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A GENERALIZATION OF THE LUPAŞ q-ANALOGUE OF THE BERNSTEIN OPERATOR

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Abstract. We introduce a Stancu type generalization of the Lupaş q-analogue of the Bernstein operator via the parameter α. The construction of our operator is based on the generalization of the Gauss identity involving q-integers. When the parameters α and q depend on n, and satisfy some additional conditions, we establish the convergence of our sequence of operators in the strong operator topology to the identity, estimating the rate of convergence by using the second order modulus of smoothness. For α and q fixed parameters, we study the existence of the limit operator of our sequence of operators taking into account the relationship between two consecutive terms of the constructed sequence of operators. The rate of convergence in the uniform norm it is also estimated with the aid of the second order modulus of smoothness.

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1. INTRODUCTION

The development of q-calculus has led to the discovery of new Bernstein type operators involving q-integers. The first example in this direction was given by Lupaş [5] in 1987. The so-called q-Bernstein operators were introduced by Phillips [9] in 1997, and they mean another generalization of Bernstein operators based on the q-integers. Nowadays, q-Bernstein operators form an area of an intensive research. A survey of the obtained main results and references in this area during the first decade of study can be found in [7]. Nowadays, there are new papers on the subject constantly coming out and generalizations of q-Bernstein operators being studied. Different types of q-integral operators, q-Bernstein type integral operators and q-summation-integral operators were introduced and studied in [1].

To present Lupaş operator, we recall some notions of the q-calculus (see [4]). Let $q > 0$. Then for each non-negative integer n, the q-integer $[n]_q$ and the q-factorial $[n]_q!$ are defined by $[n]_q = 1 + q + \ldots + q^{n-1}$ for $n = 1, 2, \ldots,$

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$[0]_q = 0$ and $[n]_q! = [1]_q[2]_q \ldots [n]_q$ for $n = 1, 2, \ldots$, $[0]_q! = 1$. For integers n and k satisfying $0 \le k \le n$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Further, we set

$$b_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} \frac{x^k(1-x)^{n-k}}{(1-x+xq)\ldots(1-x+xq^{n-1})}$$

for $k = 0, 1, \ldots, n$.

Following Lupaş [5] (see also [8]), the positive linear operator $R_{n,q} : C[0,1] \to C[0,1]$ given by

$$R_{n,q}(f;x) = \sum_{k=0}^{n} b_{n,k}(q;x) f\left(\frac{[k]_q}{[n]_q}\right) \tag{1.1}$$

is called the Lupaş q-analogue of the Bernstein operator. For $q = 1$, we recover the well-known Bernstein operator defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \tag{1.2}$$

where $f \in C[0,1]$, $x \in [0,1]$ and $n \ge 1$. The papers [8] and [3] deal with the convergence properties of the operator (1.1) and the limit Lupaş q-analogue of the Bernstein operator, which is given for $q \in (0,1)$ fixed by

$$R_{\infty,q}(f;x) = \sum_{k=0}^{\infty} f(1-q^k) \frac{q^{k(k-1)/2}(x/(1-x))^k}{(1-q)^k [k]_q! \prod_{j=0}^{\infty} (1 + q^j(x/(1-x)))},$$

where $x \in [0,1)$ and $R_{\infty,q}(f;1) = f(1)$.

For $f \in C[0,1]$, $\alpha \ge 0$, $q > 0$ and $n \ge 1$, we introduce a generalization of (1.1) as follows:

$$U^{\alpha}_{n,q}(f;x) = \sum_{k=0}^{n} b^{\alpha}_{n,k}(q;x) f\left(\frac{[k]_q}{[n]_q}\right), \tag{1.3}$$

where

$$b^{\alpha}_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} \frac{\prod_{i=0}^{k-1}(x+\alpha q^{-i}[i]_q) \prod_{j=0}^{n-k-1}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)}. \tag{1.4}$$

We note that an empty product in (1.4) denotes 1. For $\alpha = 0$, we recover the operator (1.1), and for $q = 1$, we recover the Stancu operator [11] given by

$$S^{\alpha}_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\prod_{i=0}^{k-1}(x+\alpha i) \prod_{j=0}^{n-k-1}(1-x+\alpha j)}{\prod_{i=0}^{n-1}(1+\alpha i)} f\left(\frac{k}{n}\right).$$

When $\alpha = 0$ and $q = 1$, we obtain the Bernstein operator (1.2). The parameters α and q may depend only on n. It is worth mentioning that another generalization of the Stancu operator is due to Nowak [6] involving q-integers. His generalization contains in special case the q-Bernstein operators of Phillips.

The goal of the paper is to study the approximation properties of the operators defined by (1.3)-(1.4). The construction of the new operator is based on the generalization of the well-known Gauss identity

$$\prod_{i=0}^{n-1}(x+q^i a)=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q q^{i(i-1)/2}a^i x^{n-i} \tag{1.5}$$

(see [4, p. 15, (5.5)]). We establish the uniform convergence of $U_{n,q}^{\alpha}(f;x)$ to $f(x)$ on $[0,1]$, when $\alpha = \alpha_n$ and $q = q_n$, and we give the rate of convergence by using the second order modulus of smoothness of $f \in C[0,1]$ defined by

$$\omega_2(f;\delta)=\sup_{0<h\le\delta}\ \sup_{x\in[0,1-2h]}|f(x+2h)-2f(x+h)+f(x)|. \tag{1.6}$$

Finally, for α and q fixed, we prove the existence of the limit operator $U_{\infty,q}^{\alpha} = \lim_{n\to\infty} U_{n,q}^{\alpha}$ taking into account the relationship between two consecutive terms of the sequence $\{U_{n,q}^{\alpha}(f;x)\}_{n\ge1}$. In this case the rate of convergence is also studied.

2. AUXILIARY RESULTS

In the sequel we need some useful lemmas.

Lemma 2.1. *For any $n \ge 1$, $\alpha \ge 0$, $q > 0$ and u, v real numbers, we have*

$$\prod_{i=0}^{n-1}(v+uq^i+\alpha[i]_q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{k(k-1)/2}\prod_{i=0}^{k-1}(u+\alpha q^{-i}[i]_q)\prod_{j=0}^{n-k-1}(v+\alpha[j]_q). \tag{2.1}$$

Proof. We use induction on n. The equality (2.1) is evident for $n = 1$. Let us assume that (2.1) holds for a given n. Then, by (2.1) and $[n]_q = [n-k]_q + q^{n-k}[k]_q$, we have

$$\prod_{i=0}^{n}(v+uq^i+\alpha[i]_q)=(v+uq^n+\alpha[n]_q)\prod_{i=0}^{n-1}(v+uq^i+\alpha[i]_q)=$$
$$=\sum_{k=0}^{n}(uq^n+\alpha q^{n-k}[k]_q+v+\alpha[n-k]_q)\begin{bmatrix}n\\k\end{bmatrix}_q q^{k(k-1)/2}\prod_{i=0}^{k-1}(u+\alpha q^{-i}[i]_q)$$
$$\times\prod_{j=0}^{n-k-1}(v+\alpha[j]_q)=$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} q^n \prod_{i=0}^{k} (u + \alpha q^{-i}[i]_q) \prod_{j=0}^{n-k-1} (v + \alpha[j]_q) \\
&\quad + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} \prod_{i=0}^{k-1} (u + \alpha q^{-i}[i]_q) \prod_{j=0}^{n-k} (v + \alpha[j]_q) \\
&= \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} q^n \prod_{i=0}^{k} (u + \alpha q^{-i}[i]_q) \prod_{j=0}^{n-k-1} (v + \alpha[j]_q) + q^{n(n-1)/2} q^n \\
&\quad \times \prod_{i=0}^{n} (u + \alpha q^{-i}[i]_q) \\
&\quad + \prod_{j=0}^{n} (v + \alpha[j]_q) + \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} \prod_{i=0}^{k-1} (u + \alpha q^{-i}[i]_q) \prod_{j=0}^{n-k} (v + \alpha[j]_q) \\
&= \prod_{j=0}^{n} (v + \alpha[j]_q) + \sum_{k=1}^{n} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q q^{(k-1)(k-2)/2} q^n \prod_{i=0}^{k-1} (u + \alpha q^{-i}[i]_q) \prod_{j=0}^{n-k} (v + \alpha[j]_q) \\
&\quad + \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} \prod_{i=0}^{k-1} (u + \alpha q^{-i}[i]_q) \prod_{j=0}^{n-k} (v + \alpha[j]_q) + q^{(n+1)n/2} \prod_{i=0}^{n} (u + \alpha q^{-i}[i]_q) \\
&= \prod_{j=0}^{n} (v + \alpha[j]_q) + \sum_{k=1}^{n} \left(\begin{bmatrix} n \\ k-1 \end{bmatrix}_q q^{(k-1)(k-2)/2} q^n q^{-k(k-1)/2} + \begin{bmatrix} n \\ k \end{bmatrix}_q \right) \\
&\quad \times q^{k(k-1)/2} \prod_{i=0}^{k-1} (u + \alpha q^{-i}[i]_q) \prod_{j=0}^{n-k} (v + \alpha[j]_q) + q^{(n+1)n/2} \prod_{i=0}^{n} (u + \alpha q^{-i}[i]_q) \\
&= \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q q^{k(k-1)/2} \prod_{i=0}^{k-1} (u + \alpha q^{-i}[i]_q) \prod_{j=0}^{n-k} (v + \alpha[j]_q),
\end{aligned}
$$

because

$$
\begin{bmatrix} n+1 \\ 0 \end{bmatrix}_q = \begin{bmatrix} n+1 \\ n+1 \end{bmatrix}_q = 1
$$

and

$$
\begin{bmatrix} n \\ k-1 \end{bmatrix}_q q^{(k-1)(k-2)/2} q^n q^{-k(k-1)/2} + \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q q^{n-k+1} + \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n+1 \\ k \end{bmatrix}_q
$$

for $k = 1, 2, \ldots, n$. This completes the proof of the lemma. □

Remark 2.1. For $\alpha = 0$, the identity (2.1) reduces to the Gauss identity (1.5). □

Lemma 2.2. *Let $\alpha \geq 0$ and $q > 0$. For the test functions $e_i(x) = x^i$, where $i \in \{0, 1, 2\}$ and $x \in [0, 1]$, we have $U^{\alpha}_{n,q}(e_0; x) = 1$, $U^{\alpha}_{n,q}(e_1; x) = x$ and*

$$
\begin{aligned}
U^{\alpha}_{n,q}(e_2; x) = x^2 + \frac{x(1-x)}{[n]_q} \Big\{ & 1 - q \frac{x(1-q^{n-1}) - \alpha[n-1]_q}{1-x+xq^{n-1}+\alpha[n-1]_q} \\
& + q^2 \frac{x(1-q^{n-1}) - \alpha[n-1]_q}{1-x+xq^{n-1}+\alpha[n-1]_q} \frac{x(1-q^{n-2}) - \alpha[n-2]_q}{1-x+xq^{n-2}+\alpha[n-2]_q} - \ldots + (-1)^{n-1} q^{n-1} \\
& \times \frac{x(1-q^{n-1}) - \alpha[n-1]_q}{1-x+xq^{n-1}+\alpha[n-1]_q} \frac{x(1-q^{n-2}) - \alpha[n-2]_q}{1-x+xq^{n-2}+\alpha[n-2]_q} \cdots \frac{x(1-q) - \alpha[1]_q}{1-x+xq+\alpha[1]_q} \Big\}.
\end{aligned}
$$

Proof. Choosing $u = x$ and $v = 1 - x$ in (2.1), we get, by (1.4), that $\sum_{k=0}^{n} b_{n,k}^{\alpha}(q;x) = 1$. Hence, due to (1.3), we have $U_{n,q}^{\alpha}(e_0;x) = \sum_{k=0}^{n} b_{n,k}^{\alpha}(q;x) = 1$.

Again, by (1.3) and (1.4), we obtain

$$\begin{aligned} U_{n,q}^{\alpha}(e_1;x) &= \sum_{k=0}^{n} b_{n,k}^{\alpha}(q;x)\tfrac{[k]_q}{[n]_q} \\ &= \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q q^{k(k-1)/2}\, \frac{\prod_{i=0}^{k-1}(x+\alpha q^{-i}[i]_q)\prod_{j=0}^{n-k-1}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)} \\ &= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{(k+1)k/2}\, \frac{\prod_{i=0}^{k}(x+\alpha q^{-i}[i]_q)\prod_{j=0}^{n-k-2}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)} \\ &= \sum_{k=0}^{n-1} q^k \tfrac{x+\alpha q^{-k}[k]_q}{1-x+xq^{n-1}+\alpha[n-1]_q}\, b_{n-1,k}^{\alpha}(q;x). \end{aligned}$$

But $xq^k + \alpha[k]_q = x + [k]_q(\alpha - (1-q)x)$ and $U_{n-1,q}^{\alpha}(e_0;x) = 1$, therefore we have

$$\begin{aligned} &U_{n,q}^{\alpha}(e_1;x) = \\ &= \sum_{k=0}^{n-1} \tfrac{x+[k]_q(\alpha-(1-q)x)}{1-x+xq^{n-1}+\alpha[n-1]_q}\, b_{n-1,k}^{\alpha}(q;x) \\ &= \tfrac{x}{1-x+xq^{n-1}+\alpha[n-1]_q} + \tfrac{(\alpha-(1-q)x)[n-1]_q}{1-x+xq^{n-1}+\alpha[n-1]_q} \sum_{k=0}^{n-1} \tfrac{[k]_q}{[n-1]_q}\, b_{n-1,k}^{\alpha}(q;x) \\ &= \tfrac{x}{1-x+xq^{n-1}+\alpha[n-1]_q} + \tfrac{-x+xq^{n-1}+\alpha[n-1]_q}{1-x+xq^{n-1}+\alpha[n-1]_q}\, U_{n-1,q}^{\alpha}(e_1;x). \qquad (2.2) \end{aligned}$$

For given $\alpha \geq 0$, $q > 0$ and $x \in [0,1]$, for the sake of brevity, let us set $a_i = 1 - x + xq^{i-1} + \alpha[i-1]_q$, $i = 1, 2, \ldots$ Then, by (2.2), we have

$$U_{n,q}^{\alpha}(e_1;x) = \tfrac{x}{a_n} + \left(1 - \tfrac{1}{a_n}\right) U_{n-1,q}^{\alpha}(e_1;x).$$

Solving this recurrence relation, we find that

$$\begin{aligned} U_{n,q}^{\alpha}(e_1;x) = \tfrac{x}{a_n} &+ \tfrac{x}{a_{n-1}}\left(1 - \tfrac{1}{a_n}\right) + \tfrac{x}{a_{n-2}}\left(1 - \tfrac{1}{a_n}\right)\left(1 - \tfrac{1}{a_{n-1}}\right) + \ldots \\ &+ \tfrac{x}{a_2}\left(1 - \tfrac{1}{a_n}\right)\left(1 - \tfrac{1}{a_n}\right)\ldots\left(1 - \tfrac{1}{a_3}\right) \\ &+ \left(1 - \tfrac{1}{a_n}\right)\left(1 - \tfrac{1}{a_{n-1}}\right)\ldots\left(1 - \tfrac{1}{a_2}\right) U_{1,q}^{\alpha}(e_1;x) \end{aligned}$$

Hence, by $U_{1,q}^{\alpha}(e_1;x) = x$, we get

$$U_{n,q}^{\alpha}(e_1;x) =$$

$$
\begin{aligned}
&= x\left(1-\left(1-\tfrac{1}{a_n}\right)\right) + x\left(1-\left(1-\tfrac{1}{a_{n-1}}\right)\right)\left(1-\tfrac{1}{a_n}\right)\\
&\quad + x\left(1-\left(1-\tfrac{1}{a_{n-2}}\right)\right)\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right) + \dots\\
&\quad + x\left(1-\left(1-\tfrac{1}{a_2}\right)\right)\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_3}\right)\\
&\quad + x\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_2}\right)\\
&= x - x\left(1-\tfrac{1}{a_n}\right) + x\left(1-\tfrac{1}{a_n}\right) - x\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\\
&\quad + x\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right) - x\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\left(1-\tfrac{1}{a_{n-2}}\right) + \dots\\
&\quad + x\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_3}\right) - x\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_2}\right)\\
&\quad + x\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_2}\right)\\
&= x.
\end{aligned}
$$

Further, in view of (1.3) and (1.4), we have

$$
\begin{aligned}
U_{n,q}^{\alpha}(e_2;x) &= \sum_{k=0}^{n} b_{n,k}^{\alpha}(q;x)\left(\tfrac{[k]_q}{[n]_q}\right)^2\\
&= \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q q^{k(k-1)/2}\, \frac{\prod\limits_{i=0}^{k-1}(x+\alpha q^{-i}[i]_q)\prod\limits_{j=0}^{n-k-1}(1-x+\alpha[j]_q)}{\prod\limits_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)}\, \frac{[k]_q}{[n]_q}\\
&= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{(k+1)k/2}\, \frac{\prod\limits_{i=0}^{k}(x+\alpha q^{-i}[i]_q)\prod\limits_{j=0}^{n-k-2}(1-x+\alpha[j]_q)}{\prod\limits_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)}\, \frac{[k+1]_q}{[n]_q}\\
&= \sum_{k=0}^{n-1} q^k \frac{x+\alpha q^{-k}[k]_q}{1-x+xq^{n-1}+\alpha[n-1]_q}\, \frac{1+q[k]_q}{[n]_q}\, b_{n-1,k}^{\alpha}(q;x).
\end{aligned}
$$

But $(xq^k+\alpha[k]_q)(1+q[k]_q) = \{x+(\alpha-(1-q)x)[k]_q\}(1+q[k]_q) = x + (\alpha-(1-2q)x)[k]_q + q(\alpha-(1-q)x)[k]_q^2$, therefore, by $U_{n-1,q}^{\alpha}(e_0;x)=1$ and $U_{n-1,q}^{\alpha}(e_1;x)=x$, we have

$$
\begin{aligned}
&U_{n,q}^{\alpha}(e_2;x) =\\
&= \tfrac{x}{a_n[n]_q}\, U_{n-1,q}^{\alpha}(e_0;x) + \tfrac{\alpha-(1-2q)x}{a_n}\, \tfrac{[n-1]_q}{[n]_q}\, U_{n-1,q}^{\alpha}(e_1;x)\\
&\quad + \tfrac{q(\alpha-(1-q)x)}{a_n}\, \tfrac{[n-1]_q^2}{[n]_q}\, U_{n-1,q}^{\alpha}(e_2;x)\\
&= \tfrac{x}{a_n[n]_q} + \tfrac{\alpha-(1-2q)x}{a_n}\, \tfrac{[n-1]_q}{[n]_q} x + q\tfrac{-x+xq^{n-1}+\alpha[n-1]_q}{a_n}\, \tfrac{[n-1]_q}{[n]_q}\, U_{n-1,q}^{\alpha}(e_2;x)\\
&= \tfrac{x}{a_n[n]_q} + \tfrac{\alpha-(1-2q)x}{a_n}\, \tfrac{[n-1]_q}{[n]_q} x + q\tfrac{[n-1]_q}{[n]_q}\left(1-\tfrac{1}{a_n}\right) U_{n-1,q}^{\alpha}(e_2;x).
\end{aligned}
$$

Solving this recurrence relation, we obtain

$$\begin{aligned}U_{n,q}^{\alpha}(e_2;x) &= \tfrac{x}{a_n[n]_q} + \tfrac{xq}{a_{n-1}[n]_q}\left(1-\tfrac{1}{a_n}\right) + \tfrac{xq^2}{a_{n-2}[n]_q}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right) + \dots \\ &+ \tfrac{xq^{n-2}}{a_2[n]_q}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_3}\right) \\ &+ \tfrac{x(\alpha-(1-2q)x)}{a_n}\tfrac{[n-1]_q}{[n]_q} + \tfrac{xq(\alpha-(1-2q)x)}{a_{n-1}}\tfrac{[n-2]_q}{[n]_q}\left(1-\tfrac{1}{a_n}\right) \\ &+ \tfrac{xq^2(\alpha-(1-2q)x)}{a_{n-2}}\tfrac{[n-3]_q}{[n]_q}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right) + \dots \\ &+ \tfrac{xq^{n-2}(\alpha-(1-2q)x)}{a_2}\tfrac{[1]_q}{[n]_q}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right) + \dots\left(1-\tfrac{1}{a_3}\right) \\ &+ q^{n-1}\tfrac{[1]_q}{[n]_q}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right) + \dots\left(1-\tfrac{1}{a_2}\right)U_{1,q}^{\alpha}(e_2;x).\end{aligned}$$

Taking into account that $(\alpha-(1-2q)x)[k]_q = \alpha[k]_q - (1-q^k)x + xq[k]_q = a_{k+1} - 1 + xq[k]_q$ for $k = 1, 2, \dots, n-1$ and $U_{1,q}^{\alpha}(e_2;x) = x$, we find that

$$\begin{aligned}U_{n,q}^{\alpha}(e_2;x) &= \tfrac{x}{[n]_q}\Big\{\tfrac{1}{a_n} + \tfrac{q}{a_{n-1}}\left(1-\tfrac{1}{a_n}\right) + \tfrac{q^2}{a_{n-2}}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right) + \dots \\ &+ \tfrac{q^{n-2}}{a_2}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_3}\right) \\ &+q^{n-1}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_2}\right)\Big\} + \tfrac{x}{[n]_q}\Big\{\left(1-\tfrac{1}{a_n}\right) \\ &+ q\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right) + q^2\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\left(1-\tfrac{1}{a_{n-2}}\right) + \dots \\ &+q^{n-2}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_2}\right)\Big\} + \tfrac{x^2}{[n]_q}\Big\{\tfrac{q[n-1]_q}{a_n} \\ &+ \tfrac{q^2[n-2]_q}{a_{n-1}}\left(1-\tfrac{1}{a_n}\right) + \tfrac{q^3[n-3]_q}{a_{n-2}}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right) + \dots \\ &+\tfrac{q^{n-1}[1]_q}{a_2}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_3}\right)\Big\}\end{aligned} \tag{2.3}$$

The coefficient of $\frac{x}{[n]_q}$ in (2.3) is the following:

$$\begin{aligned}&1-\left(1-\tfrac{1}{a_n}\right) + q\left(1-\left(1-\tfrac{1}{a_{n-1}}\right)\right)\left(1-\tfrac{1}{a_n}\right) \\ &+ q^2\left(1-\left(1-\tfrac{1}{a_{n-2}}\right)\right)\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots \\ &+ q^{n-2}\left(1-\left(1-\tfrac{1}{a_2}\right)\right)\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_3}\right) \\ &+ q^{n-1}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_2}\right) + \left(1-\tfrac{1}{a_n}\right) \\ &+ q\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right) + q^2\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\left(1-\tfrac{1}{a_{n-2}}\right) + \dots \\ &+ q^{n-2}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_2}\right) = \\ &= 1 + q\left(1-\tfrac{1}{a_n}\right) + q^2\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right) + \dots \\ &+ q^{n-1}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_2}\right).\end{aligned} \tag{2.4}$$

The coefficient of $\frac{x^2}{[n]_q}$ in (2.3) is the following:

$$\begin{aligned}
&q[n-1]_q\left(1-\left(1-\tfrac{1}{a_n}\right)\right)+q^2[n-2]_q\left(1-\left(1-\tfrac{1}{a_{n-1}}\right)\right)\left(1-\tfrac{1}{a_n}\right)\\
&\quad+q^3[n-3]_q\left(1-\left(1-\tfrac{1}{a_{n-2}}\right)\right)\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)+\dots\\
&\quad+q^{n-1}[1]_q\left(1-\left(1-\tfrac{1}{a_2}\right)\right)\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_3}\right)=\\
&=q[n-1]_q-(q[n-1]_q-q^2[n-2]_q)\left(1-\tfrac{1}{a_n}\right)\\
&\quad-(q^2[n-2]_q-q^3[n-3]_q)\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)-\dots\\
&\quad-(q^{n-2}[2]_q-q^{n-1}[1]_q)\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_3}\right)\\
&\quad-q^{n-1}[1]_q\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_2}\right)\\
&=[n]_q-1-q\left(1-\tfrac{1}{a_n}\right)-q^2\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)-\dots\\
&\quad-q^{n-1}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_2}\right). \qquad (2.5)
\end{aligned}$$

Combining (2.3), (2.4) and (2.5), we obtain

$$\begin{aligned}
&U_{n,q}^{\alpha}(e_2;x)=\\
&=x^2+\tfrac{x(1-x)}{[n]_q}\Big\{1+q\left(1-\tfrac{1}{a_n}\right)+q^2\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)+\dots\\
&\quad+q^{n-1}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\dots\left(1-\tfrac{1}{a_2}\right)\Big\}\\
&=x^2+\tfrac{x(1-x)}{[n]_q}\Big\{1-q\tfrac{x(1-q^{n-1})-\alpha[n-1]_q}{1-x+xq^{n-1}+\alpha[n-1]_q}\\
&\quad+q^2\tfrac{x(1-q^{n-1})-\alpha[n-1]_q}{1-x+xq^{n-1}+\alpha[n-1]_q}\tfrac{x(1-q^{n-2})-\alpha[n-2]_q}{1-x+xq^{n-2}+\alpha[n-2]_q}-\dots+(-1)^{n-1}q^{n-1}\\
&\quad\times\tfrac{x(1-q^{n-1})-\alpha[n-1]_q}{1-x+xq^{n-1}+\alpha[n-1]_q}\tfrac{x(1-q^{n-2})-\alpha[n-2]_q}{1-x+xq^{n-2}+\alpha[n-2]_q}\dots\tfrac{x(1-q)-\alpha[1]_q}{1-x+xq+\alpha[1]_q}\Big\},
\end{aligned}$$

which was to be proved. □

REMARK 2.2. Because $U_{n,q}^0(e_2;x)=R_{n,q}(e_2;x)$ and $R_{n,q}(e_2;x)=x^2+\frac{x(1-x)}{[n]_q}\frac{1-x+xq^n}{1-x+xq}$ (see [5, p. 87, (5)]), we obtain, by Lemma 2.2, the following identity:

$$\begin{aligned}
&1-q\tfrac{x(1-q^{n-1})}{1-x+xq^{n-1}}+q^2\tfrac{x(1-q^{n-1})}{1-x+xq^{n-1}}\tfrac{x(1-q^{n-2})}{1-x+xq^{n-2}}-\dots\\
&\quad+(-1)^{n-1}q^{n-1}\tfrac{x(1-q^{n-1})}{1-x+xq^{n-1}}\tfrac{x(1-q^{n-2})}{1-x+xq^{n-2}}\dots\tfrac{x(1-q)}{1-x+xq}=\tfrac{1-x+xq^n}{1-x+xq}.
\end{aligned}$$

Analogously, in view of $U_{n,1}^{\alpha}(e_2;x)=S_n^{\alpha}(e_2;x)$ and $S_n^{\alpha}(e_2;x)=x^2+\frac{x(1-x)}{n}\frac{1+n\alpha}{1+\alpha}$ (see [11, p. 1184, Lemma 4.1]), we obtain, by Lemma 2.2, the following identity:

$$\begin{aligned}
&1+\tfrac{\alpha(n-1)}{1+\alpha(n-1)}+\tfrac{\alpha(n-1)}{1+\alpha(n-1)}\tfrac{\alpha(n-2)}{1+\alpha(n-2)}+\dots\\
&\quad+\tfrac{\alpha(n-1)}{1+\alpha(n-1)}\tfrac{\alpha(n-2)}{1+\alpha(n-2)}\dots\tfrac{\alpha}{1+\alpha}=\tfrac{1+n\alpha}{1+\alpha}.
\end{aligned}$$

□

LEMMA 2.3. *Let* $U_{n,q}^{\alpha}(f;x)$ *be defined by* (1.3)-(1.4). *Then for any* $n \geq 1$, $\alpha \geq 0$ *and* $q > 0$, *we have*

$$
\begin{aligned}
&U_{n,q}^{\alpha}(f;x) - U_{n+1,q}^{\alpha}(f;x) = \\
&= \sum_{k=0}^{n-1} b_{n-1,k}^{\alpha}(q;x) \frac{(x+\alpha q^{-k}[k]_q)(1-x+\alpha[n-k-1]_q)}{(1-x+xq^n+\alpha[n]_q)(1-x+xq^{n-1}+\alpha[n-1]_q)} \\
&\times \left\{ \frac{q^n[n]_q}{[n-k]_q} f\left(\frac{[k]_q}{[n]_q}\right) + \frac{q^k[n]_q}{[k+1]_q} f\left(\frac{[k+1]_q}{[n]_q}\right) - \frac{q^k[n+1]_q}{[n-k]_q} \frac{[n]_q}{[k+1]_q} f\left(\frac{[k+1]_q}{[n+1]_q}\right) \right\}.
\end{aligned}
$$

Proof. Because of $1-x+xq^n+\alpha[n]_q = q^n(x+\alpha q^{-k}[k]_q)+(1-x+\alpha[n-k]_q)$ we have, by (1.3) and (1.4), that

$$
\begin{aligned}
&(1 - x + xq^n + \alpha[n]_q)U_{n,q}^{\alpha}(f;x) = \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} q^n \frac{\prod_{i=0}^{k}(x+\alpha q^{-i}[i]_q) \prod_{j=0}^{n-k-1}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)} f\left(\frac{[k]_q}{[n]_q}\right) \\
&\quad + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} \frac{\prod_{i=0}^{k-1}(x+\alpha q^{-i}[i]_q) \prod_{j=0}^{n-k}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)} f\left(\frac{[k]_q}{[n]_q}\right) \\
&= q^n \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} \frac{\prod_{i=0}^{k}(x+\alpha q^{-i}[i]_q) \prod_{j=0}^{n-k-1}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)} f\left(\frac{[k]_q}{[n]_q}\right) \\
&\quad + q^{(n+1)n/2} \frac{\prod_{i=0}^{n}(x+\alpha q^{-i}[i]_q)}{\prod_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)} f(1) + \frac{\prod_{j=0}^{n}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)} f(0) \\
&\quad + \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} \frac{\prod_{i=0}^{k-1}(x+\alpha q^{-i}[i]_q) \prod_{j=0}^{n-k}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)} f\left(\frac{[k]_q}{[n]_q}\right) \\
&= \frac{\prod_{j=0}^{n}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)} f(0) + \sum_{k=1}^{n} q^n \begin{bmatrix} n \\ k-1 \end{bmatrix}_q q^{(k-1)(k-2)/2} \\
&\quad \times \frac{\prod_{i=0}^{k-1}(x+\alpha q^{-i}[i]_q) \prod_{j=0}^{n-k}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)} f\left(\frac{[k-1]_q}{[n]_q}\right) +
\end{aligned}
$$

$$+\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{k(k-1)/2}\frac{\prod_{i=0}^{k-1}(x+\alpha q^{-i}[i]_q)\prod_{j=0}^{n-k}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)}f\left(\frac{[k]_q}{[n]_q}\right)$$

$$+q^{(n+1)n/2}\frac{\prod_{i=0}^{n}(x+\alpha q^{-i}[i]_q)}{\prod_{i=0}^{n-1}(1-x+xq^i+\alpha[i]_q)}f(1). \tag{2.6}$$

On the other hand, by (1.3) and (1.4),

$$U^{\alpha}_{n+1,q}(f;x)=\frac{\prod_{j=0}^{n}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n}(1-x+xq^i+\alpha[i]_q)}f(0)+\sum_{k=1}^{n}\begin{bmatrix}n+1\\k\end{bmatrix}_q q^{k(k-1)/2}$$

$$\times\frac{\prod_{i=0}^{k-1}(x+\alpha q^{-i}[i]_q)\prod_{j=0}^{n-k}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n}(1-x+xq^i+\alpha[i]_q)}f\left(\frac{[k]_q}{[n+1]_q}\right)$$

$$+q^{(n+1)n/2}\frac{\prod_{i=0}^{n}(x+\alpha q^{-i}[i]_q)}{\prod_{i=0}^{n}(1-x+xq^i+\alpha[i]_q)}f(1). \tag{2.7}$$

Taking into account that

$$\begin{bmatrix}n+1\\k\end{bmatrix}_q=\begin{bmatrix}n\\k\end{bmatrix}_q\frac{[n+1]_q}{[n+1-k]_q},\quad \begin{bmatrix}n\\k-1\end{bmatrix}_q=\begin{bmatrix}n\\k\end{bmatrix}_q\frac{[k]_q}{[n+1-k]_q},\ \text{and}\ \begin{bmatrix}n\\k+1\end{bmatrix}_q=\begin{bmatrix}n-1\\k\end{bmatrix}_q\frac{[n]_q}{[k+1]_q},$$

from (2.6) and (2.7) we obtain

$$U^{\alpha}_{n,q}(f;x)-U^{\alpha}_{n+1,q}(f;x)=$$

$$=\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{k(k-1)/2}\frac{\prod_{i=0}^{k-1}(x+\alpha q^{-i}[i]_q)\prod_{j=0}^{n-k}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n}(1-x+xq^i+\alpha[i]_q)}$$

$$\times\left\{q^{n-k+1}\frac{[k]_q}{[n+1-k]_q}f\left(\frac{[k-1]_q}{[n]_q}\right)+f\left(\frac{[k]_q}{[n]_q}\right)-\frac{[n+1]_q}{[n+1-k]_q}f\left(\frac{[k]_q}{[n+1]_q}\right)\right\}$$

$$=\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_q\frac{[n]_q}{[k+1]_q}q^{(k+1)k/2}\frac{\prod_{i=0}^{k}(x+\alpha q^{-i}[i]_q)\prod_{j=0}^{n-k-1}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n}(1-x+xq^i+\alpha[i]_q)}$$

$$\times\left\{q^{n-k}\frac{[k+1]_q}{[n-k]_q}f\left(\frac{[k]_q}{[n]_q}\right)+f\left(\frac{[k+1]_q}{[n]_q}\right)-\frac{[n+1]_q}{[n-k]_q}f\left(\frac{[k+1]_q}{[n+1]_q}\right)\right\}$$

$$=\sum_{k=0}^{n-1}b^{\alpha}_{n-1,k}(q;x)\frac{(x+\alpha q^{-k}[k]_q)(1-x+\alpha[n-k-1]_q)}{(1-x+xq^n+\alpha[n]_q)(1-x+xq^{n-1}+\alpha[n-1]_q)}$$

$$\times\left\{\frac{q^n[n]_q}{[n-k]_q}f\left(\frac{[k]_q}{[n]_q}\right)+\frac{q^k[n]_q}{[k+1]_q}f\left(\frac{[k+1]_q}{[n]_q}\right)-\frac{q^k[n+1]_q}{[n-k]_q}\frac{[n]_q}{[k+1]_q}f\left(\frac{[k+1]_q}{[n+1]_q}\right)\right\},$$

which was to be proved. □

3. MAIN RESULTS

In the next theorem we study the uniform convergence of $\{U_{n,q}^{\alpha}(f;x)\}_{n\geq 1}$.

THEOREM 3.1. *Let* $U_{n,q_n}^{\alpha_n}(f;x)$ *be defined as in* (1.3)-(1.4), *and let* $\{\alpha_n\}_{n\geq 1}$ *and* $\{q_n\}_{n\geq 1}$ *be two sequences such that* $\alpha_n \geq 0$, $0 < q_n \leq 1$, $1 \leq q_n + \alpha_n$ *for* $n \geq 1$ *and* $\alpha_n \to 0$ *as* $n \to \infty$. *Then, for all* $f \in C[0,1]$, *we have* $\lim\limits_{n\to\infty} U_{n,q_n}^{\alpha_n}(f;x) = f(x)$ *uniformly with respect to* $x \in [0,1]$.

Proof. We observe that the operators defined by (1.3) and (1.4) are positive for $\alpha_n \geq 0$ and $0 < q_n \leq 1$. Therefore, in view of Popoviciu's result given in [10] or taking into account Korovkin's theorem [2, p. 8, Theorem 3.1] and Lemma 2.2, we have to prove that $\lim\limits_{n\to\infty} U_{n,q_n}^{\alpha_n}(e_2;x) = x^2$ uniformly in $x \in [0,1]$.

For $a_i = 1 - x + xq_n^{i-1} + \alpha_n[i-1]_{q_n}$, $i = 2,3,\ldots,n$ and $x \in [0,1]$, using the hypotheses about α_n and q_n, we obtain

$$\begin{aligned} 1 - a_i &= x(1-q_n^{i-1}) - \alpha_n[i-1]_{q_n} \leq 1 - q_n^{i-1} - \alpha_n[i-1]_{q_n} \\ &= (1 - q_n - \alpha_n)[i-1]_{q_n} \leq 0 \end{aligned} \tag{3.1}$$

for $x \in [0,1]$ and $i = 2,3,\ldots,n$, and

$$\begin{aligned} a_{i+1} - a_i &= x(q_n^i - q_n^{i-1}) + \alpha_n([i]_{q_n} - [i-1]_{q_n}) = xq_n^{i-1}(q_n - 1) + \alpha_n q_n^{i-1} \\ &\geq q_n^{i-1}(q_n-1) + \alpha_n q_n^{i-1} = q_n^{i-1}(q_n + \alpha_n - 1) \geq 0 \end{aligned} \tag{3.2}$$

for $x \in [0,1]$ and $i = 2,3,\ldots,n-1$. In view of (3.1) and (3.2), we get

$$0 \leq 1 - \tfrac{1}{a_i} \leq 1 - \tfrac{1}{a_{i+1}}$$

for $i = 2,3,\ldots,n-1$ and $x \in [0,1]$. Hence, by Lemma 2.2,

$$\begin{aligned} 0 &\leq U_{n,q_n}^{\alpha_n}(e_2;x) - x^2 \\ &= \tfrac{x(1-x)}{[n]_{q_n}} \Big\{ 1 + q_n\left(1 - \tfrac{1}{a_n}\right) + q_n^2\left(1-\tfrac{1}{a_n}\right)\left(1 - \tfrac{1}{a_{n-1}}\right) + \ldots \\ &\quad + q_n^{n-1}\left(1-\tfrac{1}{a_n}\right)\left(1-\tfrac{1}{a_{n-1}}\right)\ldots\left(1-\tfrac{1}{a_2}\right)\Big\} \\ &\leq \tfrac{x(1-x)}{[n]_{q_n}} \left\{ 1 + q_n\left(1-\tfrac{1}{a_n}\right) + q_n^2\left(1-\tfrac{1}{a_n}\right)^2 + \ldots + q_n^{n-1}\left(1-\tfrac{1}{a_n}\right)^{n-1}\right\} \\ &\leq \tfrac{x(1-x)}{[n]_{q_n}} \tfrac{1}{1-q_n\left(1-\frac{1}{a_n}\right)} = \tfrac{x(1-x)}{[n]_{q_n}} \tfrac{a_n}{q_n + (1-q_n)a_n}. \end{aligned} \tag{3.3}$$

But $a_n = 1 - x(1-q_n^{n-1}) + \alpha_n[n-1]_{q_n} \leq 1 + \alpha_n[n-1]_{q_n}$ and the function $t \to \frac{t}{q_n+(1-q_n)t}$ is increasing on $(0,\infty)$, thus, by (3.3), we obtain

$$\begin{aligned} 0 \leq U_{n,q_n}^{\alpha_n}(e_2;x) - x^2 &\leq \tfrac{1}{4[n]_{q_n}} \tfrac{1+\alpha_n[n-1]_{q_n}}{1+\alpha_n(1-q_n)[n-1]_{q_n}} \\ &\leq \tfrac{1}{4}\left\{\tfrac{1}{[n]_{q_n}} + \alpha_n \tfrac{[n-1]_{q_n}}{[n]_{q_n}}\right\} \leq \tfrac{1}{4}\left\{\tfrac{1}{[n]_{q_n}} + \alpha_n\right\}. \end{aligned} \tag{3.4}$$

Because $0 \le 1 - q_n \le \alpha_n$ and $\alpha_n \to 0$ as $n \to \infty$, we get $q_n \to 1$ as $n \to \infty$. Hence $[n]_{q_n} \to \infty$ as $n \to \infty$. Indeed, for any fixed positive integer k, we have $[n]_{q_n} \ge [k]_{q_n} = 1 + q_n + \ldots + q_n^{k-1}$ when $n \ge k$. But $q_n \to 1$ as $n \to \infty$, therefore $\liminf_{n\to\infty}[n]_{q_n} \ge \liminf_{n\to\infty}[k]_{q_n} = k$. Since k has been chosen arbitrarily, it follows that $[n]_{q_n} \to \infty$ as $n \to \infty$. In conclusion (3.4) implies that $\lim_{n\to\infty} U_{n,q_n}^{\alpha_n}(e_2;x) = x^2$ uniformly in $x \in [0,1]$, which completes the proof. □

Corollary 3.1. *Let* $U_{n,q_n}^{\alpha_n}(f;x)$, $\{\alpha_n\}_{n\ge1}$ *and* $\{q_n\}_{n\ge1}$ *be defined as in Theorem* 3.1. *Then there exists* $C > 0$ *such that*

$$|U_{n,q_n}^{\alpha_n}(f;x) - f(x)| \le C\,\omega_2\left(f; \tfrac{1}{2}\sqrt{[n]_{q_n}^{-1} + \alpha_n}\right)$$

for all $f \in C[0,1]$, $x \in [0,1]$ *and* $n \ge 1$.

Proof. If $\delta > 0$ and $W^2 = \{g \in C[0,1] : g'' \in C[0,1]\}$, then the K-functional is defined as $K_2(f;\delta) = \inf\{\|f-g\| + \delta\|g''^2\}$, where $\|\cdot\|$ denotes the sup-norm on $C[0,1]$. By [2, p. 177, Theorem 2.4], there exists an absolute constant $C' > 0$ such that

$$K_2(f;\delta) \le C'\omega_2(f;\sqrt{\delta}), \quad \delta > 0, \tag{3.5}$$

where the second order modulus of smoothness is defined by (1.6).

Because of (1.3) and Lemma 2.2, we have $|U_{n,q_n}^{\alpha_n}(f;x)| \le U_{n,q_n}^{\alpha_n}(e_0;x)\|f\| = \|f\|$, therefore

$$\|U_{n,q_n}^{\alpha_n} f\| \le \|f\| \tag{3.6}$$

for all $f \in C[0,1]$. Further, for any $g \in W^2$, by Taylor's formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t g''(u)(t-u)\,du, \quad t \in [0,1],$$

and Lemma 2.2, we find that

$$\begin{aligned} &|U_{n,q_n}^{\alpha_n}(g;x) - g(x)| = \\ &= \left|U_{n,q_n}^{\alpha_n}\left(\int_x^t (t-u)g''(u)\,du; x\right)\right| \le U_{n,q_n}^{\alpha_n}\left(\Big|\int_x^t |t-u|\,|g''(u)|\,du\Big|; x\right) \\ &\le \|g''\| U_{n,q_n}^{\alpha_n}((t-x)^2;x) = \|g''\|\{U_{n,q_n}^{\alpha_n}(e_2;x) - x^2\}. \end{aligned} \tag{3.7}$$

Now, combining (3.7) and (3.4), we get

$$\|U_{n,q_n}^{\alpha_n} g - g\| \le \tfrac{1}{4}([n]_{q_n}^{-1} + \alpha_n)\|g''\|.$$

Hence, in view of (3.6),

$$\begin{aligned} \|U_{n,q_n}^{\alpha_n} f - f\| &\le \|U_{n,q_n}^{\alpha_n}(f-g) - (f-g)\| + \|U_{n,q_n}^{\alpha_n} g - g\| \\ &\le 2\|f-g\| + \tfrac{1}{4}([n]_{q_n}^{-1} + \alpha_n)\|g''\| \\ &\le 2\{\|f-g\| + \tfrac{1}{4}([n]_{q_n}^{-1} + \alpha_n)\|g''\|\}. \end{aligned}$$

Taking the infimum on the right-hand side over all $g \in W^2$, and using (3.5), we get the desired estimates. $\square$

Remark 3.1. Let $\alpha \geq 0$ and $q \in (0,1)$ be given such that $1 \leq q+\alpha$. Taking into account the equality

$$U_{n,q}^{\alpha}(e_2;x) - x^2 = \frac{x(1-x)}{[n]_q}\left\{1 + q\left(1-\frac{1}{a_n}\right) + q^2\left(1-\frac{1}{a_n}\right)\left(1-\frac{1}{a_{n-1}}\right) + \dots \right.$$
$$\left. + q^{n-1}\left(1-\frac{1}{a_n}\right)\left(1-\frac{1}{a_{n-1}}\right)\dots\left(1-\frac{1}{a_2}\right)\right\}$$

(see (3.3) and Lemma 2.2) and the inequality (3.1), we may write that $U_{n,q}^{\alpha}(e_2;x) \geq x^2 + \frac{1-q}{1-q^n}x(1-x)$. Hence

$$\lim_{n\to\infty} U_{n,q}^{\alpha}(e_2;x) \geq x^2 + (1-q)x(1-x) > x^2$$

for $x \in (0,1)$ (if the limit there exists). This means that the operators (1.3) do not satisfy the conditions of Korovkin's theorem. $\square$

In the next theorem we propose the investigation of convergence of the operators (1.3)–(1.4), when the parameters α and q are fixed.

Theorem 3.2. *Let $U_{n,q}^{\alpha}(f;x)$ be defined by* (1.3)-(1.4). *If $\alpha \geq 0$ and $q \in (0,1)$, then there exist $U_{\infty,q}^{\alpha} : C[0,1] \to C[0,1]$ positive linear operator and $C > 0$ absolute constant such that*

$$\|U_{n,q}^{\alpha} f - U_{\infty,q}^{\alpha} f\| \leq C\omega_2\left(f; \frac{q^{n/2}}{1-q^{n+1}}\right)$$

for all $f \in C[0,1]$ and $n \geq 1$.

Proof. We find for $g \in W^2$, by Taylor's formula, that

$$g\left(\frac{[k]_q}{[n]_q}\right) = g\left(\frac{[k+1]_q}{[n+1]_q}\right) + g'\left(\frac{[k+1]_q}{[n+1]_q}\right)\left(\frac{[k]_q}{[n]_q} - \frac{[k+1]_q}{[n+1]_q}\right)$$
$$+ \int_{[k+1]_q/[n+1]_q}^{[k]_q/[n]_q} \left(\frac{[k]_q}{[n]_q} - u\right) g''(u)\,du \tag{3.8}$$

and

$$g\left(\frac{[k+1]_q}{[n]_q}\right) = g\left(\frac{[k+1]_q}{[n+1]_q}\right) + g'\left(\frac{[k+1]_q}{[n+1]_q}\right)\left(\frac{[k+1]_q}{[n]_q} - \frac{[k+1]_q}{[n+1]_q}\right)$$
$$+ \int_{[k+1]_q/[n+1]_q}^{[k+1]_q/[n]_q} \left(\frac{[k+1]_q}{[n]_q} - u\right) g''(u)\,du. \tag{3.9}$$

Because

$$q^n \frac{[n]_q}{[n-k]_q} + q^k \frac{[n]_q}{[k+1]_q} = q^k \frac{[n+1]_q}{[n-k]_q}\frac{[n]_q}{[k+1]_q}$$

and

$$q^n \frac{[n]_q}{[n-k]_q}\left(\frac{[k]_q}{[n]_q} - \frac{[k+1]_q}{[n+1]_q}\right) + q^k \frac{[n]_q}{[k+1]_q}\left(\frac{[k+1]_q}{[n]_q} - \frac{[k+1]_q}{[n+1]_q}\right) = 0,$$

by combining Lemma 2.3, (3.8) and (3.9), we obtain

$$
\begin{aligned}
&|U^{\alpha}_{n,q}(g;x) - U^{\alpha}_{n+1,q}(g;x)| \leq \\
&\leq \sum_{k=0}^{n-1} b^{\alpha}_{n-1,k}(q;x) \tfrac{(x+\alpha q^{-k}[k]_q)(1-x+\alpha[n-k-1]_q)}{(1-x+xq^n+\alpha[n]_q)(1-x+xq^{n-1}+\alpha[n-1]_q)} \\
&\quad \times \left\{ q^n \tfrac{[n]_q}{[n-k]_q} \left| \int_{[k+1]_q/[n+1]_q}^{[k]_q/[n]_q} \left(\tfrac{[k]_q}{[n]_q} - u \right) g''(u)\, du \right| \right. \\
&\quad \left. + \, q^k \tfrac{[n]_q}{[k+1]_q} \left| \int_{[k+1]_q/[n+1]_q}^{[k+1]_q/[n]_q} \left(\tfrac{[k+1]_q}{[n]_q} - u \right) g''(u)\, du \right| \right\} \\
&\leq \sum_{k=0}^{n-1} b^{\alpha}_{n-1,k}(q;x) \tfrac{(x+\alpha q^{-k}[k]_q)(1-x+\alpha[n-k-1]_q)}{(1-x+xq^n+\alpha[n]_q)(1-x+xq^{n-1}+\alpha[n-1]_q)} \|g''\| \\
&\quad \times \left\{ q^n \tfrac{[n]_q}{[n-k]_q} \left(\tfrac{[k]_q}{[n]_q} - \tfrac{[k+1]_q}{[n+1]_q} \right)^2 + q^k \tfrac{[n]_q}{[k+1]_q} \left(\tfrac{[k+1]_q}{[n]_q} - \tfrac{[k+1]_q}{[n+1]_q} \right) \right\} \\
&= \sum_{k=0}^{n-1} b^{\alpha}_{n-1,k}(q;x) \tfrac{(x+\alpha q^{-k}[k]_q)(1-x+\alpha[n-k-1]_q)}{(1-x+xq^n+\alpha[n]_q)(1-x+xq^{n-1}+\alpha[n-1]_q)} \|g''\| \\
&\quad \times \left\{ q^n \tfrac{[n]_q}{[n-k]_q} \tfrac{q^{2k}[n-k]_q^2}{[n]_q^2[n+1]_q^2} + q^k \tfrac{[n]_q}{[k+1]_q} \tfrac{q^{2n}[k+1]_q^2}{[n]_q^2[n+1]_q^2} \right) \\
&= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{k(k-1)/2} \tfrac{\prod_{i=0}^{k}(x+\alpha q^{-i}[i]_q) \prod_{j=0}^{n-k-1}(1-x+\alpha[j]_q)}{\prod_{i=0}^{n}(1-x+xq^i+\alpha[i]_q)} \\
&\quad \times q^k \tfrac{[n]_q[n+1]_q}{[k+1]_q[n-k]_q} \|g''\| \left\{ q^k \tfrac{[n-k]_q}{[n]_q} + q^n \tfrac{[k+1]_q}{[n]_q} \right\} \tfrac{q^n}{[n+1]_q^2} \tfrac{[k+1]_q}{[n+1]_q} \tfrac{[n-k]_q}{[n]_q} \\
&\leq \tfrac{2q^n}{[n+1]_q^2} \|g''\| \sum_{k=0}^{n-1} b^{\alpha}_{n+1,k+1}(q;x) = \tfrac{2q^n}{[n+1]_q^2} \|g''\| \sum_{k=1}^{n} b^{\alpha}_{n+1,k}(q;x) \\
&\leq \tfrac{2q^n}{[n+1]_q^2} \|g''\| \sum_{k=0}^{n+1} b^{\alpha}_{n+1,k}(q;x) = \tfrac{2q^n}{[n+1]_q^2} \|g''\| U^{\alpha}_{n+1,q}(e_0;x) = \tfrac{2q^n}{[n+1]_q^2} \|g''\|.
\end{aligned}
$$

Hence we find for every $g \in W^2$ and $n, p \geq 1$ that

$$
\begin{aligned}
&\|U^{\alpha}_{n,q} g - U^{\alpha}_{n+p,q} g\| \leq \\
&\leq \|U^{\alpha}_{n,q} g - U^{\alpha}_{n+1,q} g\| + \|U^{\alpha}_{n+1,q} g - U^{\alpha}_{n+2,q} g\| + \ldots + \|U^{\alpha}_{n+p-1,q} g - U^{\alpha}_{n+p,q} g\| \\
&\leq 2 \left(\tfrac{q^n}{[n+1]_q^2} + \tfrac{q^{n+1}}{[n+2]_q^2} + \cdots + \tfrac{q^{n+p-1}}{[n+p]_q^2} \right) \|g''\| \\
&\leq \tfrac{2q^n}{[n+1]_q^2} (1 + q + \cdots + q^{p-1}) \|g''\| \leq \tfrac{2q^n}{(1-q^{n+1})^2} \|g''\|. \qquad (3.10)
\end{aligned}
$$

This means that the sequence $\{U^{\alpha}_{n,q} g\}_{n \geq 1}$ is a Cauchy-sequence in $C[0,1]$, and therefore converges in $C[0,1]$ for all $g \in W^2$. On the other hand, analogously to (3.6), we obtain that

$$\|U^{\alpha}_{n,q} f\| \le \|f\| \tag{3.11}$$

for $f \in C[0,1]$, which implies that $\|U^{\alpha}_{n,q}\| = \sup\{\|U^{\alpha}_{n,q} f\| : \|f\| \le 1\} \le 1$ for each $n \ge 1$. However W^2 is dense in $C[0,1]$. Then, by the well-known Banach-Steinhaus theorem (see [2, p. 29]), we obtain the convergence of $\{U^{\alpha}_{n,q} f\}_{n\ge 1}$ in $C[0,1]$ for every $f \in C[0,1]$. In conclusion there exists an operator $U^{\alpha}_{\infty,q} : C[0,1] \to C[0,1]$ such that $\|U^{\alpha}_{n,q} f - U^{\alpha}_{\infty,q} f\| \to 0$ as $n \to \infty$, for all $f \in C[0,1]$. This also implies that $U^{\alpha}_{\infty,q}$ is a positive linear operator on $C[0,1]$, because $U^{\alpha}_{n,q}$ are positive linear operators on $C[0,1]$ for $n \ge 1$.

Further, let $p \to \infty$ in (3.10). Then

$$\|U^{\alpha}_{n,q} g - U^{\alpha}_{\infty,q} g\| \le \tfrac{2q^n}{(1-q^{n+1})^2} \|g''\|. \tag{3.12}$$

Letting $n \to \infty$ in (3.11), we get

$$\|U^{\alpha}_{\infty,q} f\| \le \|f\| \tag{3.13}$$

for all $f \in C[0,1]$. Combining (3.11), (3.12) and (3.13), we find that

$$\begin{aligned} \|U^{\alpha}_{n,q} f - U^{\alpha}_{\infty,q} f\| &\le \|U^{\alpha}_{n,q} f - U^{\alpha}_{n,q} g\| + \|U^{\alpha}_{n,q} g - U^{\alpha}_{\infty,q} g\| + \|U^{\alpha}_{\infty,q} g - U^{\alpha}_{\infty,q} f\| \\ &\le 2\|f-g\| + \tfrac{2q^n}{(1-q^{n+1})^2}\|g''\|. \end{aligned}$$

Taking the infimum on the right-hand side over all $g \in W^2$, and using (3.5), we get

$$\|U^{\alpha}_{n,q} f - U^{\alpha}_{\infty,q} f\| \le C\omega_2\left(f; \tfrac{q^{n/2}}{1-q^{n+1}}\right),$$

which was to be proved. □

Remark 3.2. Ostrovska proved in [8] for $q \ne 1$ and $f \in C[0,1]$ that $R_{n,q} f$ converges uniformly to $R_{\infty,q} f$ on $[0,1]$ as $n \to \infty$, and the rate of convergence $\|R_{n,q} f - R_{\infty,q} f\|$ has been studied by Wang and Zhang in [12]. Theorem 3.2 implies for $\alpha = 0$ and $q \in (0,1)$ the following estimation: *there exists a constant $C > 0$ such that*

$$\|R_{n,q} f - R_{\infty,q} f\| \le C\omega_2\left(f; \tfrac{q^{n/2}}{1-q^{n+1}}\right),$$

where $f \in C[0,1]$ and $n \ge 1$ are arbitrary. □

Remark 3.3. If $\alpha \ge 0$ and $q \in (0,1)$ are given such that $1 \le q + \alpha$, then, in view of Remark 2.2, we have $U^{\alpha}_{\infty,q}(e_2; x) > x^2$ for all $x \in (0,1)$, where the existence of the positive linear limit operator $U^{\alpha}_{\infty,q} : C[0,1] \to C[0,1]$ is guaranteed by Theorem 3.2. On the other hand, because of Lemma 2.2, the operator $U^{\alpha}_{\infty,q}$ reproduces the linear functions. Now, applying [13, p. 1100, Theorem 9], we find that $U^{\alpha}_{\infty,q} f = f$ if and only if f is linear. □

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