

COMBINED APPROACH TO SOLVE
THE LINEAR COMPLEMENTARITY PROBLEM

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Abstract. In this paper, we present a new approach in order to solve the linear complementary problem (LCP). We have combined the ideas of Lemke’s method and its variants taking advantage of the benefits of each approach in order to improve the convergence of these algorithms.

Numerical simulations and comparative results of the new approach are provided.

Since the quadratic convex program and linear program can be written as (LCP), so it can be resolved thanks to our new approach.

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1. INTRODUCTION

Study of complementary problems began with the works of R.W. Cottle in the 1960’s [3]. Problems of this type arise frequently in engineering applications, game theory, and economics. It can be seen as a particular case of variational inequalities. Also, the KKT conditions for linear and quadratic programming problems can be written as a linear complementary problem, hence the algorithm presented in this paper can be used to solve both linear and quadratic programming problems.

An important tool existing since a long time to solve the (LCP) is the Lemke’s method (1968), based on the principle of the simplex method introduced by Dantzig in 1951. It chooses the entering variable by a complementary pivot rule, the entering variable in a step is always the complement of the dropping variable in the previous step. An artificial variable z_0 associated to the column vector $-e$ ($-e$ is the column vector of all 1’s in \mathbb{R}^n) is introduced into (LCP) and the idea is to drive the variable z_0 to 0 by series of complementary pivots, thus obtaining a solution to the (LCP). This method converges with a finite number of iterations, when the problem admits a solution.

In this paper, we present two variants of Lemke’s method:

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Variante 1: the idea is to replace the column vector associated with z_0 by any strictly positive vector d .

Variante 2: the idea is to replace the column vector associated with z_0 by the column vector $M_{.t}$ of the matrix M such that $M_{.t} > 0$.

The main difficulty of variante 1 lies in the determination of a strictly positive vector d , it is a very restrictive condition. The question is how to choose the components of d ?

Concerning variante 2, the supposed condition on the matrix M restricted classes of (LCP) treated with this variante (we can not apply this variante for linear or quadratic problems, for (LCP) with any matrix, ...) because there is no strictly positive column.

Our goal in this paper is to propose a new approach that combines the ideas of these variants and Lemke's method taking advantage of the benefits of each approach in order to improve the convergence of these algorithms.

The paper is organized as follows:

Firstly, we present Lemke's method. In section 3, we present the variants of Lemke's method. Numerical simulations and comparative experimental results between the two variants and Lemke's algorithm are given in section 4. The new approach is described in section 5 with several examples of applications. Section 6 presents some conclusions.

2. LEMKE'S METHOD

We consider the linear complementary problem

$$(LCP) \quad \text{Find } (w, z) \in \mathbb{R}^n \times \mathbb{R}^n \text{ such that}$$

- (1) $w = Mz + q,$
- (2) $(w, z) \geq 0$
- (3) $w^t z = 0.$

where M is a given square matrix of order n and q is a column vector in \mathbb{R}^n .

2.1. Preliminaries. Let w_i (resp. z_i) be the component of number i of vector w (resp. z). The component w_i (resp. z_i) is called basic variable if $w_i \geq 0$ (resp. $z_i \geq 0$). If w_i (resp. z_i) is nonbasic variable, then $w_i = 0$ (resp. $z_i = 0$).

DEFINITION 1. *The couple (w, z) is called complementarity basic feasible solution (CBFS), if it verifies the following conditions*

- 1) (w, z) is a feasible solution to (1) and (2),
- 2) one and only one component of (w_i, z_i) is a basic variable for $1 \leq i \leq n$.

REMARK 2. 1) If $q \geq 0$ then, we immediately have a complementarity basic feasible solution (CBFS) of (LCP) by letting $(w, z) = (q, 0)$.

2) If $q \not\geq 0$, an artificial variable z_0 associated with the column vector $-e$ ($-e$ is the column vector of all 1's in \mathbb{R}^n) is introduced into (LCP). \square

The idea is to drive the variable z_0 to 0 by series of complementary pivots, by solving the following problem

$$(LCP)_{z_0} \text{ Find } (w, z) \in \mathbb{R}^n \times \mathbb{R}^n \text{ such that}$$

$$(4) \quad w - Mz - ez_0 = q,$$

$$(5) \quad (w, z) \geq 0, \quad z_0 \geq 0,$$

$$(6) \quad w_i z_i = 0, \text{ for } i = 1, \dots, n.$$

Thus obtaining a solution to the (LCP). Where z_0 is defined by:

$$z_0 = \max \{-q_i : i = 1, \dots, n\}.$$

Then $(w, z) = (q + ez_0, 0)$ is a starting solution to $(LCP)_{z_0}$.

DEFINITION 3. (w, z, z_0) is called almost-complementarity basic feasible solution of $(LCP)_{z_0}$ if it verifies the following conditions:

- 1) (w, z, z_0) is a feasible solution to (4) and (5),
- 2) neither w_s nor z_s are basic for some $s \in \{1, \dots, n\}$,
- 3) z_0 is basic, and exactly one variable from each complementary pair (w_i, z_i) is basic for $i = 1, \dots, n$ and $i \neq s$.

2.2. Description of Lemke's algorithm. This algorithm was introduced by Lemke (1968), known as the complementary pivot algorithm because it chooses the entering variable by a complementary pivot rule, the entering variable in a step is always the complement of the dropping variable in the previous step. The dropping variable has to be determined according to the minimum ratio test to guarantee that the new basis obtained after the pivot step will also be a feasible basis.

ALGORITHM 1. • **Data.** $q \in \mathbb{R}^n, M \in \mathbb{R}^{n \times n}$.

• **Initialization.**

- **If** $q \geq 0$ stop. $(w, z) = (q, 0)$ is a solution of the (LCP).
- **Else** determine $s \in \{1, \dots, n\}$ to satisfy: $q_s = \min \{q_i : 1 \leq i \leq n\}$, $q_s < 0$, We perform a pivot step with the column vector of z_0 as the pivot column, and the s^{th} row as the pivot row. w_s drops from the basis vector and z_0 is the entering variable, put $y_s = z_s$.

• **Iteration.**

- **Step 1:** Let h_s be the column vector corresponding to the variable y_s then

- **If** $h_s \leq 0$ go to step 4.
- **Else** determine the set $I = \{i : h_s[i] > 0\}$, then choose r such that $\frac{\bar{q}_r}{h_s[r]} = \min_{i \in I} \left\{ \frac{\bar{q}_i}{h_s[i]} \right\}$. (the vector \bar{q} designates the second member column after pivoting)
- **If** the basic variable in the row r is z_0 go to step 3.
- **Else** go to step 2.

- **Step 2:** The basic variable in the row r is either w_k or z_k for $k \neq s$, so we perform a pivot step with the row r as the pivot row and the column of y_s as the pivot column. If w_k (respectively z_k) drops the basis, put $y_s = z_k$ (resp. $y_s = w_k$) and we return to step 1.
- **Step 3:** We perform a pivot step with the row of z_0 and the column of y_s , so z_0 leaves the basis, we obtain a solution of (LCP).
- **Step 4:** Since $h_s \leq 0$ we stop with a ray of termination (RT):

$$R = \{(w, z, z_0) + \lambda h : \lambda \geq 0\},$$
 or (w, z, z_0) is an almost complementarity basic feasible solution to $(LCP)_{z_0}$, the components of h are defined as

$$\begin{cases} 1, & \text{for the row corresponding to } y_s \\ -h_s, & \text{for rows corresponding to basic variables} \\ 0, & \text{elsewhere} \end{cases}$$
- **End.**

2.3. Convergence. This algorithm is guaranteed to work under certain non-degeneracy assumptions and when M satisfies certain properties.

THEOREM 4. We suppose that $M \in CPP$ (M is copositive plus) and each feasible almost-complementarity solution of $(LCP)_{z_0}$ is nondegenerate, then the algorithm stops after a finite number of iterations.

If the system defined by (1) and (2) is consistent, then the algorithm stops with a complementarity basic feasible solution of (LCP), else, (LCP) admits no solution.

EXAMPLE 5. It is possible that cycling occurs under degeneracy. Here we provide an example of cycling constructed by M. Kostreva [4]. Let

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

Lemke's method cycles after 7 iterations.

M is a p -matrix, so (LCP) has a unique solution, and that the complementary pivot method always terminates in a finite number of pivot steps with that solution, if it is carried out in such a way that cycling does not occur under degeneracy. So using the lexico-minimum ratio rule for choosing the pivot row in each step, cycling cannot occur, and the method gives the solution $(w, z)^t = (0, 0, 0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})^t$ after 4 iterations. \square

EXAMPLE 6. Consider the (LCP) problem, where

$$M = \begin{pmatrix} 21 & 0 & 0 \\ 28 & 14 & 0 \\ 24 & 24 & 12 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

The algorithm stops after 2 iterations with the solution: $(w, z)^t = (0, 0.33, 0.14, 0.05, 0, 0)^t$. \square

3. VARIANTS OF LEMKE'S METHOD

3.1. Variant 1. This variant is described in [7].

Principle

In Lemke's algorithm, we have the following table

w	z	z_0	second member
I	$-M$	$-e$	q

The idea of this variant is to replace the corresponding column to z_0 by a vector d such that $d > 0$, so we have the table

w	z	z_0	second member
I	$-M$	$-d$	q

and the system

Find $(w, z) \in \mathbb{R}^n \times \mathbb{R}^n$ such that:

- (7) $w - Mz - dz_0 = q,$
 (8) $(w, z) \geq 0, z_0 \geq 0,$
 (9) $w_i z_i = 0, \text{ for } i = 1 : n.$

1) If $q \geq 0$, $(w, z) = (q, 0)$ is a solution of the (LCP).

2) If $q \not\geq 0$, identify row s to satisfy $\frac{q_s}{d_s} = \min \left\{ \frac{q_i}{d_i} : 1 \leq i \leq n \right\}$. We perform a pivot step with the column vector of z_0 as the pivot column, and the s^{th} row as the pivot row. w_s drops from the basis vector and z_0 is the entering variable, put $y_s = z_s$. Then, we follow the same steps of Lemke's algorithm.

We will now illustrate this variant using the numerical examples of the previous section.

EXAMPLE 7. Consider the (LCP) problem with

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

We choose $d = (7, 3, 5)^t$, when we apply variant 1, we obtain the solution: $(w, z)^t = (0, 0, 0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})^t$ after 4 iterations. We choose $d = (15, 7, 9)^t$, then the algorithm stops after 6 iterations with the same solution. \square

EXAMPLE 8. Consider the (LCP) problem, where

$$M = \begin{pmatrix} 21 & 0 & 0 \\ 28 & 14 & 0 \\ 24 & 24 & 12 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

Let $d = (12, 14, 21)^t$, the algorithm stops after 2 iterations with the solution: $(w, z)^t = (0, 0.33, 0.14, 0.05, 0, 0)^t$.

For $d = (2, 3, 1)^t$, the algorithm stops after 8 iterations with the same solution. \square

REMARK 9. 1) The choice of the vector d affects the results obtained in number of iterations and execution time, so the question is: first, how to choose components of the vector d , and secondly, what is the ideal choice?

2) Noting that among the advantages of this variant is that it does not suffer from cycling problem (Example 7). \square

3.2. Variant 2. This variant also is described in [7].

Principle

Suppose the matrix M satisfies the condition: there exists a column vector of M in which all the entries are strictly positive ($M_{.t} > 0$). Then a variant of the Lemke's algorithm which uses no artificial variable at all, can be applied on the (LCP). So $M_{.t}$ replace the vector d and the variable z_t play the same role as that of the artificial variable z_0 .

We assume that $q \not\leq 0$. Let t be such that $M_{.t} > 0$. When a pivot step is carried out with the column of z_t as the pivot column and row s as the pivot row, the right hand side constants vector becomes nonnegative after this pivot step. Hence $(w_1, \dots, w_{s-1}, z_t, w_{s+1}, \dots, w_n)$ is feasible basic vector, and if $s = t$, it is a complementary feasible basic vector and the solution corresponding to it is a solution of the (LCP). If $s \neq t$, the feasible basic vector $(w_1, \dots, w_{s-1}, z_t, w_{s+1}, \dots, w_n)$ satisfies the following properties:

DEFINITION 10. Any feasible basic vector satisfying the following conditions is known as an almost complementary feasible basic vector:

- 1) It contains exactly one basic variable from the complementary pair (w_i, z_i) for $n - 2$ values of i (namely $i \neq s, t$ here).
- 2) It contains both the variables from a fixed complementary pair (namely (w_t, z_t) here), as basic variables.
- 3) There exists exactly one complementary pair both the variables in which are not contained in this basic vector (namely (w_s, z_s) here).

ALGORITHM 2. • **Data.** $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$.

• Initialization.

– **If** $q \geq 0$ stop. $(w, z) = (q, 0)$ is a solution of the (LCP) .

– **Else** 1) Choose t such that $M_{.t} > 0$.

2) Determine $s \in \{1, \dots, n\}$ to satisfy: $\frac{q_s}{m_{st}} = \min \left\{ \frac{q_i}{m_{it}}, i = 1 \dots n \right\}$.

We perform a pivot step with the column vector of z_t as the pivot column, and the s^{th} row as the pivot row. w_s drops from the basis vector and z_t is the entering variable, put $y_s = z_s$.

• Iteration.

• **Step 1:** Let h_s be the column vector corresponding to the variable y_s then

– **If** $h_s \leq 0$ go to step 4.

– **Else** Determine the set $I = \{i : h_s[i] > 0\}$, then choose r such that

$\frac{\bar{q}_r}{h_s[r]} = \min_{i \in I} \left\{ \frac{\bar{q}_i}{h_s[i]} \right\}$. (the vector \bar{q} designates the second member column after pivoting)

- **If** the basic variable in the row r is z_t go to step 3.
- **Else** go to step 2.

- **Step 2:** The basic variable in the row r is either w_k or z_k for $k \neq t$, so we perform a pivot step with the row r as the pivot row and the column of y_s as the pivot column. If w_k (respectively z_k) drops the basis, put $y_s = z_k$ (resp. $y_s = w_k$) and we return to step 1.
- **Step 3:** We perform a pivot step with the row of w_t or z_t and the column of y_s , so w_t or z_t leaves the basis, we obtain a solution of (LCP).
- **Step 4:** Since $h_s \leq 0$, we stop with a ray of termination:

$$R = \{(w_1, w_2, \dots, w_n, z_1, z_2, \dots, z_n) + \lambda h : \lambda \geq 0\}.$$

the components of d are defined as

$$\begin{cases} 1, & \text{for the row corresponding to } y_s \\ -h_s, & \text{for rows corresponding to basic variables} \\ 0, & \text{elsewhere} \end{cases}$$

- **End.**

We will now illustrate this variant using the same examples for variant 1.

EXAMPLE 11. Let

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

There is no strictly positive column in M , so we can not apply this variant. \square

EXAMPLE 12. Let

$$M = \begin{pmatrix} 21 & 0 & 0 \\ 28 & 14 & 0 \\ 24 & 24 & 12 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

$M_{.1} \succ 0$, so we can apply variant 2. The algorithm stops after 1 iteration with the solution: $(w, z)^t = (0, 0.33, 0.14, 0.05, 0, 0)^t$. \square

4. COMPARATIVE NUMERICAL RESULTS

In this section, we present the comparative numerical results between the two variants and Lemke's algorithm.

EXAMPLE 13. Consider a (LCP) problem with

$$M = \begin{pmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } q = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}. \quad \square$$

EXAMPLE 14. Let

$$M = \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & -\frac{1}{2} \end{pmatrix} \text{ and } q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad \square$$

EXAMPLE 15. Let

$$M = \begin{pmatrix} 21 & 0 & 0 \\ 28 & 14 & 0 \\ 24 & 24 & 12 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}. \quad \square$$

EXAMPLE 16. Consider a (LCP) problem with

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}. \quad \square$$

EXAMPLE 17. Let

$$M = \begin{pmatrix} 1 & 1 & 3 & 4 \\ 5 & 3 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 1 & 4 & 1 & 1 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 3 \end{pmatrix}. \quad \square$$

EXAMPLE 18. We consider a convex quadratic problem (CQP)

$$(CQP) \begin{cases} \min x_1^2 + 2x_2^2 - 2x_1x_2 - x_1 - 6x_2 \\ x_1 + 2x_2 \leq 4 \\ -x_1 - 2x_2 \leq 4 \\ x_1, x_2 \geq 0. \end{cases} \quad \square$$

EXAMPLE 19. We consider the following linear problem (LP)

$$(LP) \begin{cases} \min -x_1 - x_2 - x_3 \\ -x_1 + 2x_2 \leq 4 \\ x_1 - 3x_2 - x_3 \leq -3 \\ x_1 + x_2 \leq 9 \\ -x_1 - \frac{1}{2}x_2 + x_3 \leq -2 \\ -2x_2 \leq -5 \\ x_1, x_2, x_3 \geq 0. \end{cases} \quad \square$$

EXAMPLE 20. We have a (LCP) problem with

$$M[i, j] = \begin{cases} 5, & \text{if } i < j \\ 1, & \text{if } i = j \\ 0, & \text{if } i > j \end{cases} \text{ and } q[i] = -1,$$

for $i = 1, \dots, 15$ and $j = 1, \dots, 15$. □

EXAMPLE 21. We have

$$M = \begin{pmatrix} 1 & 2 & 2 & 2 & \cdot & \cdot & \cdot & 2 \\ 2 & 5 & 6 & 6 & \cdot & \cdot & \cdot & 6 \\ 2 & 6 & 9 & 10 & \cdot & \cdot & \cdot & 10 \\ 2 & 6 & 10 & 13 & 14 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 14 & 17 & 18 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 18 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 6 & 10 & \cdot & \cdot & \cdot & \cdot & 4n - 3 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} -1 \\ -1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -1 \end{pmatrix} \quad \square$$

In the following tables, n represents the problem dimension, k represents the number of iterations and t represents the calculation time in seconds for the three algorithms.

		Lemke's algor.		v. 1		v. 2	
Examples	n	k	t	k	t	k	t
Ex. 13	3	3	0.49	3	0.5	2	0.05
Ex. 14	2	RT	0.5	RT	0.5	×	×
Ex. 15	3	2	0.59	2	0.49	1	0.00
Ex. 16	3	4	0.55	6	0.5	×	×
Ex. 17	4	2	0.55	2	0.49	1	0.00
Ex. 18	4	4	0.55	4	0.49	×	×
Ex. 19	8	11	0.66	9	0.49	×	×
Ex. 20	15	30	2.11	56	2.91	2	1.51
Ex. 21	20	2	0.49	2	0.49	1	0.00

Table 1. Comparative numerical results of the Lemke's method, variant 1 and variant 2.

According to the numerical results in Table 1, the following remarks can be made:

- 1) Variant 1 difficulty lies in the determination of an ideal vector d since the choice of this vector affects the results obtained in number of iterations and execution time (See section 3).
- 2) We can not apply variant 2 for:
 - All linear programming problems (Example 18) since there is no strictly positive column. The matrix M in this case is in the form $M = \begin{pmatrix} 0 & A^t \\ -A & 0 \end{pmatrix}$ (See section 5).
 - Some convex quadratic problems (Example 17). The matrix M in this case is in the form $M = \begin{pmatrix} Q & A^t \\ -A & 0 \end{pmatrix}$ (See section 5).
 - Any matrix M which has no strictly positive column (Example 14 and Example 16).

This result does not exist for Lemke's method and variant 1.

- 3) The number of iterations and the calculation time recorded for variant 2 are better than those given by variant 1 and Lemke's method.
- 4) We must think about a new approach that treats these classes of (LCP) problems.

5. COMBINED APPROACH

Our goal in this paper is to propose a new approach that combines the ideas of these variants and Lemke's method taking advantage of the benefits of each approach in order to improve the convergence of these algorithms. We give the principle of our approach that combines the ideas of the three methods.

Principle

We assume that $q \not\leq 0$.

- If M satisfies the condition: $\exists t$ such that $M_{.t} > 0$, then we apply variant 2.
- Else, we define a vector $d \succ 0$ from any column s of M with the properties: we let all positive components of $M_{.s}$ and we put 1 for the null components and $-M[i, s]$ for the negative one.

Hence, our algorithm is defined as:

ALGORITHM 3. • **Data.** $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$.

• **Initialization.**

- **If** $q \geq 0$ stop. $(w, z) = (q, 0)$ is a solution of the (LCP) .
- **Else**
 - **If** $\exists t \in \{1, \dots, n\}$ such that: $M_{.t} > 0$ apply variant 2.
 - **Else** Choose a column vector $M_{.s} \not\leq 0$ and calculate $d \in \mathbb{R}_+^n$ as follow:

$$d[i] = \begin{cases} M[i, s], & \text{if } M[i, s] > 0, \\ 1, & \text{if } M[i, s] = 0, \\ -M[i, s], & \text{if } M[i, s] < 0, \end{cases}$$

then apply variant 1.

• **End.**

5.1. Numerical tests.

5.1.1. **Case of linear complementary problem.** We consider the (LCP) problem

$$(LCP) \left\{ \begin{array}{l} \text{Find } (w, z) \in \mathbb{R}^n \times \mathbb{R}^n \text{ such that:} \\ w = Mz + q \\ (w, z) \geq 0 \\ w^t z = 0. \end{array} \right.$$

where M is a given square matrix of order n and q is a column vector in \mathbb{R}^n .

EXAMPLE 22. We take example of Kostreva with $M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, then $d = (1, 2, 1)^t$ for $s = 3$. \square

EXAMPLE 23. Let

$$M = \begin{pmatrix} 2 & 2 & 1 & 2 \\ 3 & 3 & 2 & 3 \\ -2 & 1 & 5 & -2 \\ 1 & -2 & -1 & 2 \end{pmatrix} \text{ and } q = \begin{pmatrix} -4 \\ -6 \\ 4 \\ 4 \end{pmatrix}. \quad \square$$

EXAMPLE 24. Let

$$M = \begin{pmatrix} 2 & 2 & -1 & 3 & -3 & 2 \\ 3 & -3 & 2 & -2 & 5 & 2 \\ -2 & -1 & 5 & -2 & -2 & -1 \\ 1 & -2 & -1 & 2 & 3 & -1 \\ 2 & -1 & 2 & -3 & 1 & 0 \\ 0 & 1 & 2 & 5 & -1 & 0 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}. \quad \square$$

Examples	n	k	t
Example 22	3	4	0.49
Example 23	4	2	0.5
Example 24	6	7	0.6

Table 2. Numerical results of the combined approach in the case of a linear complementary problem.

In these examples, variant 2 is not applicable so we apply the new approach with a vector d which is well defined.

5.1.2. Case of linear programming. We consider the following (LP) problem

$$(LP) \begin{cases} \min c^t x \\ Ax \leq b \\ x \geq 0, \end{cases}$$

Where A is a given (m, n) -matrix with $\text{rank}(A) = m$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

The conditions of KKT's related to the (LP) problem are written as (LCP) where:

$$M = \begin{pmatrix} 0 & A^t \\ -A & 0 \end{pmatrix}, q = \begin{pmatrix} c \\ b \end{pmatrix} \text{ and } z^* = \begin{pmatrix} x^* \\ y^* \end{pmatrix}.$$

(x^*, y^*) is the primal-dual solution of (LP) and its dual (D).

Examples	n	k	t
Example 25	8	9	0.55
Example 26	12	13	0.82
Example 27	15	5	0.66

Table 3. Numerical results of the combined approach in the case of linear programming problems.

EXAMPLE 25. Consider the (LP) problem

$$(LP) \left\{ \begin{array}{l} \min -x_1 - x_2 - x_3 \\ -x_1 + 2x_2 \leq 4 \\ x_1 - 3x_2 - x_3 \leq -3 \\ x_1 + x_2 \leq 9 \\ -x_1 - \frac{1}{2}x_2 + x_3 \leq -2 \\ -2x_2 \leq -5 \\ x_1, x_2, x_3 \geq 0. \end{array} \right. \quad \square$$

EXAMPLE 26. We have the following (LP) problem

$$(LP) \left\{ \begin{array}{l} \min -4x_1 - 5x_2 - x_3 - 3x_4 + 5x_5 - 8x_6 \\ x_1 - 4x_3 + 3x_4 + x_5 + 3x_6 \leq 1 \\ 5x_1 + 3x_2 + x_3 - x_5 + 3x_6 \leq 4 \\ 4x_1 + 5x_2 - 3x_3 + 3x_4 - 4x_5 + x_6 \leq 4 \\ -x_2 + 2x_4 + x_5 - 5x_6 \leq 5 \\ -2x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 \leq 7 \\ 2x_1 - 3x_2 + 2x_3 - x_4 + 4x_5 + 5x_6 \leq 5 \\ x_i \geq 0, i = 1, 2, \dots, 6, \end{array} \right. \quad \square$$

EXAMPLE 27. We consider a (LP) problem where

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 5 & 4 & 3 & 2 & 1 \\ 6 & 7 & 8 & 9 & 10 & 5 & 2 & 8 & 3 & 1 \\ 11 & 12 & 13 & 14 & 15 & 6 & 7 & 80 & 90 & 10 \\ 1 & 10 & 20 & 30 & 40 & 50 & 60 & 80 & 90 & 10 \\ 3 & 9 & 27 & 60 & 45 & 60 & 75 & 8 & 9 & 46 \end{pmatrix}, b = \begin{pmatrix} 10000 \\ 10000 \\ 10000 \\ 10000 \\ 10000 \end{pmatrix}$$

and $c = (-1, -1, \dots, -1)^t$. \square

For all linear programming problems presented here, we can not apply variant 2 since there is no strictly positive column. But the new approach gives us the above results.

5.2. Case of convex quadratic programming. We consider the (CQP) problem

$$(CQP) \left\{ \begin{array}{l} \min \frac{1}{2}x^t Q x + c^t x \\ Ax \leq b \\ x \geq 0. \end{array} \right.$$

where Q is a symmetric semidefinite matrix of order n , A is a given (m, n) -matrix with $\text{rank}(A) = m$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

The conditions of KKT's related to the (CQP) problem are written as (LCP) where:

$$z^* = \begin{pmatrix} x^* \\ y^* \end{pmatrix}, q = \begin{pmatrix} c \\ b \end{pmatrix} \text{ and } M = \begin{pmatrix} Q & A^t \\ -A & 0 \end{pmatrix}.$$

(x^*, y^*) is the primal-dual solution of (CQP) and its dual (D).

EXAMPLE 28. We consider the following (CQP) problem

$$\begin{aligned} \min \quad & 2x_1^2 + 4x_2^2 + 6x_3^2 - 2x_1x_2 - 6x_1x_3 + 8x_2x_3 + 5x_1 + 6x_2 - 12x_3 \\ & -x_1 - 2x_2 - x_3 \leq -6 \\ & x_1 + x_2 + x_3 \leq 16 \\ & -x_1 + 2x_2 \leq 4 \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \quad \square$$

EXAMPLE 29. We have a (CQP) problem where

$$Q = \begin{pmatrix} 1 & 0 & -5 & 0 \\ 0 & 5 & 0 & 0 \\ -5 & 0 & 1 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix}, A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 1 & 2 & -1 \\ 0 & -1 & -4 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} 5 \\ -4 \\ \frac{3}{2} \end{pmatrix} \text{ and } c = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \end{pmatrix}. \quad \square$$

EXAMPLE 30. Consider the (CQP) problem where

$$Q = \begin{pmatrix} 30 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 21 & 0 & 1 & -1 & 1 & 0 & 1 & 0.5 & 1 \\ 1 & 0 & 15 & -0.5 & -2 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -0.5 & 30 & 3 & -1 & 1 & -1 & 0.5 & 1 \\ 1 & -1 & -2 & 3 & 27 & 1 & 0.5 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 16 & -0.5 & 0.5 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0.5 & -0.5 & 8 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 0.5 & 1 & 24 & 1 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0 & 1 & 1 & 39 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 11 \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & -1 & 1.9 & 1.25 & 1.2 & 0.4 & -0.7 & 1.06 & 1.5 & 1.05 \\ 1.3 & 1.2 & 0.15 & 2.15 & 1.25 & 1.5 & 0.4 & 1.52 & 1.3 & 1 \\ 1.5 & -1.1 & 3.5 & 1.25 & 1.8 & 2 & 1.95 & 1.2 & 1 & -1 \end{pmatrix},$$

$$b = (11.651, 16.672, 21.295)^t, \text{ and } c = (-0.5, -1, 0, 0, -0.5, 0, 0, -1, -0.5, -1)^t. \quad \square$$

In Table 4, we have applied the new approach to quadratic programming problems with any matrix M .

Examples	n	k	t
Example 28	6	4	0.55
Example 29	7	3	0.49
Example 30	13	4	0.97

Table 4. Numerical results of the combined approach in the case of quadratic programming problems.

6. CONCLUSION

In this paper, we have proposed a new efficient approach to solve linear complementary problems. We have combined the ideas of Lemke's method and its variants to introduce a new vector which is well defined, so, the algorithm obtained is able to solve any type of problems (linear problems, convex quadratic problems, any matrix which has no strictly positive column, ...) and the numerical results confirm the efficient of our algorithm.

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