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ON SZÁSZ-MIRAKYAN TYPE OPERATORS PRESERVING POLYNOMIALS

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Abstract. In this paper, a modification of Szász-Mirakyan operators is studied [1] which generalizes the Szász-Mirakyan operators with the property that the linear combination $e_2 + \alpha e_1$ of the Korovkin's test functions e_1 and e_2 are reproduced for $\alpha \geq 0$. After providing some computational results, shape preserving properties of mentioned operators are obtained. Moreover, some estimations for the rate of convergence of these operators by using different type modulus of continuity are shown. Furthermore, a Voronovskaya-type formula and an approximation result for derivative of operators are calculated.

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1. INTRODUCTION

Approximation theory is based on finding the best approximation of a function by polynomials or other type of simple functions. For many years, there have been lots of improvements about the approximation theory. In 1853, Russian mathematician Chebyshev focused on this matter. However, the big step was in 1885 when Karl Weierstrass [13] presented the theorem on approximation.

In approximation theory, positive linear operators play an essential role. The study of approximation sequences of linear positive operators was started at the beginning of the 1950s. One of the most important positive linear operators is Bernstein polynomials. Bernstein polynomials on the space $C[0,1]$ are defined by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \tbinom{n}{k} x^k (1-x)^{n-k}, \quad n \in \mathbb{N}.$$

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Up to now, there have been lots of extensions and modifications of Bernstein polynomials. It was King [9] who constructed Bernstein type linear positive operator defined on $C\,[0,1]$ having an approximation order better then the classical operators such that they reproduce the test function e_0 and e_2. This operator has an approximation order better than the classical operators on $\left[0,\frac{1}{3}\right]$.

Inspiring this fact D. Cardenas-Morales et al. [4] introduced an operator of King type, which was reproduced $e_2+\alpha e_1$ for $\alpha\geq 0$ and defined by

$$B_{n,\alpha}f(x)=\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)\left(r^{*}_{n,\alpha}(x)\right)^{k}\left(1-r^{*}_{n,\alpha}(x)\right)^{n-k}, \tag{1}$$

where

$$r^{*}_{n,\alpha}(x)=-\frac{n\alpha+1}{2(n-1)}+\sqrt{\frac{(n\alpha+1)^2}{4(n-1)^2}+\frac{n(\alpha x+x^2)}{n-1}},\quad n\in\mathbb{N}.$$

They found the shape preserving properties for $B_{n,\alpha}$ and worked on the comparison with Bernstein polynomials. Furthermore, they also showed that the sequences $B_{n,\alpha}$ for $\alpha\geq 0$ are an approximation process. Besides, for different Bernstein Durrmeyer type operators, similar results were given in [5].

Szász-Mirakyan operators are the generalizations of Bernstein polynomials on the interval $[0,\infty)$ which are defined by

$$S_nf(x)=\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)e^{-nx}\frac{(nx)^k}{k!},\quad n\in\mathbb{N}.$$

Notice that, for all functions $f:C\,[0,\infty)\to\mathbb{R}$ the series at the right hand side convergences absolutely.

There are many papers about different type of generalizations of Szász-Mirakyan operators where the basic properties of approximation are analyzed. In the recent years, the number of the articles related to this fact has increased (see [2], [7], [3] and [10]).

In order to furnish better error estimation in a certain sense than classical Szász-Mirakyan operators, in [7], authors defined the following operators,

$$D^{*}_{n}f(x)=e^{-nu^{*}_{n}(x)}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\frac{(nu^{*}_{n}(x))^k}{k!}$$

where

$$u^{*}_{n}(x)=\frac{-1+\sqrt{4n^2x^2+1}}{2n},\quad n\in\mathbb{N}.$$

Note that, both of the results of [4] and [9] were obtained in finite intervals. On the infinite interval, using similar technique, authors introduced Szász-Mirakyan operators King type by reproducing e_1 and e_2 [3].

In this paper, as in [4] for Bernstein polynomials, we consider a similar modification of the Szász–Mirakyan modified operators given in [1] using the function $r_{n,\alpha}$ which is defined by $\{S_{n,\alpha} : C[0,\infty) \to C[0,\infty)\}_{n>0}$ for $\alpha \geq 0$,

$$(2) \qquad S_{n,\alpha}f(x) := S_{n,\alpha}(f;x) = \sum_{k=0}^{\infty} f\left(\tfrac{k}{n}\right) \mathcal{P}_{n,k,\alpha}(x)$$

$$\mathcal{P}_{n,k,\alpha}(x) := e^{-nr_{n,\alpha}(x)} \frac{(nr_{n,\alpha}(x))^k}{k!}$$

$$r_{n,\alpha}(x) = -\tfrac{(\alpha n+1)}{2n} + \sqrt{\tfrac{(\alpha n+1)^2}{4n^2} + (x^2+\alpha x)}, \quad n \in \mathbb{N}, \quad x \geq 0,$$

where $\{r_{n,\alpha} : [0,\infty) \to \mathbb{R}\}_{n>0}$ is the sequence of functions.

Noting the fact that when $n \to \infty$, $r_{n,\alpha} \to x$, $S_{n,\alpha}f$ reduces to the classical Szász-Mirakyan operator. That is, classical Százs-Mirakyan operators turn out to be a limit element of $S_{n,\alpha}f$ and also if we take $\alpha = 0$, the sequence $D_n^* f$ of operators appears which is introduced in [7].

In [1], authors showed the approximation properties of Szász-Mirakyan modified operator. In the lights of the definition of the operator of (2), different kinds of results which are related to the mentioned operator are obtained.

The organization of the paper is as follows:

In section 2, shape preserving properties of the Szász-Mirakyan modified operators are investigated. Using the convexity and generalized convexity, relations between the given functions, $S_{n,\alpha}f$ and $S_n f$ operators are obtained. Then, the results of Voronovskaya-type theorem are given. Moreover, the rate of convergence properties of this operator for two different modulus of continuities are studied and a theorem which is satisfied by derivative of $S_{n,\alpha}f$ is given.

Throughout the paper, we use following definition and notations.

Definition 1. [5] *A function $f \in C^k[0,\infty)$ (the space of k times continuously differentiable functions) is said to be τ convex of order $k \in \mathbb{N}$ whenever*

$$(3) \qquad D^k(f \circ \tau^{-1}) \circ \tau \geq 0.$$

The classical convexity is obtained for $\tau = e_1$ and $k = 2$.

For $\tau(x) = e_2 + \alpha e_1$ if $f \in C[0,\infty)$ convex functions with respect to $e_2 + \alpha e_1$ (in classical sense), it fulfills

$$\begin{vmatrix} 1 & 1 & 1 \\ x_0^2 + \alpha x_0 & x_1^2 + \alpha x_1 & x_2^2 + \alpha x_2 \\ f(x_0) & f(x_1) & f(x_2) \end{vmatrix} \geq 0, \quad 0 < x_0 < x_1 < x_2 < \infty$$

or equivalently to (3) for $f \in C^2[0,\infty)$, $k = 2$

$$f''(x) - \tfrac{2}{2x+\alpha} f'(x) \geq 0, \quad x > 0$$

for $\alpha > 0$.

In this paper, mostly the name of τ convexity $\left(\tau(x) = x^2 + \alpha x\right)$ is used instead of generalized convexity.

The function space $C_2[0,\infty)$ is defined by,

$$C_2[0,\infty) = \left\{ f \in C[0,\infty) : f(x) \leq k_f(1+x^2) \right\}$$

where k_f is a constant depending on f and

$$C_2^*[0,\infty) = \left\{ f \in C[0,\infty) : \lim_{x\to\infty} \tfrac{f(x)}{1+x^2} = k_f \right\}.$$

The space $C_2[0,\infty)$ is endowed with the norm

$$\|f\|_2 = \sup_{x\geq 0} \tfrac{|f(x)|}{1+x^2}.$$

2. SHAPE PRESERVING PROPERTIES

Let $e_i(x) = x^i,\ i = 0,1,2$. For $\alpha \geq 0$ and $n > 0$, using the results for the Szász-Mirakyan operator (2), it is found that

$$S_{n,\alpha}e_0 = e_0, \qquad S_{n,\alpha}e_1 = r_{n,\alpha}, \qquad S_{n,\alpha}e_2 = (r_{n,\alpha})^2 + \tfrac{r_{n,\alpha}}{n}. \tag{4}$$

In view of the definition of $r_{n,\alpha}$ yields

$$S_{n,\alpha}\left(e_2 + \alpha e_1\right)(x) = x^2 + \alpha x. \tag{5}$$

To obtain the shape preserving properties, we need to find the first and second order derivatives of $S_{n,\alpha}f$. For Szász–Mirakyan operators similar results were first established in [12].

Lemma 2. *For any $f \in C[0,\infty),\ n \in \mathbb{N}$ and $x \in [0,\infty)$ we have*
a)

$$S'_{n,\alpha}f(x) = nr'_{n,\alpha}(x)\sum_{k=0}^{\infty}\left\{ f\left(\tfrac{k+1}{n}\right) - f\left(\tfrac{k}{n}\right)\right\}\mathcal{P}_{n,k,\alpha}(x),$$

b)

$$\begin{aligned} S''_{n,\alpha}f(x) &= nr''_{n,\alpha}(x)\sum_{k=0}^{\infty}\left\{ f\left(\tfrac{k+1}{n}\right) - f\left(\tfrac{k}{n}\right)\right\}\mathcal{P}_{n,k,\alpha}(x) \\ &\quad + n^2\left(r'_{n,\alpha}(x)\right)^2\sum_{k=0}^{\infty}\left\{ f\left(\tfrac{k+2}{n}\right) - 2f\left(\tfrac{k+1}{n}\right) + f\left(\tfrac{k}{n}\right)\right\}\mathcal{P}_{n,k,\alpha}(x). \end{aligned}$$

Calculating the first and the second order derivatives of $r_{n,\alpha}$, it is directly seen that

$$r'_{n,\alpha}(x) = \frac{1}{2}\frac{(2x+\alpha)}{\sqrt{\frac{(\alpha n+1)^2}{4n^2} + (x^2+\alpha x)}} \tag{6}$$

and

$$r''_{n,\alpha}(x) = \tfrac{1}{2}\left(\tfrac{(\alpha n+1)^2}{4n^2} + \left(x^2+\alpha x\right)\right)^{-\frac{3}{2}}\left(\tfrac{\alpha}{n} + \tfrac{1}{2n^2}\right).$$

c) For $r_{n,\alpha}(x) \in [0,\infty)/\left\{\frac{k}{n},\ k=0,1,...\right\}$, *we have*

$$S_{n,\alpha}f(x) - f(r_{n,\alpha}(x)) = \frac{r_{n,\alpha}(x)}{n}\sum_{k=0}^{\infty} f\left[r_{n,\alpha}(x), \tfrac{k}{n}, \tfrac{k+1}{n}\right]\mathcal{P}_{n,k,\alpha}(x),$$

where $f\left[x_0, x_1, x_2\right]$ *is the divided differences of* f *with respect to* x_0, x_1 *and* x_2 *such that* $0 \le x_0 < x_1 < x_2 < \infty$.

By applying first and second order derivatives of the operators, it leads to the following theorem:

THEOREM 3. *Suppose that* $f \in C[0,\infty)$, $n \in \mathbb{N}$ *and* $x \in [0,\infty)$. *Then we have*
i) If $f \in C[0,\infty)$ *is increasing, then* $S_{n,\alpha}f$ *is increasing,*
ii) If $f \in C[0,\infty)$ *is increasing and convex, then* $S_{n,\alpha}f$ *is convex.*

THEOREM 4. *If* f *is convex and decreasing, we have*

$$S_{n,\alpha}f\left(x\right) \ge f\left(x\right), \ x \in [0,\infty). \tag{7}$$

Proof. We know that a function f is convex if and only if all second order divided differences of f are nonnegative, (see [11, p. 259]). Thus, using Lemma 2, *c*) we have $S_{n,\alpha}f\left(x\right) \ge f\left(r_{n,\alpha}(x)\right), x \in [0,\infty)$. Also we know that [1, Lemma 2.2] the inequality

$$0 < r_{n,\alpha}(x) < x < \infty \tag{8}$$

holds true. Considering f is decreasing function, we have the desired result. □

As an immediate consequence of the above result, one can stated the following theorem.

THEOREM 5. *Let* $n \in \mathbb{N}$, $\alpha \in [0,\infty)$ *and* $f \in C[0,\infty)$ *be convex with respect to* $e_2 + \alpha e_1$, $\alpha > 0$. *Then we have*

$$f(x) \le S_{n,\alpha}f(x) \le S_n f(x), \quad x > 0.$$

Proof. From the remark of [14, Remark, p.426], we know that

$$S_{n,\alpha}f(x) \ge f(x), \quad x \ge 0$$

because of the f is convex with respect to $\tau = \frac{e_2+\alpha e_1}{1+\alpha}$, $\alpha > 0$. It is known that the Szász-Mirakyan operators, $n \in \mathbb{N}$, of a convex function f, satisfy

$$S_n f(x) \ge f(x)$$

for all $n \in \mathbb{N}$ and $\alpha \in [0,\infty)$ (see [14, Remark, p. 438]). Thus, since τ is convex we have

$$S_n(\tau) \ge \tau.$$

Since $\left(S_n(\tau)\right)^{-1}$ is increasing, so we get

$$\left(S_n(\tau)\right)^{-1} \circ \left(S_n(\tau)\right) \ \ \ge \ \ \left(S_n(\tau)\right)^{-1} \circ \tau.$$

Thus, we have

$$x \geq \left((S_n(\tau))^{-1} \circ \tau \right)(x).$$

Applying the operator $S_n f$ on both sides of the above inequality (S_n is monotone operator), then we obtain

$$S_n f(x) \geq S_n((S_n(\tau))^{-1} \circ \tau)(x) = S_{n,\alpha} f(x),$$

which completes the proof. □

THEOREM 6. *Let $n \in \mathbb{N}$, $\alpha \in [0, \infty)$ and $f \in C[0, \infty)$ then*

$$\lim_{\alpha \to \infty} S_{n,\alpha} f(x) = S_n f(x)$$

uniformly for any closed interval $[a, b] \subset [0, \infty)$.

Proof. It is easily checked that

$$\lim_{\alpha \to \infty} r_{n,\alpha}(x) = x$$

is uniform on $[a, b]$. This completes the proof. □

3. ASYMPTOTIC EXPRESSION

We begin by the following Voronovskaya-type theorem:

THEOREM 7. *Let $f \in C_2^*[0, \infty)$ such that f', $f'' \in C_2^*[0, \infty)$. Then we have*

$$\lim_{n \to \infty} 2n \left(S_{n,\alpha} f(x) - f(x) \right) = x \left(f''(x) - \tfrac{2}{2x+\alpha} f'(x) \right),$$

for every $x \in [0, \infty)$.

Proof. Let f, f' and $f'' \in C_2^*[0, \infty)$ and $x \in [0, \infty)$ be fixed. By the Taylor formula, we can write

$$f(t) = f(x) + f'(x)(t - x) + \tfrac{1}{2} f''(x)(t - x)^2 + \lambda_x(t)(t - x)^2, \tag{9}$$

where $\lambda_x(t) \in C_2^*[0, \infty)$ and $\lim_{t \to x} \lambda_x(t) = 0$. From (9) we have

$$\begin{aligned} 2n \left(S_{n,\alpha} f(x) - f(x) \right) &= f'(x) \, 2n S_{n,\alpha} (e_1 - x e_0)(x) \\ &\quad + \tfrac{1}{2} f''(x) \, 2n S_{n,\alpha} (e_1 - x e_0)^2 (x) \\ &\quad + 2n S_{n,\alpha} \lambda_x(\cdot) (e_1 - x e_0)^2 (x). \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$S_{n,\alpha} \lambda_x(\cdot)(e_1 - x e_0)^2 (x) \leq \sqrt{S_{n,\alpha} \lambda_x^2(\cdot)(x)} \sqrt{S_{n,\alpha} (e_1 - x e_0)^4 (x)}. \tag{10}$$

To prove Voronovskaya-type theorem, we must compute $S_{n,\alpha} e_i^x$, $i = 1, 2, 3, 4$.

Here $e_i^x(t) := (t - x)^i$. Employing the definition of $r_{n,\alpha}$, we can compute following limits.

$$\begin{aligned} \lim_{n \to \infty} 2n S_{n,\alpha} e_1^x(x) &= \lim_{n \to \infty} 2n S_{n,\alpha} (e_1 - x e_0)(x) \\ &= \lim_{n \to \infty} 2n \left(r_{n,\alpha}(x) - x \right) \\ &= -\tfrac{2x}{(\alpha + 2x)} \end{aligned}$$

and using the last limit, we get

$$\begin{aligned}
\lim_{n\to\infty} 2nS_{n,\alpha}e_2^x(x) &= \lim_{n\to\infty} 2n(S_{n,\alpha}e_2(x) - 2xS_{n,\alpha}e_1(x) + x^2) \\
&= \lim_{n\to\infty} 2n(S_{n,\alpha}\left(e_2 + \alpha e_1\right)(x) - (\alpha + 2x)\, r_{n,\alpha}(x) + x^2) \\
&= \lim_{n\to\infty} 2n\left(x^2 + \alpha x - (\alpha + 2x)\, r_{n,\alpha}(x) + x^2\right) \\
&= \lim_{n\to\infty} 2n\,(\alpha + 2x)\,(x - r_{n,\alpha}(x)) \\
&= 2x.
\end{aligned}$$

Moreover, we compute the following via Mathematica

$$\begin{aligned}
nS_{n,\alpha}e_4(x) =& \tfrac{n^4(2x^4+4x^3\alpha+6x^2\alpha^2+4x\alpha^3+\alpha^4)+n^3(10x^2\alpha+10x\alpha^2+2\alpha^3)}{2n^3} \\
& - \tfrac{n^3(2x^2\alpha+2x\alpha^2-\alpha^3)\sqrt{1+2n\alpha+n^2(2x+\alpha)^2}}{2n^3} \\
& + \tfrac{n^2(4x^2+4x\alpha+3\alpha^2)\sqrt{1+2n\alpha+n^2(2x+\alpha)^2}}{2n^3} + o(1) \\
nS_{n,\alpha}e_3(x) =& \tfrac{-n^3(3x^2\alpha+3x\alpha^2+\alpha^3)+n^2(3x^2+3x\alpha)}{2n^2} \\
& + \tfrac{n^2(x^2+\alpha x+\alpha^2)\sqrt{1+2n\alpha+n^2(2x+\alpha)^2}}{2n^2} \\
& - \tfrac{n\alpha\sqrt{1+2n\alpha+n^2(2x+\alpha)^2}}{2n^2} + o(1) \\
nS_{n,\alpha}e_2(x) =& \tfrac{1}{2}\left(\alpha + n(2x^2 + 2x\alpha + \alpha^2) - \alpha\sqrt{1 + 2n\alpha + n^2(2x+\alpha)^2}\right) \\
nS_{n,\alpha}e_1(x) =& -\tfrac{1}{2}\left(1 + n\alpha\right) + \tfrac{1}{2}\sqrt{1 + 2n\alpha + n^2(2x+\alpha)^2}.
\end{aligned}$$

Then if we use equalities which are mentioned above, we obtain

$$\begin{aligned}
nS_{n,\alpha}e_4^x(x) =& n\left(S_{n,\alpha}e_4 - 4xS_{n,\alpha}e_3 + 6x^2S_{n,\alpha}e_2 - 4x^3S_{n,\alpha}e_1 + x^4S_{n,\alpha}e_0\right)(x) \\
=& \tfrac{n^4(16x^4+32x^3\alpha+24x^2\alpha^2+8x\alpha^3+\alpha^4)}{2n^3} \\
& - \tfrac{n^3(8x^3+12x^2\alpha+6x\alpha^2+\alpha^3)\sqrt{1+2n\alpha+n^2(2x+\alpha)^2}}{2n^3} \\
& - \tfrac{n^3(8x^3+16x^2\alpha+10x\alpha^2+2\alpha^3)}{2n^3} \\
& + \tfrac{n^2(4x^2+8x\alpha+3\alpha^2)\sqrt{1+2n\alpha+n^2(2x+\alpha)^2}}{2n^3} + o(1).
\end{aligned}$$

Finally by letting $n \to \infty$, we get

$$\begin{aligned}
\lim_{n\to\infty} nS_{n,\alpha}e_4^x(x) =& -\tfrac{(8x^3+12x^2\alpha+6x\alpha^2+\alpha^3)\alpha}{2(2x+\alpha)} - (4x^3 + 8x^2\alpha + 5x\alpha^2 + \alpha^3) \\
& + \tfrac{(2x+\alpha)(4x^2+8x\alpha+3\alpha^2)}{2} \\
=& 0.
\end{aligned}$$

Putting this results in (10), we have the desired result. $\square$

4. RATE OF CONVERGENCE OF $S_{n,\alpha}f$

In this Section, the rate of convergence of $S_{n,\alpha}$ operators in terms of both the weighted modulus of continuity and classical one is obtained.

Examining relations (4) and using the fact that $r_{n,\alpha} \to x$ as $n \to \infty$, then, on the basis of Korovkin's first theorem, we observe that $S_{n,\alpha}$ is an approximation process on compact subsets included in $[0, \infty)$. Now, we want to give sufficient conditions which ensure both uniform convergence of the sequence $S_{n,\alpha}$ to the identity operator on the whole interval $[0, \infty)$ and the rate of convergence. For Bernstein type operators, a similar result was first established in [6]. This problem was further studied by de la Cal and Carcamo in [6].

THEOREM 8. *Assume that $f : [0, \infty) \to \mathbb{R}$ is bounded. Let*

$$f^*(z) = f(z^2), \quad z \in [0, \infty).$$

We have for all $x \geq 0$ and $\alpha > 0$

$$|S_{n,\alpha}f(x) - f(x)| \leq 6w\left(f^*; \tfrac{1}{\sqrt{n}}\right). \tag{11}$$

Therefore, $S_{n,\alpha}f$ converges to f uniformly on $[0, \infty)$ as $n \to \infty$, whenever f^* is uniformly continuous.

Proof. Let $x > 0$ be arbitrary fixed. Since

$$S_{n,\alpha}f(x) = S_{n,\alpha}(f^*(\sqrt{e_1}); x)$$

using the the definition of modulus of smoothness $w(f^*; \cdot)$, we have

$$\begin{aligned}
|S_{n,\alpha}f(x) - f(x)| &= |S_{n,\alpha}(f^*(\sqrt{e_1}); x) - f^*(\sqrt{x})| \\
&\leq S_{n,\alpha}\left(\left|f^*(\sqrt{e_1}) - f^*(\sqrt{x})\right|; x\right) \\
&\leq S_{n,\alpha}\left(w\left(f^*; |\sqrt{t} - \sqrt{x}|\right); x\right) \\
&= S_{n,\alpha}\left(w\left(f^*; \tfrac{|\sqrt{t}-\sqrt{x}|}{S_{n,\alpha}\left(|\sqrt{e_1}-\sqrt{x}e_0|; x\right)} S_{n,\alpha}(|\sqrt{e_1} - \sqrt{x}e_0|; x)\right)\right).
\end{aligned}$$

Further on, using the property $w(f^*;)$, we have

$$\begin{aligned}
|S_{n,\alpha}f(x) - f(x)| &\leq w\left(f^*; S_{n,\alpha}\left(\left|\sqrt{e_1} - \sqrt{x}e_0\right|; x\right)\right) \\
&\quad \times \left(1 + \tfrac{1}{S_{n,\alpha}\left(|\sqrt{e_1}-\sqrt{x}e_0|; x\right)} S_{n,\alpha}\left(\left|\sqrt{e_1} - \sqrt{x}e_0\right|; x\right)\right) \\
&= 2w\left(f^*; S_{n,\alpha}\left(\left|\sqrt{e_1} - \sqrt{x}e_0\right|; x\right)\right). \qquad (12)
\end{aligned}$$

Using Cauchy- Schwarz inequality, we have

$$\begin{aligned}
S_{n,\alpha}\left(\left|\sqrt{e_1} - \sqrt{x}e_0\right|; x\right) &= S_{n,\alpha}\left(\tfrac{|e_1 - xe_0|}{\sqrt{e_1}+\sqrt{x}}; x\right) \\
&\leq \tfrac{1}{\sqrt{x}} S_{n,\alpha}\left((e_1 - xe_0)^2; x\right)^{1/2}. \qquad (13)
\end{aligned}$$

From (4) and (5), we deduce

$$\begin{aligned} S_{n,\alpha}\left((e_1 - xe_0)^2; x\right) &= S_{n,\alpha}\left(e_2 + \alpha e_1\right)(x) - (2x+\alpha)\, S_{n,\alpha}\left(e_1\right)(x) + x^2 \\ &= (2x+\alpha)\left(x - r_{n,\alpha}(x)\right). \end{aligned}$$

Clearly

$$\begin{aligned} x - r_{n,\alpha}(x) =& x + \tfrac{(\alpha n+1)}{2n} - \sqrt{\tfrac{(\alpha n+1)^2}{4n^2} + (x^2 + \alpha x)} \\ =& \frac{x}{n\left(x + \frac{(\alpha n+1)}{2n} + \sqrt{\frac{(\alpha n+1)^2}{4n^2} + (x^2+\alpha x)}\right)} \\ \le& \tfrac{x}{nx+\alpha n+1}. \end{aligned} \tag{14}$$

According to (13) we have

$$S_{n,\alpha}\left(\left|\sqrt{e_1} - \sqrt{x} e_0\right|; x\right) \le \tfrac{\sqrt{3}}{\sqrt{n}}.$$

Thus by (12), we have the inequality (11).

Under the hypothesis of our theorem, f^* is uniformly continous on $[0,\infty)$, we know that $\lim_{\delta\to 0} w\left(f^*;\delta\right) = 0$. Since the inequality (11) valid for all $x \in [0,\infty)$ leads us to the conclusion of our theorem. □

Now, we focus on weighted space $C_2[0,\infty)$. Using (4) and (8), then we obtain

$$\begin{aligned} \frac{S_{n,\alpha}\left(e_2 + e_0\right)(x)}{1+x^2} =& \frac{1 + r^2_{n,\alpha}(x) + \frac{r_{n,\alpha}(x)}{n}}{1+x^2} \\ \le& \frac{1 + x^2 + \frac{x}{n}}{1+x^2} \\ \le& 3. \end{aligned}$$

Therefore, we can say that $S_{n,\alpha}$ acts from $C_2[0,\infty)$ to $C_2[0,\infty)$. Also, we give an estimation in terms of following weighted modulus of continuity. It is known that, if f is not uniformly continuous on the interval $[0,\infty)$, then the usual first modulus of continuity $w(f,\delta)$ does not tend to zero, as $\delta \to 0$. Here, we use the following weighted modulus of continuity to gain this property. For $f \in C_2[0,\infty)$ and for every $\delta > 0$, the weighted modulus of continuity considered in [8] is defined as follows:

$$\Omega(f,\delta) = \sup_{\substack{x\ge 0\\ |h|<\delta}} \frac{|f(x+h) - f(x)|}{(1+x^2)(1+h^2)}. \tag{15}$$

It is known that for every $f \in C_2^*[0,\infty)$, $\Omega(f,\delta)$, $\delta > 0$, the following properties hold true

$$\lim_{\delta\to 0} \Omega(f,\delta) = 0$$

and

$$\Omega(f,\lambda\delta) \le 2\left(1+\lambda\right)\left(1+\delta^2\right)\Omega(f,\delta), \qquad \lambda > 0. \tag{16}$$

For $f \in C_2[0,\infty)$, by (15) and (16), we can write

(17)
$$|f(t) - f(x)| \leq 2(1+x^2)(1+\delta^2)\left(1+\tfrac{|t-x|}{\delta}\right)\left(1+(t-x)^2\right)\Omega(f,\delta), \quad t,x \geq 0.$$

Theorem 9. *Let $f \in C_2^*[0,\infty)$, then for all $x \in [0,\infty)$ we have the following inequality*

$$\frac{|S_{n,\alpha}f(x) - f(x)|}{(1+x^2)^{5/2}} \leq C_\alpha \Omega(f, \tfrac{1}{\sqrt{n}}),$$

where C_α is a positive constant depending only on α.

Proof. Let $n \in \mathbb{N}$ and $f \in C_2^*[0,\infty)$. Considering (17) with $\lambda = |t-x|\,\delta^{-1}$, from Cauchy-Schwarz inequality we can write

$$\begin{aligned}
|S_{n,\alpha}f(x) - f(x)| &= \left|\sum_{k=0}^{\infty}\left(f\left(\tfrac{k}{n}\right) - f(x)\right)e^{-nr_{n,\alpha}(x)}\frac{(nr_{n,\alpha}(x))^k}{k!}\right| \\
&\leq \sum_{k=0}^{\infty}\left|f\left(\tfrac{k}{n}\right) - f(x)\right| e^{-nr_{n,\alpha}(x)}\frac{(nr_{n,\alpha}(x))^k}{k!} \\
&\leq 2(1+x^2)(1+\delta^2)\Omega(f,\delta)\times \\
&\quad\times \sum_{k=0}^{\infty}\left(1+\tfrac{1}{\delta}\left|\tfrac{k}{n}-x\right|\right)\left(1+\left(\tfrac{k}{n}-x\right)^2\right)e^{-nr_{n,\alpha}(x)}\tfrac{(nr_{n,\alpha}(x))^k}{k!} \\
&= 2(1+x^2)(1+\delta^2)\Omega(f,\delta)\times \\
&\quad\times\left\{1+S_{n,\alpha}e_2^x + \tfrac{1}{\delta}\left(S_{n,\alpha}e_2^x\right)^{1/2} + \tfrac{1}{\delta}\left(S_{n,\alpha}e_2^x\right)^{1/2}\left(S_{n,\alpha}e_4^x\right)^{1/2}\right\}
\end{aligned}$$

and choosing $\delta = \frac{1}{\sqrt{n}}$, then there is a constant C_α depending on α such that we have

$$\frac{|S_{n,\alpha}f(x) - f(x)|}{(1+x^2)^{5/2}} \leq C_\alpha \Omega\Big(f, \tfrac{1}{\sqrt{n}}\Big)$$

and the proof is completed. □

5. CONVERGENCE OF DERIVATIVE OF $S_{n,\alpha}f$

Before considering the main results of this section, we state the derivative of the operator (2) in following lemma.

Lemma 10. *Let f be a continuously differentiable on $[0,\infty)$. Then, we have*

$$S'_{n,\alpha}f(x) = e^{-nr_{n,\alpha}(x)}r'_{n,\alpha}(x)\sum_{k=0}^{\infty}f'\left(\tfrac{k+\phi_k}{n}\right)\frac{(nr_{n,\alpha}(x))^k}{k!}, \quad 0<\phi_k<1. \tag{18}$$

Proof. From Lemma 2, we know that

$$S'_{n,\alpha}f(x) = nr'_{n,\alpha}(x)\sum_{k=0}^{\infty}\left\{f\left(\tfrac{k+1}{n}\right) - f\left(\tfrac{k}{n}\right)\right\}\mathcal{P}_{n,k,\alpha}(x).$$

It is well known that

$$f\left(\tfrac{k+1}{n}\right)-f\left(\tfrac{k}{n}\right)=\tfrac{1}{n}f\left[\tfrac{k}{n},\tfrac{k+1}{n}\right],$$

where $f\left[\frac{k}{n},\frac{k+1}{n}\right]$ is the divided difference of the points $\frac{k}{n}$ and $\frac{k+1}{n}$.

Differentiating (2) with respect to x and applying the definition of the divided difference in first derivative of $S_{n,\alpha}$, we obtain

$$S'_{n,\alpha}f(x)=e^{-nr_{n,\alpha}(x)}r'_{n,\alpha}(x)\sum_{k=0}^{\infty}f\left[\tfrac{k}{n},\tfrac{k+1}{n}\right]\tfrac{(nr_{n,\alpha}(x))^k}{k!}.$$

Moreover, since the divided difference of f satisfies the equality

$$f\left[\tfrac{k}{n},\tfrac{k+1}{n}\right]=f'(\xi),\quad \tfrac{k}{n}<\xi<\tfrac{k+1}{n},$$

taking $\xi=\frac{k+\phi_k}{n}$, $0<\phi_k<1$, we get

$$S'_{n,\alpha}f(x)=e^{-nr_{n,\alpha}(x)}r'_{n,\alpha}(x)\sum_{k=0}^{\infty}f'\left(\tfrac{k+\phi_k}{n}\right)\tfrac{(nr_{n,\alpha}(x))^k}{k!},\quad 0<\phi_k<1$$

which completes the proof. □

First, we shall prove weighted convergence of first derivative of operator (2) for $f\in C_2^*[0,\infty)$ in Lipschitz norm. Let f be continuously differentiable function, which belongs to $C_2[0,\infty)$ and also f' satisfies the Lipschitz condition that is

$$|f'(x)-f'(t)|\le M\,|x-t|^{\beta}\ ,\ 0<\beta\le 1\ ,\ \text{for any } x,t\ge 0.$$

In this case, we write $f'\in Lip_M\beta$.

Theorem 11. *If the function f is continuously differentiable on $[0,\infty)$ and f' belongs to $Lip_M\beta$, then we obtain*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|S'_{n,\alpha}f(x)-f'(x)\right|}{1+x^{\beta+1}}=0.$$

Proof. Considering (18), we can write

$$\begin{aligned}\left|S'_{n,\alpha}f(x)-f'(x)\right| &\le \left|S'_{n,\alpha}f(x)-f'(r_{n,\alpha}(x))r'_{n,\alpha}(x)\right|\\ &\quad+\left|f'(r_{n,\alpha}(x))r'_{n,\alpha}(x)-f'(x)\right|\\ &\le r'_{n,\alpha}(x)\sum_{k=0}^{\infty}\left|f'\left(\tfrac{k+\phi_k}{n}\right)-f'(r_{n,\alpha}(x))\right|\mathcal{P}_{n,k,\alpha}(x)\\ &\quad+\left|f'(r_{n,\alpha}(x))r'_{n,\alpha}(x)-f'(x)\right|.\end{aligned}$$

By reason of $f' \in Lip_M^{\beta}$, we attain

$$\begin{aligned}\left|S'_{n,\alpha}f(x)-f'(x)\right| &\leq Mr'_{n,\alpha}(x)\sum_{k=0}^{\infty}\left|\left(\tfrac{k+\phi_k}{n}\right)-r_{n,\alpha}(x)\right|^{\beta}\mathcal{P}_{n,k,\alpha}(x)\\ &\quad+\left|f'(r_{n,\alpha}(x))r'_{n,\alpha}(x)-f'(x)\right|.\end{aligned}$$

Since $\phi_k < 1$, we deduce

$$\begin{aligned}\left|S'_{n,\alpha}f(x)-f'(x)\right| &\leq Mr'_{n,\alpha}(x)\sum_{k=0}^{\infty}\left|\left(\tfrac{k+1}{n}\right)-r_{n,\alpha}(x)\right|^{\beta}\mathcal{P}_{n,k,\alpha}(x)\\ &\quad+\left|f'(r_{n,\alpha}(x))r'_{n,\alpha}(x)-f'(x)\right|.\end{aligned}$$

Applying Holder inequality, we have

$$\begin{aligned}&\left|S'_{n,\alpha}f(x)-f'(x)\right| \leq\\ &\leq Mr'_{n,\alpha}(x)S_{n,\alpha}\left(\left(e_1(t)+\tfrac{1}{n}-r_{n,\alpha}(x)\right)^2;x\right)^{\frac{\beta}{2}}+\left|f'(r_{n,\alpha}(x))r'_{n,\alpha}(x)-f'(x)\right|\\ &= Mr'_{n,\alpha}(x)\left(S_{n,\alpha}\left((e_1(t)-r_{n,\alpha}(x))^2;x\right)+2\tfrac{1}{n}S_{n,\alpha}\left(e_1(t)-r_{n,\alpha}(x);x\right)+\tfrac{1}{n^2}\right)^{\frac{\beta}{2}}\\ &\quad+\left|f'(r_{n,\alpha}(x))r'_{n,\alpha}(x)-f'(x)\right|.\end{aligned}$$

Let us remark that

$$\begin{aligned}&\sup_{x\in[0,\infty)}\frac{r'_{n,\alpha}(x)S_{n,\alpha}\left((e_1(t)-r_{n,\alpha}(x))^2;x\right)^{\frac{\beta}{2}}}{1+x^{\beta+1}}\\ &=\sup_{x\in[0,\infty)}\frac{r_{n,\alpha}(x)^{\frac{\beta}{2}}}{(1+x^{\beta+1})\,n}\to 0,\quad \text{as } n\to\infty,\end{aligned}\tag{19}$$

$S_{n,\alpha}\left(e_1(t)-r_{n,\alpha}(x);x\right)=0$ and

$$\sup_{x\in[0,\infty)}\frac{r'_{n,\alpha}(x)}{1+x^{\beta+1}}\to 0,\quad \text{as } n\to\infty.\tag{20}$$

On the other hand,

$$f'(r_{n,\alpha}(x))r'_{n,\alpha}(x)-f'(x)=\left(f'(r_{n,\alpha}(x))-f'(x)\right)r'_{n,\alpha}(x)-f'(x)\left(1-r'_{n,\alpha}(x)\right).$$

Since $f' \in Lip_M\beta$, we can write

$$\left|f'(r_{n,\alpha}(x))-f'(x)\right|\leq M\left|r_{n,\alpha}(x)-x\right|^{\beta}$$

and

$$\left|f'(x)\right|\leq\left|f'(0)\right|+Mx^{\beta}\leq M_f\left(1+x^{\beta}\right),$$

where $M_f=\max\left\{\left|f'(0)\right|,M\right\}$.

Now by (6) and (14) , we get

$$\begin{aligned}
\frac{\left|f'(r_{n,\alpha}(x))r'_{n,\alpha}(x)-f'(x)\right|}{1+x^{\beta+1}} &\leq \frac{|f'(r_{n,\alpha}(x))-f'(x)|}{1+x^{\beta+1}}r'_{n,\alpha}(x) \\
&\quad +\frac{|f'(x)|}{1+x^{\beta+1}}\left|1-r'_{n,\alpha}(x)\right| \\
&\leq M\frac{|r_{n,\alpha}(x)-x|^{\beta}}{1+x^{\beta+1}}+\frac{|f'(x)|}{1+x^{\beta+1}}\left|1-r'_{n,\alpha}(x)\right| \\
&\leq M\frac{x^{\beta}}{(1+x^{\beta+1})\,(\alpha n+1)^{\beta}} \\
&\quad +M_f\frac{2}{\alpha^2}\frac{1+x^{\beta}}{1+x^{\beta+1}}\left(\frac{\alpha}{2n}+\frac{1}{4n^2}\right). \qquad (21)
\end{aligned}$$

Combining (19), (20) and (21), we have the the desired result. □

6. COMPARISON WITH $S_n(f)$

Let us denote $w(f,\delta)$ be the first order modulus of continuity of $f \in C_B[0,\infty)$ (the space of all bounded and continuous functions on$[0,\infty)$), where

$$w(f;\delta)=\sup_{0<|x-y|<\delta}|f(x)-f(y)|.$$

We have the following estimates for S_nf, D_n^*f and $S_{n,\alpha}f$ in terms of the modulus of continuity $w(f,\delta)$,

$$\begin{aligned}
|S_nf(x)-f(x)| &\leq 2w(f;\delta_1(x)) \\
|S_{n,\alpha}f(x)-f(x)| &\leq 2w(f,\delta_2(x))
\end{aligned}$$

where $\delta_1^2(x)=\frac{x}{n}$ and $\delta_2^2(x)=(\alpha+2x)(x-r_{n,\alpha}(x))$.

In the following theorem, we present analogues theorem for $S_{n,\alpha}f$ to show a better order of approximation.

Theorem 12. *For every $f\in C_B[0,\infty)$, $x\geq 0$ and $n\in\mathbb{N}$, we have*

$$\delta_2(x)\leq\delta_1(x)$$

and one can get the best approximation using $S_{n,\alpha}f$.

Proof. The order of approximation to a function $f\in C_B[0,\infty)$, given by the sequence $S_{n,\alpha}f$ will be at least as good as of S_nf whenever

$$2x^2+\alpha x-r_{n,\alpha}(x)(\alpha+2x)\leq\tfrac{x}{n}.$$

Let

$$K_{n,\alpha}=2x^2+\alpha x-r_{n,\alpha}(x)(\alpha+2x)-\tfrac{x}{n}.$$

For $x\in[0,\infty)$, $\alpha\in[0,\infty)$ and $n>0$, the only root is $x=0$ and one can see that $K_{n,\alpha}$ never changes the sign in $(0,\infty)$. To analyze the sign of the $K_{n,\alpha}$, we can use the first derivative of $K_{n,\alpha}$

$$K_{n,\alpha}(0)=0,\qquad K'_{n,\alpha}(x)<0$$

and so that for $x \in [0, \infty)$, we get $2x^2 + \alpha x - r_{n,\alpha}(x)(\alpha + 2x) \leq \frac{x}{n}$ that is $\delta_2 \leq \delta_1$. □

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