JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 43 (2016) no. 2, pp. 177–182
ictp.acad.ro/jnaat

ON α-CONVEX SEQUENCES OF HIGHER ORDER*

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Abstract. Many important applications of the class of convex sequences came across in several branches of mathematics as well as their generalizations. In this paper, we have introduced a new class of convex sequences, the class of α-convex sequences of higher order. In addition, the characterizations of sequences belonging to this class have been shown.

MSC 2010. 26A51, 26A48, 26D15.
Keywords. Sequence, convexity, α-convexity, $(p, q; r)$-convexity, starshaped sequence, higher order convexity.

1. INTRODUCTION

The class of convex sequences is one of most important subclass of the class of real sequences. This class is raised as a result of some efforts to solve several problems in mathematics. Naturally, the sequences that belong to that class, have useful applications in some branches of mathematics, in particular in mathematical analysis. For instance, such sequences are widely used in theory of inequalities (see [13], [7], [8]), in absolute summability of infinite series (see [1], [2]), and in theory of Fourier series, related to their uniform convergence and the integrability of their sum functions (see as example [6], page 587). Here, in this paper, we are going to introduce a new particular class of convex sequences, which indeed generalizes an another class of convex sequences introduced previously by others. In order to do this, we need first to recall some notations and notions as follows in the sequel.

Let $(a_n)_{n=0}^{\infty}$ be a real sequence. It is previously defined that

$$\triangle^0 a_n = a_n, \quad \triangle^1 a_n = a_{n+1} - a_n, \quad \triangle^m a_n = \triangle(\triangle^{m-1} a_n), \quad m, n = 0, 1, \ldots,$$

and throughout the paper we shall write $\triangle a_n$ instead of $\triangle^1 a_n$.

The following definition presents the concept of convexity of higher order.

DEFINITION 1. *A sequence $(a_n)_{n=0}^{\infty}$ is said to be convex of higher order (or m-convex) if*

$$\triangle^m a_n \geq 0,$$

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for all $n \geq 0$*.*

Next Lemma has been proved by Gh. Toader, which characterizes the convex sequences of higher order (or m-convex).

LEMMA 2. [11] *The sequence* $(a_n)_{n=0}^{\infty}$ *is convex of order* m *(*$m \in \mathbb{N}$*, fixed) iff*

$$a_n = \sum_{k=0}^{n} \binom{n+m-k-1}{m-1} b_k$$

with $b_k \geq 0$*, for* $k \geq m$*.*

Various generalizations of convexity were studied by many authors. For instance, p-convexity (see [5]), (p, q)-convexity (see [4]), and $(p, q; r)$-convexity [3].

Two other classes of sequences, the so-called, starshaped sequences and α-convex sequences have been introduced in [9] and [10].

Indeed, let throughout this paper be $\alpha \in [0, 1]$.

DEFINITION 3. *A sequence* $(a_n)_{n=0}^{\infty}$ *is called* α*-convex if the sequence*

$$\left(\alpha(a_{n+1} - a_n) + (1 - \alpha)\tfrac{a_n - a_0}{n}\right)_{n=1}^{\infty}$$

is increasing.

DEFINITION 4. *A sequence* $(a_n)_{n=0}^{\infty}$ *is called starshaped if*

$$\frac{a_{n+1} - a_0}{n+1} \geq \frac{a_n - a_0}{n}$$

for $n \geq 1$*.*

The class of starshaped sequences of higher order has been introduced in [12]:

DEFINITION 5. *A sequence* $(a_n)_{n=0}^{\infty}$ *is called starshaped of order* m *if*

$$\triangle^{m-1}\left(\frac{a_{n+1} - a_0}{n+1}\right) \geq 0$$

for all $n \geq 1$ *and* $m \in \{2, 3, \dots\}$*.*

Next Lemma characterizes the starshaped sequences of order m.

LEMMA 6. [12] *The sequence* $(a_n)_{n=0}^{\infty}$ *is starshaped sequences of order* m *(*$m \in \mathbb{N}$*, fixed) iff*

$$a_n = n \sum_{k=1}^{n} \binom{n+m-k-2}{m-2} c_k + c_0$$

with $c_k \geq 0$*, for* $k > m$*.*

Here, we introduce a new class of sequences as follows:

Definition 7. *A sequence* $(a_n)_{n=0}^{\infty}$ *is called* (m,α)*-convex (or* α*-convex of order* m*) if the sequence*

$$\left(\alpha\triangle^{m-1}(a_{n+1}-a_n)+(1-\alpha)\triangle^{m-1}\left(\tfrac{a_n-a_0}{n}\right)\right)_{n=1}^{\infty}$$

is increasing for all $n\in\{0,1,\dots\}$*, and for arbitrary fixed* m*,* $m\in\mathbb{N}$*.*

Remark 8. We note that: $(1,\alpha)$-convexity is the same with α-convexity, $(m,1)$-convexity is the same with m-convexity, $(m,0)$-convexity is the same with star-shapedness of order m, $(1,1)$-convexity is the same with ordinary convexity, and $(1,0)$-convexity is the same with star-shapedness. □

Characterizing (m,α)-convex sequences, we are going to accomplish the main aim of this paper.

2. MAIN RESULTS

First, we begin with:

Theorem 9. *The sequence* $(a_n)_{n=0}^{\infty}$ *is* (m,α)*-convex if and only if*

$$\alpha\triangle^{m+1}(a_n)+(1-\alpha)\triangle^{m}\left(\tfrac{a_n-a_0}{n}\right)\geq 0,$$

for all $n\in\{0,1,\dots\}$*, and for arbitrary fixed* m*,* $m\in\mathbb{N}$*.*

Proof. The proof of this statement is an immediate result of the Definition 7.

The proof is completed. □

For $m=1$ we obtain:

Corollary 10. [10] *The sequence* $(a_n)_{n=0}^{\infty}$ *is* α*-convex if and only if*

$$\alpha\,\triangle^2\,(a_n)+(1-\alpha)\left(\tfrac{a_{n+1}-a_0}{n+1}-\tfrac{a_n-a_0}{n}\right)\geq 0,$$

for all $n\in\{0,1,\dots\}$*.*

Theorem 11. *The sequence* $(a_n)_{n=0}^{\infty}$ *is* (m,α)*-convex if and only if*

$$(a_n-a_0+\alpha[n(a_{n+1}-a_n)-(a_n-a_0)])_{n=1}^{\infty},$$

is a starshaped sequence of order m*, for arbitrary fixed* m *and* $m\in\mathbb{N}$*.*

Proof. For the sake of brevity we denote

$$A_n:=a_n-a_0+\alpha[n(a_{n+1}-a_n)-(a_n-a_0)],\quad n\in\{1,2,\dots\},$$

which can be rewritten as

$$A_n=\alpha n(a_{n+1}-a_n)+(1-\alpha)(a_n-a_0),\quad n\in\{1,2,\dots\}.$$

For $m=1$ we have (see also [10]) that $(A_0=0)$

$$\triangle\left(\tfrac{A_n}{n}\right)\geq 0\Longleftrightarrow\alpha\,\triangle^2\,(a_n)+(1-\alpha)\triangle\left(\tfrac{a_n-a_0}{n}\right)\geq 0,$$

for all $n\in\{0,1,\dots\}$.

According to this, and since the operator $\triangle$ is a linear one, then we have:

$$\begin{aligned}\triangle^m\left(\frac{A_n}{n}\right)\geq 0 &\iff \underbrace{\triangle(\triangle(\dots(\triangle}_{m-\text{times}}(A_n/n))))\geq 0\\ &\iff \underbrace{\triangle(\triangle(\dots(\triangle}_{(m-1)-\text{times}}\left(\alpha\triangle^2(a_n)+(1-\alpha)\triangle\left(\tfrac{a_n-a_0}{n}\right)\right))))\geq 0\\ &\iff \alpha\triangle^{m-1+2}(a_n)+(1-\alpha)\triangle^{m-1+1}\left(\tfrac{a_n-a_0}{n}\right)\geq 0\\ &\iff \alpha\triangle^{m+1}(a_n)+(1-\alpha)\triangle^{m}\left(\tfrac{a_n-a_0}{n}\right)\geq 0,\end{aligned}$$

for all $n\in\{0,1,\dots\}$ and for arbitrary fixed m, $m\in\mathbb{N}$.
The proof is completed. □

For $m=1$ we obtain:

Corollary 12. [10] *The sequence* $(a_n)_{n=0}^{\infty}$ *is* α*-convex if and only if*

$$(a_n-a_0+\alpha[n(a_{n+1}-a_n)-(a_n-a_0)])_{n=1}^{\infty},$$

is a starshaped sequence.

Theorem 13. *The sequence* $(a_n)_{n=0}^{\infty}$ *is* (m,α)*-convex if and only if it may be represented by*

$$a_n=n\sum_{k=1}^{n}\frac{c_k}{k}-(n-1)c_0, \tag{1}$$

with

$$\triangle^{m-1}\left(c_{n+2}-\frac{n}{n+1}c_{n+1}\right)\geq\left(1-\frac{1}{\alpha}\right)\triangle^{m-1}\left(\frac{c_{n+1}}{n+1}\right), \tag{2}$$

$$\triangle^{m-1}\left(c_{n+2}-\frac{n}{n+1}c_{n+1}\right)\geq 0, \triangle^{m-1}\left(\frac{c_{n+1}}{n+1}\right)\geq 0,$$

$n\geq 2$, *and for arbitrary fixed* m, $m\in\mathbb{N}$.

Proof. On one hand, taking into account (1), we easy obtain

$$\triangle^2(a_n)=c_{n+2}-\frac{n}{n+1}c_{n+1}$$

and, consequently

$$\triangle^{m+1}(a_n)=\triangle^{m-1}\left(c_{n+2}-\frac{n}{n+1}c_{n+1}\right). \tag{3}$$

On the other hand, using (1) again, we also have

$$\triangle\left(\frac{a_n-a_0}{n}\right)=\frac{c_{n+1}}{n+1}$$

and, thus

$$\triangle^m\left(\frac{a_n-a_0}{n}\right)=\triangle^{m-1}\left(\frac{c_{n+1}}{n+1}\right). \tag{4}$$

From (3) and (4) we obtain

$$\alpha \triangle^{m+1}(a_n) + (1-\alpha)\triangle^m\left(\frac{a_n - a_0}{n}\right) =$$
$$= \alpha \triangle^{m-1}\left(c_{n+2} - \frac{n}{n+1}c_{n+1}\right) + (1-\alpha)\triangle^{m-1}\left(\frac{c_{n+1}}{n+1}\right).$$

Subsequently, it follows that

$$\alpha \triangle^{m+1}(a_n) + (1-\alpha)\triangle^m\left(\frac{a_n - a_0}{n}\right) \geq 0$$

if and only if

$$\triangle^{m-1}\left(c_{n+2} - \frac{n}{n+1}c_{n+1}\right) \geq \left(1 - \frac{1}{\alpha}\right)\triangle^{m-1}\left(\frac{c_{n+1}}{n+1}\right).$$

The proof is completed. □

REMARK 14. Note that representation (1) has been presented for the first time in [9]. □

COROLLARY 15 ([10]). *The sequence $(a_n)_{n=0}^{\infty}$ is α-convex if and only if it may be represented by*

$$a_n = n\sum_{k=1}^{n}\frac{c_k}{k} - (n-1)c_0,$$

with

$$c_{n+2} \geq \left(1 - \frac{1}{\alpha(n+1)}\right)c_{n+1},$$

and $c_n \geq 0$, $n \geq 2$.

THEOREM 16. *Let $m \in \mathbb{N}$ be fixed and $\alpha \in [0,1]$. If the sequence $(a_n)_{n=0}^{\infty}$ is (m,α)-convex, then it is (m,β)-convex.*

Proof. The proof follows from Theorem 13. Indeed, let the sequence $(a_n)_{n=0}^{\infty}$ be (m,α)-convex. Then, it may be represented by (1) with (2),

$$\triangle^{m-1}\left(c_{n+2} - \frac{n}{n+1}c_{n+1}\right) \geq \left(1 - \frac{1}{\alpha}\right)\triangle^{m-1}\left(\frac{c_{n+1}}{n+1}\right),$$

and

$$\triangle^{m-1}\left(c_{n+2} - \frac{n}{n+1}c_{n+1}\right) \geq 0, \quad \triangle^{m-1}\left(\frac{c_{n+1}}{n+1}\right) \geq 0,$$

$n \geq 2$.

Although, since $0 \leq \beta \leq \alpha$, we also have

$$\alpha \triangle^{m-1}\left(c_{n+2} - \frac{n}{n+1}c_{n+1}\right) \geq \left(1 - \frac{1}{\beta}\right)\triangle^{m-1}\left(\frac{c_{n+1}}{n+1}\right),$$

with same conditions as above, which shows that the sequence $(a_n)_{n=0}^{\infty}$ is (m,β)-convex as well.

The proof is completed. □

For $m = 1$, as a particular case, we obtain

COROLLARY 17 ([10]). *If the sequence $(a_n)_{n=0}^{\infty}$ is α-convex, then it is β-convex, for $0 \leq \beta \leq \alpha$.*

Acknowledgement. The author would like to thank the anonymous referee for her/his remarks which averted some inaccuracies.

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Received by the editors: August 18, 2016.