < \gamma \leq f(x) \leq \Gamma < \infty$ for μ-a.e. $x \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz's integral inequality, we have

$$\int_\Omega w \left| f - \int_\Omega wf d\mu \right| d\mu \leq \tag{1.4}$$
$$\leq \left[\int_\Omega wf^2 d\mu - \left(\int_\Omega wf d\mu \right)^2 \right]^{\frac{1}{2}} \leq \tfrac{1}{2} (\Gamma - \gamma).$$

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S.S. Dragomir obtained in 2002 [6] the following result:

Theorem 1.1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega w d\mu = 1$. Then we have the inequality:*

$$\begin{aligned} 0 &\leq \int_\Omega (\Phi \circ f) w d\mu - \Phi\left(\int_\Omega f w d\mu \right) \\ &\leq \int_\Omega (\Phi' \circ f) f w d\mu - \int_\Omega (\Phi' \circ f) w d\mu \int_\Omega w f d\mu \\ &\leq \tfrac{1}{2} \left[\Phi'(M) - \Phi'(m) \right] \int_\Omega w \left| f - \int_\Omega f w d\mu \right| d\mu. \end{aligned} \tag{1.5}$$

For a generalization of the first inequality in (1.5) without the differentiability assumption and the derivative Φ' replaced with a selection φ from the subdifferential $\partial\Phi$, see the paper [18] by C. P. Niculescu.

If $\mu(\Omega) < \infty$ and $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) \cdot f \in L(\Omega, \mu)$, then we have the inequality:

$$\begin{aligned} 0 &\leq \tfrac{1}{\mu(\Omega)} \int_\Omega (\Phi \circ f) d\mu - \Phi\left(\tfrac{1}{\mu(\Omega)} \int_\Omega f d\mu \right) \\ &\leq \tfrac{1}{\mu(\Omega)} \int_\Omega (\Phi' \circ f) f d\mu - \tfrac{1}{\mu(\Omega)} \int_\Omega (\Phi' \circ f) d\mu \cdot \tfrac{1}{\mu(\Omega)} \int_\Omega f d\mu \\ &\leq \tfrac{1}{2} \left[\Phi'(M) - \Phi'(m) \right] \tfrac{1}{\mu(\Omega)} \int_\Omega \left| f - \tfrac{1}{\mu(\Omega)} \int_\Omega f d\mu \right| d\mu. \end{aligned} \tag{1.6}$$

The following discrete inequality is of interest as well.

Corollary 1.2. *Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M). If $x_i \in [m, M]$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$,* *then one has the counterpart of Jensen's weighted discrete inequality:*

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \\
&\leq \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i \\
&\leq \tfrac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right| .
\end{aligned} \tag{1.7}
$$

REMARK 1.3. We notice that the inequality between the first and the second term in (1.7) was proved in 1994 by Dragomir & Ionescu, see [12]. □

On making use of the results (1.5) and (1.4), we can state the following sequence of reverse inequalities

$$
\begin{aligned}
0 &\leq \int_\Omega (\Phi \circ f)\, w d\mu - \Phi\left(\int_\Omega f w d\mu\right) \\
&\leq \int_\Omega (\Phi' \circ f)\, f w d\mu - \int_\Omega (\Phi' \circ f)\, w d\mu \int_\Omega f w d\mu \\
&\leq \tfrac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \int_\Omega \left| f - \int_\Omega w f d\mu \right| w d\mu \\
&\leq \tfrac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \left[\int_\Omega f^2 w d\mu - \left(\int_\Omega f w d\mu\right)^2\right]^{\frac{1}{2}} \\
&\leq \tfrac{1}{4}\left[\Phi'(M) - \Phi'(m)\right](M - m),
\end{aligned} \tag{1.8}
$$

provided that $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. on Ω with $\int_\Omega w d\mu = 1$.

REMARK 1.4. We notice that the inequality between the first, second and last term from (1.8) was proved in the general case of positive linear functionals in 2001 by S.S. Dragomir in [5]. □

The following reverse of the Jensen's inequality holds [9], [10]:

THEOREM 1.5. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ is the interior of I. If $f : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$
-\infty < m \leq f(x) \leq M < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega \tag{1.9}
$$

and such that $f, \Phi \circ f \in L_w(\Omega, \mu)$, *where* $w \geq 0$ μ*-a.e. on* Ω *with* $\int_\Omega w d\mu = 1$, *then*

$$0 \leq \int_\Omega (\Phi \circ f)\, w d\mu - \Phi\left(\bar{f}_{\Omega,w}\right) \tag{1.10}$$
$$\leq \frac{(M-\bar{f}_{\Omega,w})(\bar{f}_{\Omega,w}-m)}{M-m} \sup_{t\in(m,M)} \Psi_\Phi(t; m, M)$$
$$\leq \left(M - \bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w} - m\right) \frac{\Phi'_-(M)-\Phi'_+(m)}{M-m}$$
$$\leq \tfrac{1}{4}(M-m)\left[\Phi'_-(M) - \Phi'_+(m)\right],$$

where $\bar{f}_{\Omega,w} := \int_\Omega w(x) f(x) d\mu(x) \in [m, M]$ *and* $\Psi_\Phi(\cdot; m, M) : (m, M) \to \mathbb{R}$ *is defined by*

$$\Psi_\Phi(t; m, M) = \frac{\Phi(M) - \Phi(t)}{M - t} - \frac{\Phi(t) - \Phi(m)}{t - m}.$$

We also have the inequality

$$0 \leq \int_\Omega (\Phi \circ f)\, w d\mu - \Phi\left(\bar{f}_{\Omega,w}\right) \leq \tfrac{1}{4}(M-m)\,\Psi_\Phi\left(\bar{f}_{\Omega,w}; m, M\right) \tag{1.11}$$
$$\leq \tfrac{1}{4}(M-m)\left[\Phi'_-(M) - \Phi'_+(m)\right],$$

provided that $\bar{f}_{\Omega,w} \in (m, M)$.

For a real function $g : [m, M] \to \mathbb{R}$ and two distinct points $\alpha, \beta \in [m, M]$ we recall that the *divided difference* of g in these points is defined by

$$[\alpha, \beta; g] := \frac{g(\beta) - g(\alpha)}{\beta - \alpha}.$$

In what follows, we assume that $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. $x \in \Omega$, is a μ-measurable function with $\int_\Omega w d\mu = 1$.

The following result holds [11]:

Theorem 1.6. *Let* $\Phi : I \to \mathbb{R}$ *be a continuous convex function on the interval of real numbers* I *and* $m, M \in \mathbb{R}$, $m < M$ *with* $[m, M] \subset \mathring{I}$, $\mathring{I}$ *the interior of* I. *If* $f : \Omega \to \mathbb{R}$, *is* μ*-measurable, satisfying the bounds (1.9) and such that* $f, \Phi \circ f \in L_w(\Omega, \mu)$, *then by denoting*

$$\overline{f}_{\Omega,w} := \int_\Omega w f d\mu \in [m, M]$$

and assuming that $\overline{f}_{\Omega,w} \neq m, M,$ *we have*

$$\left|\int_{\Omega}\left|\Phi(f)-\Phi\left(\overline{f}_{\Omega,w}\right)\right|\operatorname{sgn}\left[f-\overline{f}_{\Omega,w}\right]wd\mu\right| \leq \tag{1.12}$$
$$\leq \int_{\Omega}(\Phi\circ f)\,wd\mu-\Phi\left(\overline{f}_{\Omega,w}\right)$$
$$\leq \tfrac{1}{2}\left(\left[\overline{f}_{\Omega,w},M;\Phi\right]-\left[m,\overline{f}_{\Omega,w};\Phi\right]\right)D_w(f)$$
$$\leq \tfrac{1}{2}\left(\left[\overline{f}_{\Omega,w},M;\Phi\right]-\left[m,\overline{f}_{\Omega,w};\Phi\right]\right)D_{w,2}(f)$$
$$\leq \tfrac{1}{4}\left(\left[\overline{f}_{\Omega,w},M;\Phi\right]-\left[m,\overline{f}_{\Omega,w};\Phi\right]\right)(M-m),$$

where

$$D_w(f) := \int_{\Omega} w\left|f-\overline{f}_{\Omega,w}\right|d\mu$$

and

$$D_{w,2}(f) := \left[\int_{\Omega} wf^2d\mu-\left(\overline{f}_{\Omega,w}\right)^2\right]^{\frac{1}{2}}$$

The constant $\frac{1}{2}$ *in the second inequality from (1.10) is the best possible.*

For recent results related to Jensen's inequality, see [1]–[8], [13]–[23] and the references therein.

Motivated by the above results, in this paper we establish some Jensen type inequalities for the class of *AH-convex (concave)* functions. Some applications for power and logarithmic functions are provided as well. Some inequalities for functions of selfadjoint operators in Hilbert spaces are also established.

2. *AH*-CONVEX FUNCTIONS

Let X be a linear space and C a convex subset in X. A function $\Phi : C \to \mathbb{R}\setminus\{0\}$ is called *AH-convex (concave)* on the convex set C if the following inequality holds

(AH)

$$\Phi((1-\lambda)x+\lambda y) \leq (\geq) \frac{1}{(1-\lambda)\frac{1}{\Phi(x)}+\lambda\frac{1}{\Phi(y)}} = \frac{\Phi(x)\Phi(y)}{(1-\lambda)\Phi(y)+\lambda\Phi(x)}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

An important case which provides many examples is that one in which the function is assumed to be positive for any $x \in C$. In that situation the inequality (AH) is equivalent to

$$(1-\lambda)\frac{1}{\Phi(x)}+\lambda\frac{1}{\Phi(y)} \leq (\geq) \frac{1}{\Phi((1-\lambda)x+\lambda y)}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

Therefore we can state the following fact:

Criterion 2.1. *Let X be a linear space and C a convex subset in X. The function $\Phi : C \to (0,\infty)$ is AH-convex (concave) on C if and only if $\frac{1}{\Phi}$ is concave (convex) on C in the usual sense.*

In what follows, we assume that $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. $x \in \Omega$, is a μ-measurable function with $\int_\Omega w d\mu = 1$.

If $\Delta : I \to \mathbb{R}$ is continuous concave function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ the interior of I, then by taking $\Phi = -\Delta$ in (1.12) we get

$$\begin{aligned}
&\left|\int_\Omega \left|\Delta(f) - \Delta\left(\overline{f}_{\Omega,w}\right)\right| \operatorname{sgn}\left[f - \overline{f}_{\Omega,w}\right] w d\mu\right| \leq \\
&\leq \Delta\left(\overline{f}_{\Omega,w}\right) - \int_\Omega (\Delta \circ f) w d\mu \\
&\leq \tfrac{1}{2}\left(\left[m, \overline{f}_{\Omega,w}; \Delta\right] - \left[\overline{f}_{\Omega,w}, M; \Delta\right]\right) D_w(f) \\
&\leq \tfrac{1}{2}\left(\left[m, \overline{f}_{\Omega,w}; \Delta\right] - \left[\overline{f}_{\Omega,w}, M; \Delta\right]\right) D_{w,2}(f) \\
&\leq \tfrac{1}{4}\left(\left[m, \overline{f}_{\Omega,w}; \Delta\right] - \left[\overline{f}_{\Omega,w}, M; \Delta\right]\right) (M - m).
\end{aligned} \tag{2.1}$$

If $\Phi : I \to (0,\infty)$ is continuous AH-convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, then by taking $\Delta = \frac{1}{\Phi}$ in (2.1) we get

$$\begin{aligned}
&\left|\int_\Omega \left|\tfrac{\Phi(\overline{f}_{\Omega,w})}{\Phi(f)} - 1\right| \operatorname{sgn}\left[f - \overline{f}_{\Omega,w}\right] w d\mu\right| \leq \\
&\leq 1 - \Phi\left(\overline{f}_{\Omega,w}\right) \int_\Omega \tfrac{w d\mu}{\Phi \circ f} \\
&\leq \tfrac{1}{2}\left(\tfrac{1}{\Phi(M)}\left[\overline{f}_{\Omega,w}, M; \Phi\right] - \tfrac{1}{\Phi(m)}\left[m, \overline{f}_{\Omega,w}; \Phi\right]\right) D_w(f) \\
&\leq \tfrac{1}{2}\left(\tfrac{1}{\Phi(M)}\left[\overline{f}_{\Omega,w}, M; \Phi\right] - \tfrac{1}{\Phi(m)}\left[m, \overline{f}_{\Omega,w}; \Phi\right]\right) D_{w,2}(f) \\
&\leq \tfrac{1}{4}\left(\tfrac{1}{\Phi(M)}\left[\overline{f}_{\Omega,w}, M; \Phi\right] - \tfrac{1}{\Phi(m)}\left[m, \overline{f}_{\Omega,w}; \Phi\right]\right) (M - m),
\end{aligned}$$

provided that $f : \Omega \to \mathbb{R}$, is μ-measurable, satisfying the bounds (1.9) and such that $f, (\Phi \circ f)^{-1} \in L_w(\Omega, \mu)$. As above

$$\overline{f}_{\Omega,w} := \int_\Omega w f d\mu \in [m, M]$$

and we assume that $\overline{f}_{\Omega,w} \neq m, M$,

In the case of functions defined on real line we have:

Proposition 2.2. *Let $\Phi : I \to (0,\infty)$ be defined on the interval I. The following statements are equivalent:*

(i) The function Φ is AH-convex (concave) on I;

(ii) For any $x, y \in \mathring{I}$*, the interior of* I*, then there exists* $\varphi(y) \in [\Phi'_-(y), \Phi'_+(y)]$ *such that*

$$\frac{\Phi(y)}{\Phi(x)} - 1 \leq (\geq) \frac{\varphi(y)}{\Phi(y)}(y-x) \tag{2.2}$$

holds.

Proof. (i) $\Rightarrow$ (ii). Let $x, y \in \mathring{I}$. Since the function $\frac{1}{\Phi}$ is concave (convex) then the lateral derivatives $\Phi'_-(y), \Phi'_+(y)$ exists for $y \in \mathring{I}$ and $\left(\frac{1}{\Phi}\right)'_{-(+)}(y) = -\frac{\Phi'_{-(+)}(y)}{\Phi^2(y)}$.

Since $\frac{1}{\Phi}$ is concave (convex) then we have the gradient inequality

$$\frac{1}{\Phi(y)} - \frac{1}{\Phi(x)} \geq (\leq) \lambda(y)(y-x) = -\lambda(y)(x-y)$$

with $\lambda(y) \in \left[-\frac{\Phi'_+(y)}{\Phi^2(y)}, -\frac{\Phi'_-(y)}{\Phi^2(y)}\right]$, which is equivalent to

$$\frac{1}{\Phi(y)} - \frac{1}{\Phi(x)} \geq (\leq) \frac{\varphi(y)}{\Phi^2(y)}(x-y) \tag{2.3}$$

with $\varphi(y) \in \left[\Phi'_-(y), \Phi'_+(y)\right]$.

The inequality (2.3) can be also written as

$$1 - \frac{\Phi(y)}{\Phi(x)} \geq (\leq) \frac{\varphi(y)}{\Phi(y)}(x-y)$$

or as

$$\frac{\Phi(y)}{\Phi(x)} - 1 \leq (\geq) \frac{\varphi(y)}{\Phi(y)}(y-x)$$

and the inequality (2.2) is proved.

"(ii) $\Rightarrow$ (i)" Let $x, y \in I$ and $\lambda \in (0,1)$. Then $(1-\lambda)x + \lambda y \in \mathring{I}$. From (2.3) we have

$$\begin{aligned} \frac{1}{\Phi(x)} - \frac{1}{\Phi((1-\lambda)x+\lambda y)} &\geq (\leq) \frac{\varphi((1-\lambda)x+\lambda y)}{\Phi^2((1-\lambda)x+\lambda y)}((1-\lambda)x+\lambda y - x) \\ &= \lambda(y-x)\frac{\varphi((1-\lambda)x+\lambda y)}{\Phi^2((1-\lambda)x+\lambda y)} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \frac{1}{\Phi(y)} - \frac{1}{\Phi((1-\lambda)x+\lambda y)} &\geq (\leq) \frac{\varphi((1-\lambda)x+\lambda y)}{\Phi((1-\lambda)x+\lambda y)}((1-\lambda)x+\lambda y - y) \\ &= -(1-\lambda)(y-x)\frac{\varphi((1-\lambda)x+\lambda y)}{\Phi^2((1-\lambda)x+\lambda y)}. \end{aligned} \tag{2.5}$$

If we multiply (2.4) by $1-\lambda$ and (2.5) by λ and add the obtained results, we get

$$\frac{1-\lambda}{\Phi(x)}+\frac{\lambda}{\Phi(y)}-\frac{1}{\Phi((1-\lambda)x+\lambda y)}\geq(\leq)\,0$$

for any $\lambda\in[0,2]$, which shows that Φ is AH-convex (concave) on I. □

Corollary 2.3. *Let $\Phi: I\to(0,\infty)$ be differentiable on $\mathring{I}$. Then Φ is AH-convex (concave) on I if and only if for any $x,y\in\mathring{I}$, we have*

$$\frac{\Phi(y)}{\Phi(x)}-1\leq(\geq)\,\frac{\Phi'(y)}{\Phi(y)}(y-x). \tag{2.6}$$

If $\Delta: I\to\mathbb{R}$ is differentiable concave function on the interval of real numbers I and $m,M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\mathring{I}$, and $f:\Omega\to[m,M]$ then by taking $\Phi=-\Delta$ in (1.8) we get

$$\begin{aligned}0&\leq\Delta\left(\overline{f}_{\Omega,w}\right)-\int_\Omega(\Delta\circ f)\,wd\mu\\&\leq\overline{f}_{\Omega,w}\int_\Omega(\Delta'\circ f)\,wd\mu-\int_\Omega(\Delta'\circ f)\,fwd\mu\\&\leq\tfrac{1}{2}\left[\Delta'(m)-\Delta'(M)\right]D_w(f)\\&\leq\tfrac{1}{2}\left[\Delta'(m)-\Delta'(M)\right]D_{w,2}(f)\\&\leq\tfrac{1}{4}\left[\Delta'(m)-\Delta'(M)\right](M-m).\end{aligned}\tag{2.7}$$

If $\Phi: I\to(0,\infty)$ is continuous AH-convex function on the interval of real numbers I and $m,M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\mathring{I}$, then by taking $\Delta=\frac{1}{\Phi}$, which is differentiable concave, in (2.7) we get

$$\begin{aligned}0&\leq\left[\Phi\left(\overline{f}_{\Omega,w}\right)\right]^{-1}-\int_\Omega\tfrac{wd\mu}{\Phi\circ f}\\&\leq\int_\Omega\tfrac{\Phi'\circ f}{\Phi^2(f)}fwd\mu-\overline{f}_{\Omega,w}\int_\Omega\tfrac{\Phi'\circ f}{\Phi^2(f)}wd\mu\\&\leq\tfrac{1}{2}\left[\tfrac{\Phi'(M)}{\Phi^2(M)}-\tfrac{\Phi'(m)}{\Phi^2(m)}\right]D_w(f)\\&\leq\tfrac{1}{2}\left[\tfrac{\Phi'(M)}{\Phi^2(M)}-\tfrac{\Phi'(m)}{\Phi^2(m)}\right]D_{w,2}(f)\\&\leq\tfrac{1}{4}\left[\tfrac{\Phi'(M)}{\Phi^2(M)}-\tfrac{\Phi'(m)}{\Phi^2(m)}\right](M-m),\end{aligned}\tag{2.8}$$

provided $f:\Omega\to[m,M]$ so that $\frac{1}{\Phi\circ f}$, f, $\frac{\Phi'\circ f}{\Phi^2(f)}f$, $\frac{\Phi'\circ f}{\Phi^2(f)}\in L_w(\Omega,\mu)$.

If $w\geq0$ a.e. on Ω with $\int_\Omega wd\mu=1$ and $\Phi: I\to(0,\infty)$ is continuous AH-convex on I then

$$p:=\frac{(\Phi\circ f)\,w}{\int_\Omega(\Phi\circ f)\,wd\mu}$$

is a weight with $\int_\Omega pd\mu=1$.

We have

$$\overline{f}_{\Omega,p}=\int_\Omega pfd\mu=\frac{\int_\Omega(\Phi\circ f)\,fwd\mu}{\int_\Omega(\Phi\circ f)\,wd\mu}$$

and

$$\int_{\Omega} \frac{pd\mu}{\Phi \circ f} = \frac{1}{\int_{\Omega} (\Phi \circ f) \, wd\mu}.$$

If we use the first inequality in (2.8) for the weight p we get

$$0 \leq \left[\Phi \left(\frac{\int_{\Omega} (\Phi \circ f) \, f wd\mu}{\int_{\Omega} (\Phi \circ f) \, wd\mu} \right) \right]^{-1} - \frac{1}{\int_{\Omega} (\Phi \circ f) \, wd\mu},$$

which is equivalent to

$$\int_{\Omega} (\Phi \circ f) \, wd\mu \geq \Phi \left(\frac{\int_{\Omega} (\Phi \circ f) \, f wd\mu}{\int_{\Omega} (\Phi \circ f) \, wd\mu} \right), \tag{2.9}$$

where $f : \Omega \to I$ and $\Phi : I \to (0, \infty)$ is continuous AH-convex so that $(\Phi \circ f)$, f, $(\Phi \circ f) \, f$, $(\Phi \circ f) \in L_w (\Omega, \mu)$.

Now, if we use the inequality between the first and last term in (2.8) we get

$$\begin{aligned} 0 &\leq \left[\Phi \left(\frac{\int_{\Omega} (\Phi \circ f) \, f wd\mu}{\int_{\Omega} (\Phi \circ f) \, wd\mu} \right) \right]^{-1} - \frac{1}{\int_{\Omega} (\Phi \circ f) \, wd\mu} \\ &\leq \frac{1}{4} \left[\frac{\Phi'_{-} (M)}{\Phi^2 (M)} - \frac{\Phi'_{+} (m)}{\Phi^2 (m)} \right] (M - m), \end{aligned} \tag{2.10}$$

where $f : \Omega \to [m, M] \subset I$ and $\Phi : I \to (0, \infty)$ is continuous AH-convex so that $(\Phi \circ f)$, f, $(\Phi \circ f) \, f$, $(\Phi \circ f) \in L_w (\Omega, \mu)$.

Similar results may be stated by using various reverses of Jensen's inequality as stated in the introduction.

Let $\Phi : [m, M] \to \mathbb{R}$ be a AH-convex function on $[m, M]$. If $x_i \in [m, M]$ and $w_i \geq 0 \ \ (i = 1, \dots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then from (2.9) one has the weighted discrete inequality:

$$\Phi \left(\frac{\sum_{i=1}^n w_i x_i \Phi(x_i)}{\sum_{i=1}^n w_i \Phi(x_i)} \right) \leq \sum_{i=1}^n w_i \Phi (x_i) \tag{2.11}$$

while from (2.10) we have

$$\begin{aligned} 0 &\leq \left[\Phi \left(\frac{\sum_{i=1}^n w_i x_i \Phi(x_i)}{\sum_{i=1}^n w_i \Phi(x_i)} \right) \right]^{-1} - \frac{1}{\sum_{i=1}^n w_i \Phi(x_i)} \\ &\leq \frac{1}{4} \left[\frac{\Phi'_{-}(M)}{\Phi^2(M)} - \frac{\Phi'_{+}(m)}{\Phi^2(m)} \right] (M - m). \end{aligned} \tag{2.12}$$

3. NEW RESULTS

In what follows, we assume that $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. $x \in \Omega$, is a μ-measurable function with $\int_{\Omega} wd\mu = 1$.

The following result holds:

THEOREM 3.1. *Let $\Phi : I \to (0, \infty)$ be differentiable on $\mathring{I}$. If Φ is AH-convex (concave) on I and $f : \Omega \to I$ so that $\Phi \circ f$, $\frac{(\Phi' \circ f) f}{\Phi \circ f}$, $\frac{\Phi' \circ f}{\Phi \circ f} \in L_w (\Omega, \mu)$, then*

$$\frac{1}{\Phi(x)} \int_\Omega (\Phi \circ f) \, w d\mu - 1 \le (\ge) \le (\ge) \int_\Omega \frac{(\Phi' \circ f) f w}{\Phi \circ f} d\mu - x \int_\Omega \frac{(\Phi' \circ f) w}{\Phi \circ f} d\mu \tag{3.1}$$

for any $x \in I$.

Moreover, if

$$\frac{\int_\Omega \frac{(\Phi' \circ f) f w}{\Phi \circ f} d\mu}{\int_\Omega \frac{(\Phi' \circ f) w}{\Phi \circ f} d\mu} \in I, \tag{3.2}$$

then

$$\int_\Omega (\Phi \circ f) \, w d\mu \le (\ge) \, \Phi \left(\frac{\int_\Omega \frac{(\Phi' \circ f) f w}{\Phi \circ f} d\mu}{\int_\Omega \frac{(\Phi' \circ f) w}{\Phi \circ f} d\mu} \right). \tag{3.3}$$

Proof. From (2.6) we have for any $x \in I$, that

$$\frac{\Phi \circ f}{\Phi(x)} - 1 \le (\ge) \, \frac{\Phi' \circ f}{\Phi \circ f} (f - x) \tag{3.4}$$

almost everywhere on Ω.

If we multiply this by $w \ge 0$ a.e. on Ω we get

$$\frac{(\Phi \circ f) w}{\Phi(x)} - w \le (\ge) \, \frac{(\Phi' \circ f) f w}{\Phi \circ f} - \frac{(\Phi' \circ f) w}{\Phi \circ f} x, \tag{3.5}$$

almost everywhere on Ω.

Integrating the inequality (3.5) on Ω and taking into account that $\int_\Omega w d\mu = 1$, we get

$$\frac{1}{\Phi(x)} \int_\Omega (\Phi \circ f) \, w d\mu - 1 \le (\ge) \le (\ge) \int_\Omega \frac{(\Phi' \circ f) f w}{\Phi \circ f} d\mu - x \int_\Omega \frac{(\Phi' \circ f) w}{\Phi \circ f} d\mu \tag{3.6}$$

and the inequality (3.1) is proved.

The inequality (3.3) follows by (3.1) by taking

$$x = \frac{\int_\Omega \frac{(\Phi' \circ f) f w}{\Phi \circ f} d\mu}{\int_\Omega \frac{(\Phi' \circ f) w}{\Phi \circ f} d\mu} \in I.$$

□

COROLLARY 3.2. *With the assumptions of Theorem 3.1 and if $f : \Omega \to [m, M] \subset I$ and Φ is monotonic nondecreasing or nonincreasing on $[m, M]$, then the inequality (3.3) holds.*

Proof. Since $m \le f \le M$ a.e. on Ω, then for Φ monotonic nondecreasing we have

$$m\frac{(\Phi'\circ f)w}{\Phi\circ f} \le \frac{(\Phi'\circ f)w}{\Phi\circ f}f \le M\frac{(\Phi'\circ f)w}{\Phi\circ f}$$

a.e. on Ω.

Integrating on Ω we get

$$m\int_\Omega \frac{(\Phi'\circ f)w}{\Phi\circ f}d\mu \le \int_\Omega \frac{(\Phi'\circ f)w}{\Phi\circ f}fd\mu \le M\int_\Omega \frac{(\Phi'\circ f)w}{\Phi\circ f}d\mu,$$

which shows that the condition (3.2) is satisfied.

The case of Φ is nonincreasing goes likewise and the statement is proved. $\square$

Remark 3.3. If $f : \Omega \to [m, M] \subset I$, then by taking

$$x = \int_\Omega fwd\mu \in [m, M]$$

in (3.1) we get the inequality

$$\frac{1}{\Phi\left(\int_\Omega fwd\mu\right)}\int_\Omega (\Phi\circ f)\,wd\mu - 1 \le (\ge) \le (\ge)\int_\Omega \frac{(\Phi'\circ f)fw}{\Phi\circ f}d\mu - \int_\Omega fwd\mu\int_\Omega \frac{(\Phi'\circ f)w}{\Phi\circ f}d\mu. \tag{3.7}$$ $\square$

Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable AH-convex (concave) function on $[m, M]$. If $x_i \in [m, M]$ and $w_i \ge 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then from (3.1) one has the weighted discrete inequality:

$$\frac{1}{\Phi(x)}\sum_{i=1}^n w_i\Phi(x_i) - 1 \le (\ge) \le (\ge)\sum_{i=1}^n \frac{\Phi'(x_i)x_iw_i}{\Phi(x_i)} - x\sum_{i=1}^n \frac{\Phi'(x_i)w_i}{\Phi(x_i)} \tag{3.8}$$

for any $x \in [m, M]$.

In particular, we have

$$\frac{1}{\Phi\left(\sum_{i=1}^n w_ix_i\right)}\sum_{i=1}^n w_i\Phi(x_i) - 1 \le (\ge) \le (\ge)\sum_{i=1}^n \frac{\Phi'(x_i)x_iw_i}{\Phi(x_i)} - \sum_{i=1}^n w_ix_i\sum_{i=1}^n \frac{\Phi'(x_i)w_i}{\Phi(x_i)}. \tag{3.9}$$

Moreover, if Φ is monotonic nondecreasing or nonincreasing on $[m, M]$, then we have the inequality

$$\sum_{i=1}^n w_i\Phi(x_i) \le (\ge)\,\Phi\left(\frac{\sum_{i=1}^n \frac{\Phi'(x_i)x_iw_i}{\Phi(x_i)}}{\sum_{i=1}^n \frac{\Phi'(x_i)w_i}{\Phi(x_i)}}\right).$$

The following result also holds:

Theorem 3.4. *Let* $\Phi : I \to (0,\infty)$ *be differentiable on* $\mathring{I}$. *If* Φ *is* AH*-convex (concave) on* I *and* $f : \Omega \to I$ *so that* $\Phi \circ f$, $\frac{(\Phi' \circ f) f}{\Phi \circ f}$, $\frac{\Phi' \circ f}{\Phi \circ f} \in L_w(\Omega,\mu)$, *then*

$$\frac{1}{\Phi(x)} \int_\Omega (\Phi \circ f)^2 w d\mu - \int_\Omega (\Phi \circ f) w d\mu \leq (\geq) \tag{3.10}$$
$$\leq (\geq) \int_\Omega (\Phi' \circ f) f w d\mu - x \int_\Omega (\Phi' \circ f) w d\mu$$

for any $x \in I$.

Moreover, if

$$\frac{\int_\Omega (\Phi' \circ f) f w d\mu}{\int_\Omega (\Phi' \circ f) w d\mu} \in I, \tag{3.11}$$

then

$$\frac{\int_\Omega (\Phi \circ f)^2 w d\mu}{\int_\Omega (\Phi \circ f) w d\mu} \leq (\geq) \Phi \left(\frac{\int_\Omega (\Phi' \circ f) f w d\mu}{\int_\Omega (\Phi' \circ f) w d\mu} \right). \tag{3.12}$$

Proof. From (2.6) we have for any $x \in I$, that

$$\frac{1}{\Phi(x)} (\Phi \circ f)^2 - \Phi \circ f \leq (\geq) (\Phi' \circ f)(f - x)$$

almost everywhere on Ω.

If we multiply this by $w \geq 0$ a.e. on Ω we get

$$\frac{1}{\Phi(x)} (\Phi \circ f)^2 w - (\Phi \circ f) w \leq (\geq) (\Phi' \circ f) f w - (\Phi' \circ f) w x \tag{3.13}$$

almost everywhere on Ω.

Integrating the inequality (3.13) on Ω and taking into account that $\int_\Omega w d\mu = 1$, we get

$$\frac{1}{\Phi(x)} \int_\Omega (\Phi \circ f)^2 w d\mu - \int_\Omega (\Phi \circ f) w d\mu \leq (\geq) \tag{3.14}$$
$$\leq (\geq) \int_\Omega (\Phi' \circ f) f w d\mu - x \int_\Omega (\Phi' \circ f) w d\mu$$

and the inequality (3.1) is proved.

If we take

$$x = \frac{\int_\Omega (\Phi' \circ f) f w d\mu}{\int_\Omega (\Phi' \circ f) w d\mu}$$

in (3.10) and do the required calculation, we get the desired result (3.12). □

Corollary 3.5. *With the assumptions of Theorem 3.4 and if* $f : \Omega \to [m, M] \subset I$ *and* Φ *is monotonic nondecreasing (nonincreasing) on* $[m, M]$, *then the inequality (3.12) holds.*

Remark 3.6. If $f : \Omega \to [m, M] \subset I$, then by taking

$$x = \int_\Omega f w d\mu \in [m, M]$$

in (3.10) we get the inequality

$$\frac{1}{\Phi\left(\int_\Omega f w d\mu\right)} \int_\Omega (\Phi \circ f)^2 \, w d\mu - \int_\Omega (\Phi \circ f) \, w d\mu \le (\ge) \le (\ge) \int_\Omega (\Phi' \circ f) \, f w d\mu - \int_\Omega f w d\mu \int_\Omega (\Phi' \circ f) \, w d\mu. \tag{3.15}$$

□

Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable AH-convex (concave) function on $[m, M]$. If $x_i \in [m, M]$ and $w_i \ge 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then from (3.10) one has the weighted discrete inequality:

$$\frac{1}{\Phi(x)} \sum_{i=1}^n w_i \Phi^2 (x_i) - \sum_{i=1}^n w_i \Phi (x_i) \le (\ge) \le (\ge) \sum_{i=1}^n w_i \Phi' (x_i) \, x_i - x \sum_{i=1}^n w_i \Phi' (x_i) \tag{3.16}$$

for any $x \in [m, M]$.

In particular we have

$$\frac{1}{\Phi\left(\sum_{i=1}^n w_i x_i\right)} \sum_{i=1}^n w_i \Phi^2 (x_i) - \sum_{i=1}^n w_i \Phi (x_i) \le (\ge) \le (\ge) \sum_{i=1}^n w_i \Phi' (x_i) \, x_i - \sum_{i=1}^n w_i x_i \sum_{i=1}^n w_i \Phi' (x_i). \tag{3.17}$$

Moreover, if Φ is monotonic nondecreasing or nonincreasing on $[m, M]$, then we have the inequality

$$\frac{\sum_{i=1}^n w_i \Phi^2 (x_i)}{\sum_{i=1}^n w_i \Phi (x_i)} \le (\ge) \, \Phi \left(\frac{\sum_{i=1}^n w_i \Phi' (x_i) x_i}{\sum_{i=1}^n w_i \Phi' (x_i)} \right). \tag{3.18}$$

4. SOME EXAMPLES

Consider the function

$$\Phi_p (t) = t^p = \frac{1}{t^{-p}}$$

if $-p > 1$ or $-p < 0$, i.e. $p \in (-\infty, -1) \cup (0, \infty)$ then the function $\Phi_p (t) = t^p, t > 0$ is AM-concave. If $p \in (-1, 0)$ then the function $\Phi_p (t) = t^p, t > 0$ is AM-convex.

If we apply the inequality (3.3) for Φ_p, then we get

$$\int_\Omega f^p w d\mu \left(\int_\Omega \frac{w}{f} d\mu \right)^p \le (\ge) \, 1, \tag{4.1}$$

where $p \in (-1, 0) \, (p \in (-\infty, -1) \cup (0, \infty))$, and $f : \Omega \to (0, \infty)$ is such that $f^p, \frac{1}{f} \in L_w (\Omega, \mu)$.

If we use the inequality (3.12) for Φ_p, then we get

$$\frac{\int_\Omega f^{2p} w d\mu}{\int_\Omega f^p w d\mu} \le (\ge) \left(\frac{\int_\Omega f^p w d\mu}{\int_\Omega f^{p-1} w d\mu} \right)^p, \tag{4.2}$$

where $p \in (-1,0)\,(p \in (-\infty,-1) \cup (0,\infty))$, and $f : \Omega \to (0,\infty)$ is such that $f^{2p},\ f^{p},\ f^{p-1} \in L_w(\Omega,\mu)$.

If we use the inequality (3.7) for Φ_p, then we have

$$\frac{1}{\left(\int_\Omega fwd\mu\right)^p}\int_\Omega f^p wd\mu - 1 \le (\ge)\, p\left[1 - \int_\Omega fwd\mu \int_\Omega \tfrac{w}{f}d\mu\right], \tag{4.3}$$

where $p \in (-1,0)\,(p \in (-\infty,-1) \cup (0,\infty))$, and $f : \Omega \to (0,\infty)$ is such that $f^{p},\ f,\ \frac{1}{f} \in L_w(\Omega,\mu)$.

By Čebyšev inequality for asynchronous functions we have

$$1 \le \int_\Omega fwd\mu \int_\Omega \tfrac{w}{f}d\mu$$

and then by (4.2) for $p \in (-\infty,-1)$ we have

$$\frac{1}{\left(\int_\Omega fwd\mu\right)^p}\int_\Omega f^p wd\mu - 1 \ge p\left[1 - \int_\Omega fwd\mu \int_\Omega \tfrac{w}{f}d\mu\right] \ge 0. \tag{4.4}$$

If we denote $r = -p \in (1,\infty)$, then the inequality (4.4) can be written as

$$\left(\int_\Omega fwd\mu\right)^r \int_\Omega \tfrac{w}{f^r}d\mu - 1 \ge r\left[\int_\Omega fwd\mu \int_\Omega \tfrac{w}{f}d\mu - 1\right] \ge 0 \tag{4.5}$$

provided $\frac{1}{f^r},\ f,\ \frac{1}{f} \in L_w(\Omega,\mu)$.

Also, if we use the inequality (3.15) for Φ_p, then we have

$$\begin{aligned} &\frac{1}{\left(\int_\Omega fwd\mu\right)^p}\int_\Omega f^{2p} wd\mu - \int_\Omega f^p wd\mu \le (\ge) \\ &\le (\ge)\, p\left[\int_\Omega f^p wd\mu - \int_\Omega fwd\mu \int_\Omega f^{p-1}wd\mu\right], \end{aligned} \tag{4.6}$$

where $p \in (-1,0)\,(p \in (-\infty,-1) \cup (0,\infty))$, and $f : \Omega \to (0,\infty)$ is such that $f^{2p},\ f^{p},\ f^{p-1},\ f \in L_w(\Omega,\mu)$.

We observe that by Čebyšev inequality for synchronous functions we have for $p \ge 1$

$$\int_\Omega f^p wd\mu \ge \int_\Omega fwd\mu \int_\Omega f^{p-1}wd\mu$$

and by (4.6) we have

$$\begin{aligned} &\frac{1}{\left(\int_\Omega fwd\mu\right)^p}\int_\Omega f^{2p} wd\mu - \int_\Omega f^p wd\mu \ge \\ &\ge p\left[\int_\Omega f^p wd\mu - \int_\Omega fwd\mu \int_\Omega f^{p-1}wd\mu\right] \ge 0 \end{aligned} \tag{4.7}$$

for $p \ge 1$.

Now consider the function $\Phi_{\ln}(t) = \ln t,\ t > 1$. The function

$$g(t) := \tfrac{1}{\ln t},\ \ t > 1$$

is convex on $(1,\infty)$. Therefore $\Phi_{\ln}(t) = \ln t,\ t > 1$ is a AM-concave function on $(1,\infty)$.

Let $f:\Phi\rightarrow(1,\infty)$ be so that $\ln f,\ \frac{1}{\ln f},\ \frac{1}{f\ln f}\in L_{w}(\Omega,\mu)$, then by using the inequality (3.3) for $\Phi_{\ln}$ we have

$$\int_{\Omega}w\ln f d\mu\geq\ln\left(\frac{\int_{\Omega}\frac{w}{\ln f}d\mu}{\int_{\Omega}\frac{w}{f\ln f}d\mu}\right). \tag{4.8}$$

Let $f:\Phi\rightarrow(1,\infty)$ be so that $(\ln f)^{2},\ \ln f,\ \frac{1}{f}\in L_{w}(\Omega,\mu)$, then by (3.12) for $\Phi_{\ln}$ we get

$$\frac{\int_{\Omega}w(\ln f)^{2}d\mu}{\int_{\Omega}w\ln f d\mu}\geq\ln\left(\frac{1}{\int_{\Omega}\frac{w}{f}d\mu}\right). \tag{4.9}$$

5. APPLICATIONS FOR FUNCTIONS OF SELFADJOINT OPERATORS

Let A be a selfadjoint operator on the complex Hilbert space $(H,\langle .,.\rangle)$ with the spectrum $Sp(A)$ included in the interval $[m,M]$ for some real numbers $m<M$ and let $\{E_{\lambda}\}_{\lambda}$ be its *spectral family.* Then for any continuous function $f:[m,M]\rightarrow[a,b]$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral* (see for instance [13, p. 257]):

$$\langle f(A)x,y\rangle=\int_{m-0}^{M}f(\lambda)\,d(\langle E_{\lambda}x,y\rangle), \tag{5.1}$$

and

$$\|f(A)x\|^{2}=\int_{m-0}^{M}|f(\lambda)|^{2}\,d\|E_{\lambda}x\|^{2}, \tag{5.2}$$

for any $x,y\in H$.

The function $g_{x,y}(\lambda):=\langle E_{\lambda}x,y\rangle$ is of *bounded variation* on the interval $[m,M]$ and

$$g_{x,y}(m-0)=0 \text{ while } g_{x,y}(M)=\langle x,y\rangle$$

for any $x,y\in H$. It is also well known that $g_{x}(\lambda):=\langle E_{\lambda}x,x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m,M]$ for any $x\in H$.

Now, assume that $\Phi:[a,b]\subset I\rightarrow(0,\infty)$ is continuous AH-convex function on the interval of real numbers I, $f:[m,M]\rightarrow[a,b]$, $p:[m,M]\rightarrow(0,\infty)$ are continuous functions on $[m,M]$ and $g:[m,M]\rightarrow\mathbb{R}$ is monotonic nondecreasing on $[m,M]$.

By (2.9) and (2.10) we have the following inequalities for the Riemann-Stieltjes integral:

$$\frac{\int_{m}^{M}\Phi(f(t))\,p(t)\,dg(t)}{\int_{m}^{M}p(t)\,dg(t)}\geq\Phi\left(\frac{\int_{m}^{M}\Phi(f(t))\,f(t)\,p(t)\,dg(t)}{\int_{m}^{M}\Phi(f(t))\,p(t)\,dg(t)}\right), \tag{5.3}$$

and

$$0 \le \left[\Phi\left(\frac{\int_m^M \Phi(f(t))f(t)p(t)dg(t)}{\int_m^M \Phi(f(t))p(t)dg(t)}\right)\right]^{-1} - \frac{\int_m^M p(t)\,dg(t)}{\int_m^M \Phi(f(t))\,p(t)\,dg(t)} \le \frac{1}{4}\left[\frac{\Phi'_-(M)}{\Phi^2(M)} - \frac{\Phi'_+(m)}{\Phi^2(m)}\right](M-m). \tag{5.4}$$

Now, if we apply the inequalities (5.3) and (5.4) for the monotonic nondecreasing function $g_x(\lambda) := \langle E_\lambda x, x\rangle$, $x \in H$, where $\{E_\lambda\}_\lambda$ is the spectral family of A, then we get

$$\frac{\langle \Phi(f(A))\,p(A)\,x, x\rangle}{\langle p(A)\,x, x\rangle} \ge \Phi\left(\frac{\langle \Phi(f(A))\,f(A)\,p(A)\,x, x\rangle}{\langle \Phi(f(A))\,p(A)\,x, x\rangle}\right), \tag{5.5}$$

and

$$0 \le \left[\Phi\left(\frac{\langle \Phi(f(A))\,f(A)\,p(A)\,x, x\rangle}{\langle \Phi(f(A))\,p(A)\,x, x\rangle}\right)\right]^{-1} - \frac{\langle p(A)\,x, x\rangle}{\langle \Phi(f(A))\,p(A)\,x, x\rangle} \le \frac{1}{4}\left[\frac{\Phi'_-(M)}{\Phi^2(M)} - \frac{\Phi'_+(m)}{\Phi^2(m)}\right](M-m), \tag{5.6}$$

for any $x \in H$, $x \neq 0$.

In particular, if p is taken to be the constant 1, then for any $x \in H$, $\|x\| = 1$, we have

$$\langle \Phi(f(A))\,x, x\rangle \ge \Phi\left(\frac{\langle \Phi(f(A))\,f(A)\,x, x\rangle}{\langle \Phi(f(A))\,x, x\rangle}\right), \tag{5.7}$$

and

$$0 \le \left[\Phi\left(\frac{\langle \Phi(f(A))\,f(A)\,x, x\rangle}{\langle \Phi(f(A))\,x, x\rangle}\right)\right]^{-1} - \frac{1}{\langle \Phi(f(A))\,x, x\rangle} \le \frac{1}{4}\left[\frac{\Phi'_-(M)}{\Phi^2(M)} - \frac{\Phi'_+(m)}{\Phi^2(m)}\right](M-m). \tag{5.8}$$

Moreover, if $[a,b] = [m,M]$ and $f(t) = t$, then from (5.7) and (5.8) we get

$$\langle \Phi(A)\,x, x\rangle \ge \Phi\left(\frac{\langle \Phi(A)\,Ax, x\rangle}{\langle \Phi(A)\,x, x\rangle}\right), \tag{5.9}$$

and

$$0 \le \left[\Phi\left(\frac{\langle \Phi(A)\,Ax, x\rangle}{\langle \Phi(A)\,x, x\rangle}\right)\right]^{-1} - \frac{1}{\langle \Phi(A)\,x, x\rangle} \le \frac{1}{4}\left[\frac{\Phi'_-(M)}{\Phi^2(M)} - \frac{\Phi'_+(m)}{\Phi^2(m)}\right](M-m). \tag{5.10}$$

for any $x \in H$, $\|x\| = 1$.

Further on, assume that $\Phi : [a,b] \subset I \to (0,\infty)$ is continuously differentiable AH-convex and monotonic nondecreasing (nonincreasing) function on the interval of real numbers I, $f : [m,M] \to [a,b]$, $p : [m,M] \to (0,\infty)$ are continuous functions on $[m, M]$ and $g : [m, M] \to \mathbb{R}$ is monotonic nondecreasing on $[m, M]$.

If we write the inequalities (3.3) and (3.12) for the Riemann-Stieltjes integral, then we have

$$\frac{\int_m^M \Phi(f(t)) p(t) dg(t)}{\int_m^M p(t) dg(t)} \le \Phi\left(\frac{\int_m^M \frac{\Phi'(f(t)) f(t) p(t)}{\Phi(f(t))} dg(t)}{\int_m^M \frac{\Phi'(f(t)) p(t)}{\Phi(f(t))} dg(t)}\right) \tag{5.11}$$

and

$$\frac{\int_m^M (\Phi(f(t)))^2 p(t) dg(t)}{\int_m^M \Phi(f(t)) p(t) dg(t)} \le \Phi\left(\frac{\int_m^M \Phi'(f(t)) f(t) p(t) dg(t)}{\int_m^M \Phi'(f(t)) p(t) dg(t)}\right). \tag{5.12}$$

The inequalities (5.11) and (5.12) imply the following operator inequalities

$$\frac{\langle \Phi(f(A)) p(A) x, x\rangle}{\langle p(A) x, x\rangle} \le \Phi\left(\frac{\left\langle \frac{\Phi'(f(A)) f(A) p(A)}{\Phi(f(A))} x, x\right\rangle}{\left\langle \frac{\Phi'(f(A)) p(A)}{\Phi(f(A))} x, x\right\rangle}\right) \tag{5.13}$$

and

$$\frac{\left\langle (\Phi(f(A)))^2 p(A) x, x\right\rangle}{\langle \Phi(f(A)) p(A) x, x\rangle} \le \Phi\left(\frac{\langle \Phi'(f(A)) f(A) p(A) x, x\rangle}{\langle \Phi'(f(A)) p(A) x, x\rangle}\right) \tag{5.14}$$

for any $x \in H$, $x \ne 0$.

In particular, if p is taken to be the constant 1, then for any $x \in H$, $\|x\| = 1$, we have

$$\langle \Phi(f(A)) x, x\rangle \le \Phi\left(\frac{\left\langle \frac{\Phi'(f(A)) f(A)}{\Phi(f(A))} x, x\right\rangle}{\left\langle \frac{\Phi'(f(A))}{\Phi(f(A))} x, x\right\rangle}\right) \tag{5.15}$$

and

$$\frac{\left\langle (\Phi(f(A)))^2 x, x\right\rangle}{\langle \Phi(f(A)) x, x\rangle} \le \Phi\left(\frac{\langle \Phi'(f(A)) f(A) x, x\rangle}{\langle \Phi'(f(A)) x, x\rangle}\right). \tag{5.16}$$

Moreover, if $[a, b] = [m, M]$ and $f(t) = t$, then from (5.15) and (5.16) we get

$$\langle \Phi(A) x, x\rangle \le \Phi\left(\frac{\left\langle \frac{\Phi'(A) A}{\Phi(A)} x, x\right\rangle}{\left\langle \frac{\Phi'(A)}{\Phi(A)} x, x\right\rangle}\right) \tag{5.17}$$

and

$$\frac{\left\langle (\Phi(A))^2 x, x\right\rangle}{\langle \Phi(A) x, x\rangle} \le \Phi\left(\frac{\langle \Phi'(A) A x, x\rangle}{\langle \Phi'(A) x, x\rangle}\right), \tag{5.18}$$

for any $x \in H$, $\|x\| = 1$.

If $\Phi : [a, b] \subset I \to (0, \infty)$ is AH-concave, then the reverse versions of the inequalities above hold. We omit the details.

The interested reader may state various power inequalities for positive definite operators by choosing $\Phi_p(t) = t^p$ which is AH-convex for $p \in (-1, 0)$ and AM-concave for $p \in (-\infty, -1) \cup (0, \infty)$.

For instance, if we take $p = -r$, with $r \in (0, 1)$ then from (5.7), and (5.18) we get

$$\langle A^{-r}x, x\rangle \geq \left(\frac{\langle A^{1-r}x, x\rangle}{\langle A^{-r}x, x\rangle}\right)^{-r},$$

and

$$\frac{\langle A^{-2r}x, x\rangle}{\langle A^{-r}x, x\rangle} \leq \left(\frac{\langle A^{-r}x, x\rangle}{\langle A^{-r-1}x, x\rangle}\right)^{-r},$$

which can be written as

$$\langle A^{-r}x, x\rangle^{1-r} \left\langle A^{1-r}x, x\right\rangle^{r} \geq 1, \tag{5.19}$$

and

$$\left\langle A^{-2r}x, x\right\rangle \leq \left\langle A^{-r-1}x, x\right\rangle^{r} \langle A^{-r}x, x\rangle^{1-r}, \tag{5.20}$$

for any $x \in H$, $\|x\| = 1$, where A is a positive definite operator on H and $r \in (0, 1)$.

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