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# CHARACTERIZATION OF MIXED MODULUS OF SMOOTHNESS IN WEIGHTED $L^p$ SPACES[*,a]

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**Abstract.** The paper is concerned with estimates for the mixed modulus of smoothness in Lebesgue spaces with Muckenhoupt weights, Steklov type averages.

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**Keywords.** Modulus of smoothness, Lebesgue spaces, Muckenhoupt weights, partial de la Vallée Poussin means, Fourier series.

## 1. INTRODUCTION

The main aim of this research is to investigate the approximation properties of some means of two dimensional Fourier series in Lebesgue spaces $L^p_\omega(\mathbb{T}^2)$ with weights $\omega$ in the Muckenhoupt's class $A_p(\mathbb{T}^2, \mathbb{J})$, where $\mathbb{J}$ is the set of rectangles in $\mathbb{T}^2 := \mathbb{T} \times \mathbb{T}$, $\mathbb{T} := [0, 2\pi]$ with sides parallel to coordinate axes. Trigonometric approximation by "angle" and mixed $K$-functional will be the main tools. We obtain the main properties of the weighted mixed modulus of smoothness $\Omega_r(f, \delta_1, \delta_2)_{p,\omega}$ in $L^p_\omega(\mathbb{T}^2)$, $\omega \in A_p(\mathbb{T}^2, \mathbb{J})$, $1 < p < \infty$. Note that, in general, in weighted spaces, such as $L^p_\omega(\mathbb{T}^2)$, the classical translation operators are not bounded. Instead of classical translation operators we use Steklov type operators to define the weighted mixed modulus of smoothness $\Omega_r(f, \delta_1, \delta_2)_{p,\omega}$ in $L^p_\omega(\mathbb{T}^2)$ (see [2]). Starting from 70s, in the classical nonweighted Lebesgue spaces $L^p(\mathbb{T}^2)$ (defined on $\mathbb{T}^2$ or $\mathbb{T}^d$, $d \geq 1$), some problems related to the classical nonweighted mixed modulus of smoothness $\omega_r(f, \delta_1, \delta_2)_p$ have been actively studied by mathematicians: M. K. Potapov [14, 17], [18], [15, 16]; Potapov, Simonov, Lakovich [20]; Potapov, Simonov, Tikhonov [22], [19, 21]; A. F. Timan [27]; M. F. Timan [28, Chapter 2]. Among

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these problems we mention direct and inverse theorems of angular approximation [14, 17], [18]; Hardy-Littlewood, Marcinkiewicz-Littlewood-Paley and embedding results [15, 16]; transformed Fourier series; embedding results of the Besov-Nikolski and Weyl-Nikolskii classes [19, 21], Ulyanov type inequalities [23]; mixed $K$-functionals [6], [25]; fractional order classical mixed modulus of smoothness [24].

In what follows, $A \lesssim B$ will mean that, there exists a positive constant $C_{u,v,\ldots}$, depending only on the parameters $u, v, \ldots$ and can be different in different places, such that the inequality $A \leq CB$ holds. If $A \lesssim B$ and $B \lesssim A$ we will write $A \approx B$.

It is well known that the main property of modulus of smoothness $\Omega_r(\cdot, \delta_1, \delta_2)_{p,w}$ is that it decreases to zero as $\max\{\delta_1, \delta_2\} \to 0$. This rate can be characterized by some class $\Phi_{a_1,a_2}$ defined below: the class $\Phi_{a_1,a_2}$ $(a_1, a_2 \in \mathbb{R} \times \mathbb{R})$ consists of functions $\psi(\cdot, \cdot)$ satisfying conditions

(a) $\psi(t_1, t_2) \geq 0$ bounded on $(0, \infty) \times (0, \infty)$,
(b) $\psi(t_1, t_2) \to 0$ as $\max\{t_1, t_2\} \to 0$,
(c) $\psi(t_1, t_2)$ is non-decreasing in $t_1$ and $t_2$, (d) $t_i^{-a_i}\psi(t_i)$ is non-increasing in $t_i$ $(i = 1, 2)$.

We suppose that $\mathbb{J}$ is the set of rectangles in $\mathbb{T}^2$ with the sides parallel to coordinate axes. A function $\omega : \mathbb{T}^2 \to \mathbb{R}^{\geq} := [0, \infty)$ is called a weight on $\mathbb{T}^2$ if $\omega(x_1, x_2)$ is measurable and positive almost everywhere on $\mathbb{T}^2$. We denote by $A_p(\mathbb{T}^2, \mathbb{J})$, $(1 < p < \infty)$ the collection of locally integrable weights $\omega : \mathbb{T}^2 \to \mathbb{R}^{\geq}$ such that $\omega(x_1, x_2)$ is $2\pi$-periodic with respect to each variable $x, y$ and

(1)
$$C{:=}\sup_{G\in\mathbb{J}}\left(\frac{1}{|G|}\int\limits_G \omega(x_1, x_2)\, dx_1 dx_2\right)\left(\frac{1}{|G|}\int\limits_G [\omega(x_1, x_2)]^{\frac{1}{1-p}}\, dx_1 dx_2\right)^{p-1} < \infty.$$

The least constant $C$ in (1) will be called the Muckenhoupt's constant of $\omega$ and denoted by $[\omega]_{A_p}$.

The main result of this work is the characterization of the modulus of smoothness, given in the following theorem.

Theorem 1. *Let $r \in \mathbb{N}$, $p \in (1, \infty)$ and $w \in A_p(\mathbb{T}^2, \mathbb{J})$.*

(a) *If $f \in L^p_\omega(\mathbb{T}^2)$, then there exists $\psi \in \Phi_{2r,2r}$ such that*

$$\Omega_r(f, t_1, t_2)_{p,w} \approx \psi(t_1, t_2) \tag{2}$$

*holds for all $t_1, t_2 \in (0, \infty) \times (0, \infty)$ with equivalence constants depending only on $r$ and $[w]_{A_p}$.*

(b) *If $\psi \in \Phi_{2r,2r}$ then there exist $f_0 \in L^p_\omega(\mathbb{T}^2)$ and the positive real numbers $t_0, t_3$ such that*

$$\Omega_r(f_0, \delta_1, \delta_2)_{p,w} \approx \psi(\delta_1, \delta_2) \tag{3}$$

*holds for all* $\delta_1, \delta_2 \in (0, t_0) \times (0, t_3)$ *with equivalence constants depending only on* $r$ *and* $[w]_{A_p}$.

For functions in $L^p_\omega(\mathbb{T})$, $p \in (1,\infty)$, $\omega \in A_p(\mathbb{T})$ Theorem 1 was obtained by the author in [1]. In this work we simplify the (long) proof given in [1].

This type characterization theorem was proved in [26] (one dimensional case) for the spaces $L^p(\mathbb{T})$, $p \in [1,\infty)$, with classical moduli of smoothness of fractional order. The class $\Phi_\varrho$ describes completely the class of all majorants for the moduli of smoothness $\omega_r(\cdot,\delta)_p$ in the space $L^p(\mathbb{T})$, $p \in [1,\infty)$. For $\omega_r(\cdot,\delta)_p$, $r \in \mathbb{N}$ the characterization problem was investigated by O. V. Besov, S. B. Stechkin [4], V. I. Kolyada [12]; for $\omega_r(\cdot,\delta)_p$, $r > 0$ the characterization theorem was obtained by S. Tikhonov [26].

## 2. PRELIMINARIES

Let $L^1(\mathbb{T}^2)$ be the collection of Lebesgue integrable functions $f(x_1,x_2) : \mathbb{T}^2 \to \mathbb{R}$ such that $f(x_1,x_2)$ is $2\pi$-periodic with respect to each variable $x_1, x_2$. Let $1 < p < \infty$, $\omega(x_1,x_2) \in A_p(\mathbb{T}^2,\mathbb{J})$, and let $L^p_\omega(\mathbb{T}^2)$ be the collection of Lebesgue integrable functions $f(x_1,x_2) : \mathbb{T}^2 \to \mathbb{R}$ such that $f(x_1,x_2)$ is $2\pi$-periodic with respect to each variable $x_1, x_2$ and

$$\|f\|_{p,\omega} := \left( \iint\limits_{\mathbb{T}^2} |f(x_1,x_2)|^p \,\omega(x_1,x_2)\, dx_1 dx_2 \right)^{1/p} < \infty.$$

When $\omega(x_1,x_2) \equiv 1$ we denote $\|f\|_{p,1} =: \|f\|_p$ and $L^p_1(\mathbb{T}^2) =: L^p(\mathbb{T}^2)$ for $1 \le p < \infty$; $L^\infty_1(\mathbb{T}^2) =: L^\infty(\mathbb{T}^2)$.

We define Steklov type averages by

$$\sigma_{h_1,h_2} f(x_1,x_2) = \tfrac{1}{4h_1h_2} \int_{x_1-h_1}^{x_1+h_1} \int_{x_2-h_2}^{x_2+h_2} f(t,\tau)\, dt d\tau.$$
$$\sigma_{h_1,\circ} f(x_1,x_2) = \tfrac{1}{2h} \int_{x_1-h_1}^{x_1+h_1} f(t,\tau)\, dt,$$
$$\sigma_{\circ,h_2} f(x_1,x_2) = \tfrac{1}{2k} \int_{x_2-h_2}^{x_2+h_2} f(t,\tau)\, d\tau.$$

Lemma 2. [8, Theorem 3.3],[2] *If* $1 < p < \infty$, $\omega \in A_p(\mathbb{T}^2,\mathbb{J})$, $f \in L^p_\omega(\mathbb{T}^2)$, *then*

$$\left\{ \|\sigma_{h_1,h_2} f\|_{p,\omega}, \|\sigma_{h_1,\circ} f\|_{p,\omega}, \|\sigma_{\circ,h_2} f\|_{p,\omega} \right\} \lesssim \|f\|_{p,\omega}, \tag{4}$$

*uniformly in* $h_1, h_2$, *where the constants depend only on* $[\omega]_{A_p}$ *and* $p$.

For $1<p<\infty$, $\omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, $f\in L^p_\omega\left(\mathbb{T}^2\right)$, $h_1,h_2$, $r\in\mathbb{N}$, we define the mixed differences by

$$\begin{aligned}\bigtriangledown_{h_1,\circ}^{r,\circ}f\left(x_1,x_2\right) &= \left(\mathbb{I}-\sigma_{h_1,\circ}\right)^r f\left(x_1,x_2\right),\\ \bigtriangledown_{\circ,h_2}^{\circ,r}f\left(x_1,x_2\right) &= \left(\mathbb{I}-\sigma_{\circ,h_2}\right)^r f\left(x_1,x_2\right),\\ \bigtriangledown_{h_1,h_2}^{r,r}f\left(x_1,x_2\right) &= \bigtriangledown_{h_1,\circ}^{r,\circ}\left(\bigtriangledown_{\circ,h_2}^{\circ,r}f\right)\left(x_1,x_2\right),\end{aligned}$$

where $\mathbb{I}$ is identity operator on $\mathbb{T}^2$. Using the inequalities (4) we get

$$\left\{\left\|\bigtriangledown_{h,\circ}^{r,\circ}f\right\|_{p,\omega},\left\|\bigtriangledown_{\circ,k}^{\circ,r}f\right\|_{p,\omega},\left\|\bigtriangledown_{h,k}^{r,r}f\right\|_{p,\omega}\right\}\lesssim\|f\|_{p,\omega}, \tag{5}$$

for $1<p<\infty$, $\omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, $f\in L^p_\omega\left(\mathbb{T}^2\right)$, $r\in\mathbb{N}$, with constants depending only on $[\omega]_{A_p}$ and $p,r$.

The mixed modulus of smoothness of $f\in L^p_\omega\left(\mathbb{T}^2\right)$, $1<p<\infty$, $\omega\left(x,y\right)\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, $r\in\{0\}\cup\mathbb{N}$, can be defined as

$$\Omega_r\left(f,\delta_1,\delta_2\right)_{p,\omega}=\begin{cases}\sup\limits_{\substack{0\leq h_1\leq\delta_1\\0\leq h_2\leq\delta_2}}\left\|\bigtriangledown_{h_1,h_2}^{r,r}f\right\|_{p,\omega} & ,\ r\in\mathbb{N},\\ \|f\|_{p,\omega} & ,\ r=0.\end{cases} \tag{6}$$

If $1<p<\infty$, $\omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, $f\in L^p_\omega\left(\mathbb{T}^2\right)$, $r\in\mathbb{N}$, then from (6) and (5) $\Omega_r\left(f,\delta_1,\delta_2\right)_{p,\omega}\lesssim\|f\|_{p,\omega}$ with constant depending only on $[\omega]_{A_p}$ and $p,r$.

Note that from the definition of $\Omega_r\left(f,\cdot,\cdot\right)_{p,\omega}$, it has the following properties when $1<p<\infty$, $\omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, $f\in L^p_\omega\left(\mathbb{T}^2\right)$, $r\in\mathbb{N}$:

**(1)** $\Omega_r\left(f,0,0\right)_{p,\omega}=0$.
**(2)** $\Omega_r\left(f,\delta_1,\delta_2\right)_{p,\omega}$ is subadditive with respect to $f$.
**(3)** $\Omega_r\left(f,\delta_1,\delta_2\right)_{p,\omega}\leq\Omega_r\left(f,t_1,t_2\right)_{p,\omega}$ for $0\leq\delta_i\leq t_i;\quad i=1,2$.

When $\omega\left(x_1,x_2\right)\equiv1$ we donote $\Omega_r\left(f,\delta_1,\delta_2\right)_{p,1}=:\Omega_r\left(f,\delta_1,\delta_2\right)_p$ for $1\leq p<\infty$; $\Omega_r\left(f,\delta_1,\delta_2\right)_{\infty,1}=:\Omega_r\left(f,\delta_1,\delta_2\right)_\infty$.

Let $1<p<\infty$, $\omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, and $f\in L^p_\omega\left(\mathbb{T}^2\right)$, then there is $\lambda\in(1,\infty)$ such that $f\in L^\lambda\left(\mathbb{T}^2\right)$, namely, we have $L^p_\omega\left(\mathbb{T}^2\right)\subset L^\lambda\left(\mathbb{T}^2\right)$ and this gives possibility to define the corresponding Fourier series of $f$.

Lemma 3. [2] *If $1<p<\infty$, $\omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, and $f\in L^p_\omega\left(\mathbb{T}^2\right)$, then we have*

$$L^\infty\left(\mathbb{T}^2\right)\subset L^p_\omega\left(\mathbb{T}^2\right)\subset L^\lambda\left(\mathbb{T}^2\right) \tag{7}$$

*for some $\lambda>1$.*

We define $\mathcal{T}_{m,n}$ as the set of all trigonometric polynomials of degree at most $m$ with respect to variable $x_1$ and of degree at most $n$ with respect to variable $x_2$. Then

$$Y_{m_1,m_2}(f)_{p,\omega}=\inf\left\{\left\|f-\sum_{i=1}^{2}T_i\right\|_{p,\omega}:T_i\in\mathcal{T}_{m_i}\right\},$$

where $\mathcal{T}_{m_i}$ is the set of all two dimensional trigonometric polynomials of degree at most $m_i$ with respect to variable $x_i$ $(i = 1, 2)$

Let $1 < p < \infty$, $\omega \in A_p(\mathbb{T}^2, \mathbb{J})$ and $\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} A_{n_1,n_2}(x_1, x_2)$ be the corresponding Fourier series for $f \in L^p_\omega(\mathbb{T}^2)$. We define the partial sums of Fourier series of $f \in L^p_\omega(\mathbb{T}^2)$, $1 < p < \infty$, $\omega \in A_p(\mathbb{T}^2, \mathbb{J})$ as

$$\begin{aligned} S_{m,\circ}(f)(x_1, x_2) &= \sum_{n_1=0}^{m}\sum_{n_2=0}^{\infty} A_{n_1,n_2}(x_1, x_2, f), \\ S_{\circ,n}(f)(x_1, x_2) &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n} A_{n_1,n_2}(x_1, x_2, f), \\ S_{m_1,m_2}(f)(x_1, x_2) &= \sum_{n_1=0}^{m_1}\sum_{n_2=0}^{m_2} A_{n_1,n_2}(x_1, x_2, f). \end{aligned}$$

Define the partial de la Valleè Poussin means of $f$ as

$$V_{m,\circ}(f)(x_1, x_2) = \tfrac{1}{m+1}\sum_{k=m}^{2m-1} S_{k,\circ}(f), \tag{8}$$

$$V_{\circ,n}(f)(x_1, x_2) = \tfrac{1}{n+1}\sum_{l=n}^{2n-1} S_{\circ,l}(f), \tag{9}$$

$$V_{m_1,m_2}(f)(x_1, x_2) = \tfrac{1}{(n+1)(m+1)}\sum_{k=m_1}^{2m_1-1}\sum_{l=m_2}^{2m_2-1} S_{k,l}(f). \tag{10}$$

Lemma 4. [2] *If* $1 < p < \infty$, $\omega \in A_p(\mathbb{T}^2, \mathbb{J})$, $f \in L^p_\omega(\mathbb{T}^2)$, *then*

$$\begin{aligned} \left\{\|S_{m,\circ}(f)\|_{p,\omega}, \|S_{\circ,n}(f)\|_{p,\omega}, \|S_{m_1,m_2}(f)\|_{p,\omega}\right\} &\lesssim \|f\|_{p,\omega}, \\ \left\{\|V_{m,\circ}(f)\|_{p,\omega}, \|V_{\circ,n}(f)\|_{p,\omega}, \|V_{m_1,m_2}(f)\|_{p,\omega}\right\} &\lesssim \|f\|_{p,\omega}, \\ \|f - W_{m_1,m_2}f\|_{p,\omega} &\lesssim Y_{m_1,m_2}(f)_{p,\omega} \end{aligned}$$

*where* $W_{m_1,m_2}f(x_1, x_2) := (V_{m_1,\circ}(f) + V_{\circ,m_2}(f) - V_{m_1,m_2}(f))(x_1, x_2)$ *with all constants depending only on* $[\omega]_{A_p}$ *and* $p$.

By Theorem 6 of [13]

$$\left\|f - C^\alpha_{m_1,m_2}f\right\|_{p,\omega} \to 0 \tag{11}$$

as $m_1, m_2 \to \infty$ where $C^\alpha_{m_1,m_2}f$ is $\alpha$th Cesàro mean of $f$. From this we can deduce that $C(\mathbb{T}^2)$ is a dense subset of $L^p_\omega(\mathbb{T}^2)$ for $1 < p < \infty$, $\omega \in A_p(\mathbb{T}^2, \mathbb{J})$. Then $Y_{m_1,m_2}(f)_{p,\omega} \lesssim \left\|f - C^\alpha_{m_1,m_2}f\right\|_{p,\omega} \to 0$ and $Y_{m_1,m_2}(f)_{p,\omega} \to 0$ as $m_1, m_2 \to \infty$.

Let $W^{r,s}_{p,\omega}$, $r,s\in\mathbb{N}$, (respectively $W^{r,\circ}_{p,\omega}$; $W^{\circ,s}_{p,\omega}$) denote the collection of functions $f\in L^1\left(\mathbb{T}^d\right)$ such that $f^{(r,s)}\in L^p_\omega\left(\mathbb{T}^d\right)$ (respectively $f^{(r,\circ)}\in L^p_\omega\left(\mathbb{T}^d\right)$; $f^{(\circ,s)}\in L^p_\omega\left(\mathbb{T}^d\right)$).

The following inequalities can be obtained by the method given in [2]. For $1<p<\infty$, $\omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, $r\in\mathbb{N}$, there exist constants depending only on $[\omega]_{A_p}$ and $p,r$ so that

(i) (Jackson inequalities of Favard type)

$$Y_{m_1,m_2}\left(g_1\right)_{p,\omega}\lesssim\frac{1}{(m_1+1)^{2r}}\left\|g_1^{(2r,\circ)}\right\|_{p,\omega},\quad g_1\in W^{2r,\circ}_{p,\omega},\tag{12}$$

$$Y_{m_1,m_2}\left(g_2\right)_{p,\omega}\lesssim\frac{1}{(m_2+1)^{2r}}\left\|g_2^{(\circ,2r)}\right\|_{p,\omega},\quad g_2\in W^{\circ,2r}_{p,\omega},\tag{13}$$

$$Y_{m_1,m_2}\left(g\right)_{p,\omega}\lesssim\frac{1}{(m_1+1)^{2r}(m_2+1)^{2r}}\left\|g^{(2r,2r)}\right\|_{p,\omega},\quad g\in W^{2r,2r}_{p,\omega}.\tag{14}$$

(ii) if $\delta_1,\delta_2>0$ then

$$\Omega_r\left(g_1,\delta,\cdot\right)_{p,\omega}\lesssim\delta^2\Omega_{r-1}\left(g_1^{(2,\circ)},\delta,\cdot\right)_{p,\omega},\quad g_1\in W^{2,\circ}_{p,\omega},\tag{15}$$

$$\Omega_r\left(g_2,\cdot,\xi\right)_{p,\omega}\lesssim\xi^2\Omega_{r-1}\left(g_2^{(\circ,2)},\cdot,\xi\right)_{p,\omega},\quad g_2\in W^{\circ,2}_{p,\omega},\tag{16}$$

$$\Omega_r\left(g,\delta_1,\delta_2\right)_{p,\omega}\lesssim\delta_1^2\delta_2^2\Omega_{r-1}\left(g^{(2,2)},\delta_1,\delta_2\right)_{p,\omega},\quad g\in W^{2,2}_{p,\omega}.\tag{17}$$

and hence

$$\begin{aligned}\Omega_r\left(f,\delta,\cdot\right)_{p,\omega}&\lesssim\delta^{2r}\left\|f^{(2r,\circ)}\right\|_{p,\omega},\\ \Omega_r\left(f,\cdot,\xi\right)_{p,\omega}&\lesssim\xi^{2r}\left\|f^{(\circ,2r)}\right\|_{p,\omega},\\ \Omega_r\left(f,\delta_1,\delta_2\right)_{p,\omega}&\lesssim\delta_1^{2r}\delta_2^{2r}\left\|f^{(2r,2r)}\right\|_{p,\omega}.\end{aligned}$$

Definition 5. *The mixed $K$-functional is defined as*

$$K(f,\delta_1,\delta_2,p,\omega,r,s):=$$
$$:=\inf_{g_1,g_2,g}\left\{\|f-g_1-g_2-g\|_{p,\omega}+\delta_1^r\left\|\frac{\partial^r g_1}{\partial x^r}\right\|_{p,\omega}+\delta_2^s\left\|\frac{\partial^s g_2}{\partial y^s}\right\|_{p,\omega}+\delta_1^r\delta_2^s\left\|\frac{\partial^{r+s}g}{\partial x^r\partial y^s}\right\|_{p,\omega}\right\}$$

*where the infimum is taken for all $g_1$, $g_2$, $g$ so that $g_1\in W^{r,\circ}_{p,\omega}$, $g_2\in W^{\circ,s}_{p,\omega}$, $g\in W^{r,s}_{p,\omega}$ where $r,s\in\mathbb{N}$, $1<p<\infty$, $\omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, $f\in L^p_\omega\left(\mathbb{T}^2\right)$.*

(iii) If $1<p<\infty$, $\omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, $f\in L^p_\omega\left(\mathbb{T}^2\right)$ and $r\in\mathbb{N}$, then there exist constants depending only on Muckenhoupt's constant $[\omega]_{A_p}$ of $\omega$ and $p,r$ so that the equivalence

$$\Omega_r\left(f,\delta_1,\delta_2\right)_{p,\omega}\approx K(f,\delta_1,\delta_2,p,\omega,2r)\tag{18}$$

and the properties

$$\Omega_r\left(f,\lambda\delta_1,\eta\delta_2\right)_{p,\omega}\lesssim\left(1+\lfloor\lambda\rfloor\right)^{2r}\left(1+\lfloor\eta\rfloor\right)^{2r}\Omega_r\left(f,\delta_1,\delta_2\right)_{p,\omega},$$
$$\frac{\Omega_r\left(f,\delta_1,\delta_2\right)_{p,\omega}}{\delta_1^{2r}\delta_2^{2r}}\lesssim\frac{\Omega_r\left(f,t_1,t_2\right)_{p,\omega}}{t_1^{2r}t_2^{2r}},\qquad 0<t_i\le\delta_i;\quad i=1,2,$$

hold for $\delta_1,\delta_2>0$, where $\lfloor x\rfloor:=\max\{z\in\mathbb{Z}:z\le x\}$.

(iv) (11) can be refined by the inequality (19) below [2]. If $1<p<\infty$, $\omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, $f\in L_\omega^p\left(\mathbb{T}^2\right)$ and $r\in\mathbb{N}$, then there exists $C_{[\omega]_{A_p},p,r}$ depending only on Muckenhoupt's constant $[\omega]_{A_p}$ of $\omega$ and $p,r$ so that

$$Y_{m_1,m_2}\left(f\right)_{p,\omega}\le C_{[\omega]_{A_p},p,r}\Omega_r\left(f,\tfrac{1}{m_1},\tfrac{1}{m_2}\right)_{p,\omega} \tag{19}$$

where $m_1,m_2\in\mathbb{N}$.

## 3. PROOF OF THEOREM 1

Let $\omega_r\left(\cdot,\delta_1,\delta_2\right)_p$, $1\le p\le\infty$, be the usual nonweighted mixed modulus of smoothness:

$$\omega_r\left(g,\delta_1,\delta_2\right)_p:=\sup_{0\le h_1\le\delta_1,0\le h_2\le\delta_2}\left\|\left(\mathbb{I}-T_{h_1,\circ}\right)^r\left(\mathbb{I}-T_{\circ,h_2}\right)^r g\right\|_p,\quad g\in L^p\left(\mathbb{T}^2\right),$$

where $T_{h_1,\circ}g\left(x_1,x_2\right):=g(x_1+h_1,x_2)$; $T_{\circ,h_2}g\left(x_1,x_2\right):=g(x_1,x_2+h_2)$. From [25] $(1\le p<\infty)$ and [6] $(p=\infty)$ and (18) there exist positive constants, depending only $r,p$, such that

$$\omega_{2r}\left(g,\delta_1,\delta_2\right)_p\approx\Omega_r\left(g,\delta_1,\delta_2\right)_p \tag{20}$$

holds for $1\le p\le\infty$ and $g\in L^p\left(\mathbb{T}^2\right)$.

Theorem 2.5 of [26] give that: Let $r\in\mathbb{N}$, $p\in[1,\infty]$.

(a) If $f\in L^p\left(\mathbb{T}^2\right)$, then there exists $\psi\in\Phi_{r,r}$ such that

$$\omega_r\left(f,t_1,t_2\right)_p\approx\psi\left(t_1,t_2\right) \tag{21}$$

holds for all $t_1,t_2\in(0,\infty)\times(0,\infty)$ with equivalence constants depending only on $r$.

(b) If $\psi\in\Phi_{r,r}$ then there exist $f_0\in L^p\left(\mathbb{T}^2\right)$ and the positive real numbers $t_0,t_3$ such that

$$\omega_r\left(f_0,\delta_1,\delta_2\right)_p\approx\psi\left(\delta_1,\delta_2\right) \tag{22}$$

holds for all $\delta_1,\delta_2\in(0,t_0)\times(0,t_3)$ with equivalence constants depending only on $r$.

*Proof of Theorem 1.* (i) Note that if $F\in C\left(\mathbb{T}^2\right)$ then from (7)

$$\left\|\triangledown_{h_1,h_2}^{r,r}F\right\|_{p,w}\le C_{p,[w]_{A_p}}\left\|\triangledown_{h_1,h_2}^{r,r}F\right\|_{C(\mathbb{T}^2)}. \tag{23}$$

Using Theorem 2.5 (A) of [26], (7), (20), (18), (23) there exists $\psi \in \Phi_{2r}$ such that

$$\begin{aligned}\Omega_r(F,\delta_1,\delta_2)_{p,w} &\le C_{p,[w]_{A_p}}\Omega_r(F,\delta_1,\delta_2)_\infty \le C_{p,[w]_{A_p}}\omega_{2r}(F,\delta_1,\delta_2)_\infty \\ &\le C_{r,p,[w]_{A_p}}\psi(\delta_1,\delta_2).\end{aligned}$$

If $p \in (1,\infty)$, $A_p(\mathbb{T}^2,\mathbb{J})$, $f \in L^p_\omega(\mathbb{T}^2)$, then, by (11), for any $\varepsilon > 0$ there exists $F \in C(\mathbb{T}^2)$ such that $\|f-F\|_{p,w} < \varepsilon$. Thus

$$\begin{aligned}\Omega_r(f,\delta_1,\delta_2)_{p,w} &\le \Omega_r(f-F,\delta_1,\delta_2)_{p,w} + \Omega_r(F,\delta_1,\delta_2)_{p,w} \\ &\le C_{r,p,[w]_{A_p}}\|f-F\|_{p,w} + C_{r,p,[w]_{A_p}}\psi(\delta_1,\delta_2).\end{aligned}$$

Letting $\varepsilon \to 0^+$ we get

$$\Omega_r(f,\delta_1,\delta_2)_{p,w} \le C_{r,p,[w]_{A_p}}\psi(\delta_1,\delta_2).$$

On the other hand, from (18), (20), (7) and Theorem 2.5 (A) of [26]

$$\psi(\delta_1,\delta_2) \le C_{r,p,[w]_{A_p}}\omega_{2r}(f,\delta_1,\delta_2)_1 \le C_{r,p,[w]_{A_p}}\Omega_r(f,\delta_1,\delta_2)_{p,w}$$

and the equivalence (2) is established.

(ii) For the equivalence (3) let $\psi \in \Phi_{2r}$. By Theorem 2.5 (B) and Remark 2.7 (1) of [26] there exist $f \in L^\infty(\mathbb{T}^2)$ and the positive real numbers $t_0, t_3$ such that

$$\omega_{2r}(f,\delta_1,\delta_2)_p \approx \psi(\delta_1,\delta_2), \quad p = 1,\infty$$

holds for all $\delta_1,\delta_2 \in (0,t_0)\times(0,t_3)$ with equivalence constants depending only on $r$. Then by (18), (20) we get

$$\begin{aligned}\psi(\delta_1,\delta_2) &\le C_r\omega_{2r}(f,\delta_1,\delta_2)_1 \le C_r\Omega_r(f,\delta_1,\delta_2)_1 \le C_{r,p,[w]_{A_p}}\Omega_r(f,\delta_1,\delta_2)_{p,w} \\ &\le C_{r,p,[w]_{A_p}}\Omega_r(f,\delta_1,\delta_2)_\infty \le C_{r,p,[w]_{A_p}}\omega_{2r}(f,\delta_1,\delta_2)_\infty \\ &\le C_{r,p,[w]_{A_p}}\psi(\delta_1,\delta_2)\end{aligned}$$

for all $\delta_1,\delta_2 \in (0,t_0)\times(0,t_3)$. □

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