CHARACTERIZATION OF MIXED MODULUS OF SMOOTHNESS IN WEIGHTED $L^p$ SPACES$^{*,a}$

RAMAZAN AKGÜN$^{**}$

Abstract. The paper is concerned with estimates for the mixed modulus of smoothness in Lebesgue spaces with Muckenhoupt weights, Steklov type averages.

MSC 2010. 42A10, 41A17.

Keywords. Modulus of smoothness, Lebesgue spaces, Muckenhoupt weights, partial de la Vallée Poussin means, Fourier series.

1. INTRODUCTION

The main aim of this research is to investigate the approximation properties of some means of two dimensional Fourier series in Lebesgue spaces $L^p_ω(\mathbb{T}^2)$ with weights $ω$ in the Muckenhoupt’s class $A_p(\mathbb{T}^2, J)$, where $J$ is the set of rectangles in $\mathbb{T}^2 := \mathbb{T} \times \mathbb{T}$, $\mathbb{T} := [0, 2\pi]$ with sides parallel to coordinate axes. Trigonometric approximation by “angle” and mixed $K$-functional will be the main tools. We obtain the main properties of the weighted mixed modulus of smoothness $Ω_r(f, δ_1, δ_2)_{p,ω}$ in $L^p_ω(\mathbb{T}^2)$, $ω \in A_p(\mathbb{T}^2, J)$, $1 < p < \infty$. Note that, in general, in weighted spaces, such as $L^p_ω(\mathbb{T}^2)$, the classical translation operators are not bounded. Instead of classical translation operators we use Steklov type operators to define the weighted mixed modulus of smoothness $Ω_r(f, δ_1, δ_2)_{p,ω}$ in $L^p_ω(\mathbb{T}^2)$ (see [2]). Starting from 70s, in the classical nonweighted Lebesgue spaces $L^p(\mathbb{T}^2)$ (defined on $\mathbb{T}^2$ or $\mathbb{T}^d$, $d \geq 1$), some problems related to the classical nonweighted mixed modulus of smoothness $ω_r(f, δ_1, δ_2)_p$ have been actively studied by mathematicians: M. K. Potapov [14, 17, 18, 15, 16]; Potapov, Simonov, Lakovich [20]; Potapov, Simonov, Tikhonov [22, 19, 21]; A. F. Timan [27]; M. F. Timan [28, Chapter 2]. Among

$^*$This work has been supported by Balıkesir University Scientific Research Project 2016/58.

$^*$Results of this work were presented in the International Conference on Analysis and Its Applications (ICAA-2016), which held in 12-15 of July at the Ahi Evran University, Kirsehir, Turkey.

$^{**}$Department of Mathematics, Faculty of Arts and Sciences, University of Balıkesir, Cagısp Yeşilkesesi, 10145, Balıkesir, Turkey, e-mail: rakgun@balikesir.edu.tr.
these problems we mention direct and inverse theorems of angular approximation \[14, 17, 18\]; Hardy-Littlewood, Marcinkiewicz-Littlewood-Paley and embedding results \[15, 16\]; transformed Fourier series; embedding results of the Besov-Nikolski and Weyl-Nikolski classes \[19, 21\], Ulyanov type inequalities \[23\]; mixed K-functionals \[6, 25\]; fractional order classical mixed modulus of smoothness \[24\].

In what follows, \( A \lesssim B \) will mean that, there exists a positive constant \( C_{u,v,...} \), depending only on the parameters \( u,v,... \) and can be different in different places, such that the inequality \( A \leq CB \) holds. If \( A \lesssim B \) and \( B \lesssim A \) we will write \( A \approx B \).

It is well known that the main property of modulus of smoothness \( \Omega_r (\cdot,\delta_1,\delta_2)_{p,w} \) is that it decreases to zero as \( \max \{\delta_1,\delta_2\} \to 0 \). This rate can be characterized by some class \( \Phi_{a_1,a_2} \) defined below: the class \( \Phi_{a_1,a_2} \) consists of functions \( \psi (\cdot,\cdot) \) satisfying conditions

\[
\begin{align*}
(a) & \quad \psi (t_1,t_2) \geq 0 \text{ bounded on } (0,\infty) \times (0,\infty), \\
(b) & \quad \psi (t_1,t_2) \to 0 \text{ as } \max \{t_1,t_2\} \to 0, \\
(c) & \quad \psi (t_1,t_2) \text{ is non-decreasing in } t_1 \text{ and } t_2, \quad (d) \quad t_i^{-a_i} \psi (t_i) \text{ is non-increasing in } t_i \, (i = 1, 2).
\end{align*}
\]

We suppose that \( J \) is the set of rectangles in \( \mathbb{T}^2 \) with the sides parallel to coordinate axes. A function \( \omega : \mathbb{T}^2 \to \mathbb{R}^+ := [0, \infty) \) is called a weight on \( \mathbb{T}^2 \) if \( \omega (x_1,x_2) \) is measurable and positive almost everywhere on \( \mathbb{T}^2 \). We denote by \( A_p (\mathbb{T}^2, J) \), \( 1 < p < \infty \) the collection of locally integrable weights \( \omega : \mathbb{T}^2 \to \mathbb{R}^+ \) such that \( \omega (x_1,x_2) \) is \( 2\pi \)-periodic with respect to each variable \( x,y \) and

\[
\text{(1)} \quad C := \sup_{G \in J} \left( \frac{1}{|G|} \int_G \omega (x_1,x_2) \, dx_1 dx_2 \right) \left( \frac{1}{|G|} \int_G \left[ \omega (x_1,x_2) \right]^{\frac{1}{p-1}} \, dx_1 dx_2 \right)^{p-1} < \infty.
\]

The least constant \( C \) in (1) will be called the Muckenhoupt’s constant of \( \omega \) and denoted by \( [\omega]_{A_p} \).

The main result of this work is the characterization of the modulus of smoothness, given in the following theorem.

**Theorem 1.** Let \( r \in \mathbb{N}, \ p \in (1, \infty) \) and \( w \in A_p (\mathbb{T}^2, J) \).

(a) If \( f \in L^p_w (\mathbb{T}^2) \), then there exists \( \psi \in \Phi_{2r,2r} \) such that

\[
\Omega_r (f,t_1,t_2)_{p,w} \approx \psi (t_1,t_2)
\]

holds for all \( t_1,t_2 \in (0,\infty) \times (0,\infty) \) with equivalence constants depending only on \( r \) and \([w]_{A_p}\).

(b) If \( \psi \in \Phi_{2r,2r} \), then there exist \( f_0 \in L^p_w (\mathbb{T}^2) \) and the positive real numbers \( t_0, t_3 \) such that

\[
\Omega_r (f_0,\delta_1,\delta_2)_{p,w} \approx \psi (\delta_1,\delta_2)
\]

\( (2) \quad \Omega_r (f,t_1,t_2)_{p,w} \approx \psi (t_1,t_2) \)

holds for all \( t_1,t_2 \in (0,\infty) \times (0,\infty) \) with equivalence constants depending only on \( r \) and \([w]_{A_p}\).
holds for all \( \delta_1, \delta_2 \in (0, t_0) \times (0, t_3) \) with equivalence constants depending only on \( r \) and \( [w]_{A_p} \).

For functions in \( L^p_\omega (\mathbb{T}) \), \( p \in (1, \infty), \omega \in A_p (\mathbb{T}) \) Theorem 1 was obtained by the author in \cite{1}. In this work we simplify the (long) proof given in \cite{1}.

This type characterization theorem was proved in \cite{26} (one dimensional case) for the spaces \( L^p (\mathbb{T}), p \in [1, \infty) \), with classical moduli of smoothness of fractional order. The class \( \Phi^\omega \), described in \cite{4}, V. I. Kolyada \cite{12}; for \( \omega \) theorem was obtained by S. Tikhonov \cite{26}. For \( \omega_r (\cdot, \delta)_p \), \( r \in \mathbb{N} \) the characterization problem was investigated by O. V. Besov, S. B. Stechkin \cite{4}, V. I. Kolyada \cite{12}; for \( \omega_r (\cdot, \delta)_p, r > 0 \) the characterization theorem was obtained by S. Tikhonov \cite{26}.

2. PRELIMINARIES

Let \( L^1 (\mathbb{T}^2) \) be the collection of Lebesgue integrable functions \( f (x_1, x_2) : \mathbb{T}^2 \rightarrow \mathbb{R} \) such that \( f (x_1, x_2) \) is \( 2\pi \)-periodic with respect to each variable \( x_1, x_2 \). Let \( 1 < p < \infty, \omega (x_1, x_2) \in A_p (\mathbb{T}^2, [\omega]) \), and let \( L^p_\omega (\mathbb{T}^2) \) be the collection of Lebesgue integrable functions \( f (x_1, x_2) : \mathbb{T}^2 \rightarrow \mathbb{R} \) such that \( f (x_1, x_2) \) is \( 2\pi \)-periodic with respect to each variable \( x_1, x_2 \) and

\[
\|f\|_{p, \omega} := \left( \int_{\mathbb{T}^2} |f (x_1, x_2)|^p \omega (x_1, x_2) dx_1 dx_2 \right)^{1/p} < \infty.
\]

When \( \omega (x_1, x_2) \equiv 1 \) we denote \( \|f\|_{p, 1} =: \|f\|_p \) and \( L^p_1 (\mathbb{T}^2) =: L^p (\mathbb{T}^2) \) for \( 1 \leq p < \infty; L^\infty_1 (\mathbb{T}^2) =: L^\infty (\mathbb{T}^2) \).

We define Steklov type averages by

\[
\sigma_{h_1, h_2} f (x_1, x_2) = \frac{1}{h_1 h_2} \int_{x_1-h_1}^{x_1+h_1} \int_{x_2-h_2}^{x_2+h_2} f (t, \tau) dt d\tau.
\]

\[
\sigma_{h_1, 0} f (x_1, x_2) = \frac{1}{h_1} \int_{x_1-h_1}^{x_1+h_1} f (t, \tau) dt.
\]

\[
\sigma_{0, h_2} f (x_1, x_2) = \frac{1}{h_2} \int_{x_2-h_2}^{x_2+h_2} f (t, \tau) d\tau.
\]

**Lemma 2.** \cite{3} Theorem 3.3,\cite{2} If \( 1 < p < \infty, \omega \in A_p (\mathbb{T}^2, [\omega]), f \in L^p_\omega (\mathbb{T}^2) \), then

\[
\left\{ \|\sigma_{h_1, h_2} f\|_{p, \omega}, \|\sigma_{h_1, 0} f\|_{p, \omega}, \|\sigma_{0, h_2} f\|_{p, \omega} \right\} \lesssim \|f\|_{p, \omega},
\]

uniformly in \( h_1, h_2 \), where the constants depend only on \( [\omega]_{A_p} \) and \( p \).
For \(1 < p \leq \infty\), \(\omega \in A_p(T^2, \mathfrak{J})\), \(f \in L^p_{\omega}(T^2)\), \(h_1, h_2, r \in \mathbb{N}\), we define the mixed differences by
\[
\nabla^{r,\sigma}_{h_1,\sigma} f(x_1, x_2) \quad (1 - \sigma h_1, \sigma) f(x_1, x_2), \\
\nabla^{r,\rho}_{h_1,\rho} f(x_1, x_2) \quad (1 - \sigma h_1, \sigma) f(x_1, x_2), \\
\nabla^{r,\sigma}_{h_1,\sigma} f(x_1, x_2) = \nabla^{r,\rho}_{h_1,\rho} f(x_1, x_2),
\]
where \(I\) is identity operator on \(T^2\). Using the inequalities \([5]\) we get
\[
\left\{ \left\| \nabla^{r,\sigma}_{h_1,\sigma} f \right\|_{p,\omega}, \left\| \nabla^{r,\rho}_{h_1,\rho} f \right\|_{p,\omega}, \left\| \nabla^{r,\sigma}_{h_1,\sigma} f \right\|_{p,\omega} \right\} \lesssim \|f\|_{p,\omega},
\]
for \(1 < p < \infty\), \(\omega \in A_p(T^2, \mathfrak{J})\), \(f \in L^p_{\omega}(T^2)\), \(r \in \mathbb{N}\), with constants depending only on \([\omega]_{A_p}\) and \(p, r\).

The mixed modulus of smoothness of \(f \in L^p_{\omega}(T^2)\), \(1 < p < \infty\), \(\omega(x, y) \in A_p(T^2, \mathfrak{J})\), \(r \in \{0\} \cup \mathbb{N}\), can be defined as
\[
\Omega_r (f, \delta_1, \delta_2)_{p,\omega} = \left\{ \begin{array}{ll}
\sup_{0 \leq h_1 \leq \delta_1, 0 \leq h_2 \leq \delta_2} \left\| \nabla^{r,\rho}_{h_1,\rho} f \right\|_{p,\omega}, & r \in \mathbb{N}, \\
\|f\|_{p,\omega}, & r = 0.
\end{array} \right.
\]
If \(1 < p < \infty\), \(\omega \in A_p(T^2, \mathfrak{J})\), \(f \in L^p_{\omega}(T^2)\), \(r \in \mathbb{N}\), then from \([6]\) and \([5]\) \(\Omega_r (f, \delta_1, \delta_2)_{p,\omega} \lesssim \|f\|_{p,\omega}\) with constant depending only on \([\omega]_{A_p}\) and \(p, r\).

Note that from the definition of \(\Omega_r (f, \cdot, \cdot)_{p,\omega}\), it has the following properties when \(1 < p < \infty\), \(\omega \in A_p(T^2, \mathfrak{J})\), \(f \in L^p_{\omega}(T^2)\), \(r \in \mathbb{N}\):

1. \(\Omega_r (f, 0, 0)_{p,\omega} = 0\).
2. \(\Omega_r (f, \delta_1, \delta_2)_{p,\omega}\) is subadditive with respect to \(f\).
3. \(\Omega_r (f, \delta_1, \delta_2)_{p,\omega} \leq \Omega_r (f, t_1, t_2)_{p,\omega}\) for \(0 \leq t_1 \leq \delta_1; \quad i = 1, 2\).

When \(\omega(x_1, x_2) \equiv 1\) we denote \(\Omega_r (f, \delta_1, \delta_2)_{p,1} =: \Omega_r (f, \delta_1, \delta_2)_{p}\) for \(1 \leq p < \infty\); \(\Omega_r (f, \delta_1, \delta_2)_{\infty,1} =: \Omega_r (f, \delta_1, \delta_2)_{\infty}\).

Let \(1 < p < \infty\), \(\omega \in A_p(T^2, \mathfrak{J})\), and \(f \in L^p_{\omega}(T^2)\), then there is \(\lambda \in (1, \infty)\) such that \(f \in L^\lambda(T^2)\), namely, we have \(L^p_{\omega}(T^2) \subset L^\lambda(T^2)\) and this gives possibility to define the corresponding Fourier series of \(f\).

**Lemma 3.** \([2]\) If \(1 < p < \infty\), \(\omega \in A_p(T^2, \mathfrak{J})\), and \(f \in L^p_{\omega}(T^2)\), then we have
\[
L^\infty(T^2) \subset L^p_{\omega}(T^2) \subset L^\lambda(T^2)
\]
for some \(\lambda > 1\).

We define \(T_{m,n}\) as the set of all trigonometric polynomials of degree at most \(m\) with respect to variable \(x_1\) and of degree at most \(n\) with respect to variable \(x_2\). Then
\[
Y_{m_1,m_2}(f)_{p,\omega} = \inf \left\{ \left\| f - \sum_{i=1}^2 T_i \right\|_{p,\omega} : T_i \in T_{m_i} \right\}.
\]
where $T_m$ is the set of all two dimensional trigonometric polynomials of degree at most $m_i$ with respect to variable $x_i$ ($i = 1, 2$).

Let $1 < p < \infty,$ $\omega \in A_p (T^2, \mathcal{J})$ and \( \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} A_{n_1, n_2} (x_1, x_2) \) be the corresponding Fourier series for $f \in L^p_\omega (T^2)$. We define the partial sums of Fourier series of $f \in L^p_\omega (T^2)$, $1 < p < \infty,$ $\omega \in A_p (T^2, \mathcal{J})$ as

$$S_{m, o} (f) (x_1, x_2) = \sum_{n_1=0}^{m} \sum_{n_2=0}^{m} A_{n_1, n_2} (x_1, x_2, f),$$

$$S_{o, n} (f) (x_1, x_2) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} A_{n_1, n_2} (x_1, x_2, f),$$

$$S_{m_1, m_2} (f) (x_1, x_2) = \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} A_{n_1, n_2} (x_1, x_2, f).$$

Define the partial de la Vallée Poussin means of $f$ as

$$V_{m, o} (f) (x_1, x_2) = \frac{1}{m+1} \sum_{k=m}^{2m-1} S_{k, o} (f),$$

$$V_{o, n} (f) (x_1, x_2) = \frac{1}{n+1} \sum_{l=n}^{2n-1} S_{o, l} (f),$$

$$V_{m_1, m_2} (f) (x_1, x_2) = \frac{1}{(n+1)(m+1)} \sum_{k=m_1}^{2m_1-1} \sum_{l=m_2}^{2m_2-1} S_{k, l} (f).$$

**Lemma 4.** [2] If $1 < p < \infty,$ $\omega \in A_p (T^2, \mathcal{J})$, $f \in L^p_\omega (T^2)$, then

\[
\begin{align*}
\left\| S_{m, o} (f) \right\|_{p, \omega} & , \left\| S_{o, n} (f) \right\|_{p, \omega} ; \left\| S_{m_1, m_2} (f) \right\|_{p, \omega} \lesssim \left\| f \right\|_{p, \omega}, \\
\left\| V_{m, o} (f) \right\|_{p, \omega} & , \left\| V_{o, n} (f) \right\|_{p, \omega} ; \left\| V_{m_1, m_2} (f) \right\|_{p, \omega} \lesssim \left\| f \right\|_{p, \omega}, \\
\left\| f - W_{m_1, m_2} f \right\|_{p, \omega} & \lesssim Y_{m_1, m_2} (f)_{p, \omega}
\end{align*}
\]

where $W_{m_1, m_2} f (x_1, x_2) := (V_{m_1, o} (f) + V_{o, m_2} (f) - V_{m_1, m_2} (f)) (x_1, x_2)$ with all constants depending only on $[\omega]_A_p$ and $p$.

By Theorem 6 of [13]

$$\left\| f - C_{m_1, m_2}^\alpha f \right\|_{p, \omega} \to 0$$

as $m_1, m_2 \to \infty$ where $C_{m_1, m_2}^\alpha f$ is $\alpha$th Cesàro mean of $f$. From this we can deduce that $C (T^2)$ is a dense subset of $L^p_\omega (T^2)$ for $1 < p < \infty,$ $\omega \in A_p (T^2, \mathcal{J})$. Then $Y_{m_1, m_2} (f)_{p, \omega} \lesssim \left\| f - C_{m_1, m_2}^\alpha f \right\|_{p, \omega} \to 0$ and $Y_{m_1, m_2} (f)_{p, \omega} \to 0$ as $m_1, m_2 \to \infty$. 

Let $W^r_{p,\omega}, r, s \in \mathbb{N}$, respectively $W^r, W^o$ denote the collection of functions $f \in L^1(\mathbb{T}^d)$ such that $f^{(r,s)} \in L^p(\mathbb{T}^d)$ (respectively $f^{(r,o)} \in L^p(\mathbb{T}^d)$).

The following inequalities can be obtained by the method given in [2]. For $1 < p < \infty$, $\omega \in A_p(\mathbb{T}^2, J)$, $r \in \mathbb{N}$, there exist constants depending only on $[\omega]_{A_p}$ and $p, r$ so that

(i) (Jackson inequalities of Favard type)

\begin{align}
(12) & \quad Y_{m_1,m_2} (g_1)_{p,\omega} \lesssim \frac{1}{(m_1+1)^{2r}} \left\| g_1^{(2r,o)} \right\|_{p,\omega}, \quad g_1 \in W^2_{p,\omega}, \\
(13) & \quad Y_{m_1,m_2} (g_2)_{p,\omega} \lesssim \frac{1}{(m_2+1)^{2r}} \left\| g_2^{(2r,2r)} \right\|_{p,\omega}, \quad g_2 \in W^{2r}_{p,\omega}, \\
(14) & \quad Y_{m_1,m_2} (g) g_{p,\omega} \lesssim \frac{1}{(m_1+1)^{2r}(m_2+1)^{2r}} \left\| g^{(2r,2r)} \right\|_{p,\omega}, \quad g \in W^{2r,2r}_{p,\omega}.
\end{align}

(ii) if $\delta_1, \delta_2 > 0$ then

\begin{align}
(15) & \quad \Omega_r (g_1, \delta_1, \cdot)_{p,\omega} \lesssim \delta_1^2 \Omega_{r-1} (g_1^{(2,o)}, \delta_1, \cdot)_{p,\omega}, \quad g_1 \in W^2_{p,\omega}, \\
(16) & \quad \Omega_r (g_2, \cdot, \xi)_{p,\omega} \lesssim \xi^2 \Omega_{r-1} (g_2^{(2,o)}, \cdot, \xi)_{p,\omega}, \quad g_2 \in W^2_{p,\omega}, \\
(17) & \quad \Omega_r (g, \delta_1, \delta_2)_{p,\omega} \lesssim \delta_1^2 \delta_2^2 \Omega_{r-1} (g^{(2,o)}, \delta_1, \delta_2)_{p,\omega}, \quad g \in W^{2,2}_{p,\omega}.
\end{align}

and hence

\begin{align*}
\Omega_r (f, \delta_1, \cdot)_{p,\omega} & \lesssim \delta_1^2 \left\| f^{(2r,o)} \right\|_{p,\omega}, \\
\Omega_r (f, \cdot, \xi)_{p,\omega} & \lesssim \xi^2 \left\| f^{(o,2r)} \right\|_{p,\omega}, \\
\Omega_r (f, \delta_1, \delta_2)_{p,\omega} & \lesssim \delta_1^2 \delta_2^2 \left\| f^{(2r,2r)} \right\|_{p,\omega}.
\end{align*}

**Definition 5.** The mixed $K$-functional is defined as

\[ K(f, \delta_1, \delta_2, p, \omega, r, s) := \inf_{g_1, g_2 \in g} \left\{ \left\| f - g_1 - g_2 \right\|_{p,\omega} + \delta_1^2 \left\| \frac{\partial g_1}{\partial x^r} \right\|_{p,\omega} + \delta_2^2 \left\| \frac{\partial g_2}{\partial y^r} \right\|_{p,\omega} + \delta_1^2 \delta_2^2 \left\| \frac{\partial^{s+2r}}{\partial x^s \partial y^r} \right\|_{p,\omega} \right\} \]

where the infimum is taken for all $g_1, g_2, g$ so that $g_1 \in W^r_{p,\omega}, g_2 \in W^s_{p,\omega}, g \in W^{r,s}_{p,\omega}$, $r, s \in \mathbb{N}, 1 < p < \infty$, $\omega \in A_p(\mathbb{T}^2, J)$, $f \in L^p_{\omega}(\mathbb{T}^2)$.

(iii) If $1 < p < \infty$, $\omega \in A_p(\mathbb{T}^2, J)$, $f \in L^p_{\omega}(\mathbb{T}^2)$ and $r \in \mathbb{N}$, then there exist constants depending only on Muckenhoupt’s constant $[\omega]_{A_p}$ of $\omega$ and $p, r$ so that the equivalence

\[ \Omega_r (f, \delta_1, \delta_2)_{p,\omega} \approx K(f, \delta_1, \delta_2, p, \omega, 2r) \]
and the properties
\[ \Omega_r(f, \lambda_1, \eta_1) \lesssim (1 + [\lambda])^{2r} (1 + [\eta])^{2r} \Omega_r(f, \lambda_2, \eta_2), \]
\[ \Omega_r(f, \lambda_1, \eta_1) \lesssim \Omega_r(f, t_1, t_2), \quad 0 < t_1 \leq \delta_i; \quad i = 1, 2, \]
hold for \( \delta_1, \delta_2 > 0 \), where \( x := \max \{ z \in \mathbb{Z} : z \leq x \} \).

(iv) \( [11] \) can be refined by the inequality \( [19] \) below \( [2] \). If \( 1 < p < \infty \), \( \omega \in A_p(\mathbb{T}^2, \mathbb{S}) \), \( f \in L_p^p(\mathbb{T}^2) \) and \( r \in \mathbb{N} \), then there exists \( C_{\omega} A_p \) depending only on Muckenhoupt’s constant \( \{\omega\} A_p \) of \( \omega \) and \( p, r \) so that
\[ Y_{m_1, m_2}(f) \leq C_{\omega} A_p \Omega_r \left( f, \frac{1}{m_1}, \frac{1}{m_2} \right) \]
where \( m_1, m_2 \in \mathbb{N} \).

3. PROOF OF THEOREM 1

Let \( \omega_r(\cdot, \delta_1, \delta_2) \), \( 1 \leq p \leq \infty \), be the usual nonweighted mixed modulus of smoothness:
\[ \omega_r(g, \delta_1, \delta_2) := \sup_{0 \leq h_1 \leq \delta_1, 0 \leq h_2 \leq \delta_2} \| (I - T_{h_1, \omega}^r ) (I - T_{h_2, \omega}^r ) g \|_p, \quad g \in L_p^p(\mathbb{T}^2), \]
where \( T_{h_1, \omega} g(x_1, x_2) := g(x_1 + h_1, x_2) \); \( T_{h_2, \omega} g(x_1, x_2) := g(x_1, x_2 + h_2) \). From \( [25] \) \((1 \leq p < \infty)\) and \( [6] \)(\( p = \infty \)) and \( [18] \) there exist positive constants, depending only \( r, p \), such that
\[ \omega_{2r}(g, \delta_1, \delta_2) \approx \Omega_r(g, \delta_1, \delta_2) \]
holds for \( 1 \leq p \leq \infty \) and \( g \in L_p^p(\mathbb{T}^2) \).

Theorem 2.5 of \( [26] \) give that: Let \( r \in \mathbb{N} \), \( p \in [1, \infty] \).
(a) If \( f \in L_p^p(\mathbb{T}^2) \), then there exists \( \psi \in F_{r, r} \) such that
\[ \omega_r(f, t_1, t_2) \approx \psi(t_1, t_2) \]
holds for all \( t_1, t_2 \in (0, \infty) \times (0, \infty) \) with equivalence constants depending only on \( r \).
(b) If \( \psi \in F_{r, r} \) then there exist \( f_0 \in L_p^p(\mathbb{T}^2) \) and the positive real numbers \( t_0, t_3 \) such that
\[ \omega_r(f_0, \delta_1, \delta_2) \approx \psi(\delta_1, \delta_2) \]
holds for all \( \delta_1, \delta_2 \in (0, t_0) \times (0, t_3) \) with equivalence constants depending only on \( r \).

Proof of Theorem 1 (i) Note that if \( F \in C(\mathbb{T}^2) \) then from \( [7] \)
\[ \| \nabla_{h_1, h_2}^r F \|_{p, \omega} \leq C_{p, \omega} A_p \| \nabla_{h_1, h_2}^r F \|_{C(\mathbb{T}^2)}. \]
Using Theorem 2.5 (A) of [26], [7], [20], [18], [23] there exists $\psi \in \Phi_{2r}$ such that
\[
\Omega_r(F, \delta_1, \delta_2)_{p,w} \leq C_{p,[w]_{Ap}} \Omega_r(F, \delta_1, \delta_2)_\infty \leq C_{p,[w]_{Ap}} \omega_{2r}(F, \delta_1, \delta_2)_\infty \\
\leq C_{r,p,[w]_{Ap}} \psi(\delta_1, \delta_2).
\]

If $p \in (1, \infty)$, $A_p(T^2, \mathbb{J})$, $f \in L^p_c(T^2)$, then, by (11), for any $\varepsilon > 0$ there exists $F \in C(T^2)$ such that for all $\varepsilon > 0$ there exists $F \in C(T^2)$ such that $\|f - F\|_{p,w} < \varepsilon$. Thus
\[
\Omega_r(f, \delta_1, \delta_2)_{p,w} \leq \Omega_r(f - F, \delta_1, \delta_2)_{p,w} + \Omega_r(F, \delta_1, \delta_2)_{p,w} \\
\leq C_{r,p,[w]_{Ap}} \|f - F\|_{p,w} + C_{r,p,[w]_{Ap}} \psi(\delta_1, \delta_2).
\]

Letting $\varepsilon \to 0^+$ we get
\[
\Omega_r(f, \delta_1, \delta_2)_{p,w} \leq C_{r,p,[w]_{Ap}} \psi(\delta_1, \delta_2).
\]

On the other hand, from (18), (20), (7) and Theorem 2.5 (A) of [26] there exist $C_{r,p,[w]_{Ap}} \omega_{2r}(f, \delta_1, \delta_2)_{1} \leq C_{r,p,[w]_{Ap}} \Omega_r(f, \delta_1, \delta_2)_{p,w}$ and the equivalence (2) is established.

(ii) For the equivalence (3) let $\psi \in \Phi_{2r}$. By Theorem 2.5 (B) and Remark 2.7 (1) of [26] there exist $f \in L^\infty(T^2)$ and the positive real numbers $t_0, t_3$ such that
\[
\omega_{2r}(f, \delta_1, \delta_2)_p \approx \psi(\delta_1, \delta_2), \quad p = 1, \infty
\]
holds for all $\delta_1, \delta_2 \in (0,t_0) \times (0,t_3)$ with equivalence constants depending only on $r$. Then by (18), (20) we get
\[
\psi(\delta_1, \delta_2) \leq C_r \omega_{2r}(f, \delta_1, \delta_2)_{1} \leq C_r \Omega_r(f, \delta_1, \delta_2)_{1} \leq C_{r,p,[w]_{Ap}} \Omega_r(f, \delta_1, \delta_2)_{p,w} \\
\leq C_{r,p,[w]_{Ap}} \Omega_r(f, \delta_1, \delta_2)_\infty \leq C_{r,p,[w]_{Ap}} \omega_{2r}(f, \delta_1, \delta_2)_\infty \\
\leq C_{r,p,[w]_{Ap}} \psi(\delta_1, \delta_2)
\]
for all $\delta_1, \delta_2 \in (0,t_0) \times (0,t_3)$. \hfill $\square$

ACKNOWLEDGEMENTS. The author wish to express his sincere gratitude to the referee(s) for his/her valuable suggestions.

REFERENCES


10 Characterization of mixed modulus of smoothness


Received by the editors: August 22nd, 2016.