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THE SECOND ZOLOTAREV CASE IN THE ERDÖS-SZEGÖ SOLUTION TO A MARKOV-TYPE EXTREMAL PROBLEM OF SCHUR

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Abstract. Schur's [21] Markov-type extremal problem is to determine (i) $M_n = \sup_{-1\leq\xi\leq 1}\sup_{P_n\in\mathbf{B}_{n,\xi,2}}(|P_n^{(1)}(\xi)|/n^2)$, where $\mathbf{B}_{n,\xi,2} = \{P_n \in \mathbf{B}_n : P_n^{(2)}(\xi) = 0\} \subset \mathbf{B}_n = \{P_n : |P_n(x)| \leq 1 \text{ for } |x| \leq 1\}$ and P_n is an algebraic polynomial of degree $\leq n$. Erdös and Szegö [4] found that for $n \geq 4$ this maximum is attained if $\xi = \pm 1$ and $P_n \in \mathbf{B}_{n,\pm 1,2}$ is a (unspecified) member of the one-parameter family of hard-core Zolotarev polynomials. An extremal such polynomial as well as the constant M_n we have explicitly specified for $n = 4$ in [18], and in this paper we strive to obtain an analogous amendment to the Erdös - Szegö solution for $n = 5$. The cases $n > 5$ still remain arcane. Our approach is based on the quite recently discovered explicit algebraic power form representation [6], [7] of the quintic hard-core Zolotarev polynomial, $Z_{5,t}$, to which we add here explicit descriptions of its critical points, the explicit form of Pell's (aka: Abel's) equation, as well as an alternative proof for the range of the parameter, t. The optimal $t = t^*$ which yields $M_5 = |Z_{5,t^*}^{(1)}(1)|/25$ we identify as the negative zero with smallest modulus of a minimal P_{10}. We then turn to an extension of (i), to higher derivatives as proposed by Shadrin [23], and we provide an analogous solution for $n = 5$. Finally, we describe, again for $n = 5$, two new algebraic approaches towards a solution to Zolotarev's so-called first problem [2], [25] which was originally solved by means of elliptic functions.

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1. INTRODUCTION

The famous A.A. Markov inequality of 1889 [12] asks for an estimate on the size of the first derivative $|P_n^{(1)}(x)|$ of an algebraic polynomial P_n of degree $\leq n$ when x varies in the unit interval $\mathbf{I} := [-1, 1]$ and P_n varies in the unit ball $\mathbf{B}_n := \{P_n : |P_n(x)| \leq 1 \text{ for } x \in \mathbf{I}\}$, see also G.V. Milovanović *et al.* [15, p. 529], Th.J. Rivlin [19, p. 123]. Markov showed that $|P_n^{(1)}(x)|/n^2 \leq 1$ holds

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and equality will be attained only for $x = \pm 1$ and for $P_n = \pm T_n$ (Chebyshev polynomial of the first kind relative to $\mathbf{I}$), defined by

$$(1) \qquad T_n(x) := 2xT_{n-1}(x) - T_{n-2}(x), \quad \text{with } T_1(x) = x, \quad T_0(x) = 1.$$

In 1919 I. Schur [21, § 2], inspired by Markov's problem, was led to the problem of finding the maximum of $|P_n^{(1)}(\xi)|/n^2$ under the additional restriction $P_n^{(2)}(\xi) = 0$ where $\xi \in \mathbf{I}$ is a given number.

In 1942 P. Erdös and G. Szegö addressed Schur's problem and they showed in [4, Th. 2] that under the said restriction the value $|P_n^{(1)}(\xi)|/n^2, n \geq 4$, will attain the maximum, M_n, only if $\xi = \pm 1$ and $P_n \in \mathbf{B}_n$ coincides with an n-th degree proper *Zolotarev polynomial* relative to $\mathbf{I}$. Such a polynomial is, for each n, a member of some one-parameter family of polynomials and an extremal one among that family was therefore coined *Schur polynomial* by F. Peherstorfer and K. Schiefermayr [16, Section 5d], see Section 3 for details.

Although both solutions to the stated problems have in common that the maximum is attained at the endpoints ± 1 of $\mathbf{I}$, they differ greatly when it comes to exhibit an explicit extremal polynomial from $\mathbf{B}_n$: The algebraic power form representation of an extremizer $\pm T_n$ in Markov' problem is explicitly known [19]. On the contrary, a (parameterized) algebraic power form representation of a proper Zolotarev polynomial is not known for a general n, nor is the optimal parameter known which singles out the Schur polynomial. Rather, proper Zolotarev polynomials are usually expressed by means of elliptic functions (see N. I. Achieser [1, p. 280], B. C. Carlson and J. Todd [2], [15, p. 407], E. I. Zolotarev [25]), a presentation form which is considered as *very complicated* [2].

The purpose of this note is threefold: To describe, for $n = 5$, in more detail the Erdös-Szegö solution [4] to Schur's problem, the A. Shadrin solution [23] to the generalized Schur's problem, and to provide new algebraic solutions to Zolotarev's so-called *first problem* [2], [25]. To this end, we take advantage of a quite recently published explicit (parameterized) power form representation of the proper quintic Zolotarev polynomial [6], [7].

In Section 2 we introduce the (parameterized) proper Zolotarev polynomial and in particular the said novel power form representation for the degree $n = 5$ due to G. Grasegger and N. Th. Vo. As a supplement to their result we provide explicit formulas for its critical points (to be defined below), the explicit form of Pell's (aka: Abel's) equation and an alternative proof for the range of its parameter.

In Section 3 we turn to the Erdös-Szegö solution [4] of Schur's problem for $n = 5$. The optimal parameter of the Schur polynomial we describe as a zero of a dedicated P_{10}. From this we deduce numerical approximations for the coefficients of the Schur polynomial as well as for the sought-for constant M_5.

In Section 4 we consider a generalization of Schur's problem due to Shadrin [23]. It is based on V. A. Markov's inequality of 1892 [13] for the higher derivatives of P_n. Again we will exemplify the degree $n = 5$, now making use of Proposition 4.4 in [23].

In Section 5 we describe, taking recourse to results from Section 2, two new algebraic approaches to Zolotarev's first problem of 1877 [2], [25] (for $n = 5$) which asks to determine a P_n, with prescribed values for its first and second leading coefficient, that deviates least (in the uniform norm) from the zero function in $\mathbf{I}$.

The explicit quartic Schur polynomial (first Zolotarev case, see [18]) and the here introduced approximate quintic Schur polynomial (second Zolotarev case), may well serve as illustrating instances of the result in [4, Th. 2], which is referred to in S. R. Finch's book [5, Section 3.9].

2. THE QUINTIC HARD-CORE ZOLOTAREV POLYNOMIAL

We distinguish between the *hard-core* or *proper* Zolotarev polynomial introduced in this Section, and the *improper* Zolotarev polynomial which will be introduced in Section 4, see also [1], [2], [4], [15], [23], [25].

According to [4, p. 453], [23, p. 1190], a proper Zolotarev polynomial belongs to $\mathbf{B}_n$, is of exact degree n, and equioscillates n times in $\mathbf{I}$. The *equioscillation points* $-1 \leq z_0 < z_1 < z_2 < ... < z_{n-2} < z_{n-1} \leq 1$, at which the values ± 1 are attained alternately, include both endpoints of $\mathbf{I}$, that is, $-1 = z_0$ and $z_{n-1} = 1$. To be compliant with [4], we assume that at the endpoint $-1 = z_0$ the value $(-1)^{n-1}$ is attained. Thus a proper Zolotarev polynomial has $n - 1$ roots in the interior of $\mathbf{I}$, and it is furthermore required that it has one additional root outside of $\mathbf{I}$, and we assume, again following [4], that this root is to the right of $\mathbf{I}$. According to the quoted references above, it is more specifically required that there exist three points A_n, B_n and C_n with $1 < A_n < B_n < C_n$, which we call *Zolotarev points*, having the property that the proper Zolotarev polynomial of degree n attains the value 1 at $x = B_n$ and the value -1 at $x = C_n$ (so that its n-th root is sandwiched between B_n and C_n) and that its first derivative vanishes at $x = A_n$.

But all these stated requirements do not uniquely determine a polynomial of degree n; rather, there are infinitely many polynomials which fulfill these conditions. Therefore we will denote a proper Zolotarev polynomial of degree n by $Z_{n,t}$. The additional parameter t indicates that the coefficients of $Z_{n,t}$ are not constant but vary with t, which in turn varies in some interval of $\mathbb{R}$ (which may be different for different n's). The equioscillation points of $Z_{n,t}$ in the interior of $\mathbf{I}$ also depend on t, so that we will denote them more precisely by $z_1(t) < z_2(t) < ... < z_{n-2}(t)$, and the Zolotarev points $A_n < B_n < C_n$ we will likewise denote more precisely by $A_n(t) < B_n(t) < C_n(t)$. These $n + 1$ parameterized points on the x-axis which characterize $Z_{n,t}$ (together with the identities $Z_{n,t}(-1) = (-1)^{n-1}$ and $Z_{n,t}(1) = 1$) will be called the *critical points* of $Z_{n,t}$. Besides $Z_{n,t}$, the polynomials $-Z_{n,t}$ as well as $\pm Q_{n,t}$, where $Q_{n,t}(x) = Z_{n,t}(-x)$, are also considered as proper Zolotarev polynomials.

When trying to represent $Z_{n,t}$ in the usual (parameterized) algebraic power form as a linear combination of monomials,

$$Z_{n,t}(x) = \sum_{i=0}^{n} a_i(t)x^i, \tag{2}$$

one encounters severe difficulties. According to [11, p. 932], *A. A. Markov himself tried to find an algebraic solution, but he was not fully successful, because an algebraic solution requires an amazing amount of calculations.* To the best of our knowledge, the current situation concerning the algebraic power form representation of hardcore Zolotarev polynomials relative to **I** can be delineated as follows:

$\mathbf{n = 2}$: An algebraic representation is readily found, *e.g.*, $Z_{2,t}(x) = \frac{1+2tx-x^2}{2t}$, $t > 1$, see also [2, pp. 2]. But it is *unexpectedly complicated* to derive it from the elliptic representation, see [2, pp. 11].

$\mathbf{n = 3}$: The task to determine $Z_{3,t}$ is posed as a problem in [19, p. 94]. A solution has been provided by several authors and is given by $Z_{3,t}(x) = \frac{-1+4t^2+t^4-4tx-2(-1+3t^2)x^2+4tx^3}{(-1+t^2)^2}$, where $-\frac{1}{3} < t < 0$; see also [2, pp. 4].

$\mathbf{n = 4}$: Algebraic representations have been provided by the present author [17, p. 357] and by Shadrin [22, p. 242], and can be traced back to a result of V. A. Markov [13, p. 73], which is not contained in the abridged German translation [14] of [13], see [18] for details.

According to [23, p. 1185], *there is no explicit expression for* (proper) *Zolotarev polynomials of degree* $n > 4$. But only quite recently it was claimed by Grasegger and Vo [6] that such an expression has been obtained for $5 \leq n \leq 6$ by making use of symbolic computation (they also treat $n \leq 4$). We focus here on the representation for the degree $n = 5$ in [6] and may leave aside the degree $n = 6$, see Remark 9 below. In order to be compliant with the assumptions made about $Z_{n,t}$, we transform the term $y(x) \equiv y_{5,t}(x)$ as given in [6, p. 12] to $Y_{5,t}(x) := -y(-x)$. In this way we get

$$Y_{5,t}(x) := \sum_{i=0}^{5} a_i(t)x^i, \tag{3}$$

where the parameterized coefficients $a_i(t)$ are defined as follows:

$$\begin{aligned}
a_0(t) &:= \kappa(1 + 10t + 17t^2 - 56t^3 - 174t^4 - 500t^5 - 966t^6 \\
&\quad +1128t^7 + 6221t^8 + 8122t^9 + 2581t^{10}) && (4)\\
a_1(t) &:= -8\kappa\sqrt{2}t^5 v_1^3(1 + 2t + 5t^2)(-1 - 2t + 11t^2) && (5)\\
a_2(t) &:= -4\kappa(-1 + 5t^2)(-2 - 17t - 21t^2 + 43t^3 + 83t^4 \\
&\quad +237t^5 + 625t^6 + 825t^7 + 275t^8) && (6)
\end{aligned}$$

$$(7) \qquad a_3(t) := 8\kappa\sqrt{2}t^5v_1^3(-3-10t+6t^2+40t^3+85t^4+10t^5)$$
$$(8) \qquad a_4(t) := 8\kappa(1+t)^3(1-5t^2)^2(1+5t-5t^2+15t^3)$$
$$(9) \qquad a_5(t) := -16\kappa\sqrt{2}(1+t)^4v_1(t-5t^3)^2,$$

with

$$(10) \qquad \kappa := \kappa(t) = \tfrac{1}{(-1+t)^6(1+3t)^4} \quad \text{and } v_1 := v_1(t) = \sqrt{\tfrac{(1+t)(-1+5t^2)}{t^3}}.$$

It is readily verified that for certain values of the parameter t the monomial representation $Y_{5,t}$ is not defined (*e.g.*, for $t = 1$), is complex-valued (*e.g.*, for $t = -0.5$), or does not belong to $\mathbf{B}_5$ (*e.g.*, for $t = 2$ (and $x = 0$)). In order that $Y_{5,t}$ should represent a quintic hard-core Zolotarev polynomial, it is therefore mandatory to appropriately restrict the range of the parameter t. This need we have communicated to one of the authors of [6], and in [7] Grasegger was able to provide the maximal range for the parameter t appearing in $y(x) \equiv y_{5,t}(x)$, respectively in $Y_{5,t}$. We concede that an explicit algebraic power form representation of the quintic hard-core Zolotarev polynomial constitutes a major breakthrough in the long history of these intricate polynomials.

Proposition 1. *(see* [6], [7]*). The (parameterized) algebraic power form of the quintic hard-core Zolotarev polynomial on* $\mathbf{I}, Z_{5,t}$*, coincides with that of* $Y_{5,t}$ *as given in* (3)-(10)*, provided the parameter* t *belongs to the open interval*

$$(11) \qquad J_5 := (t^\circ, 0), \ \textit{where } t^\circ := -\tan^2(\tfrac{\pi}{10}) = \tfrac{2}{\sqrt{5}} - 1 = -0.1055728090\ldots .$$

From now on we will identify $Z_{5,t}$ with $Y_{5,t}$ but will assume that the parameter t belongs to J_5. Below we will provide an alternative proof for the range (11) of t. The algebraic power form representation of $y(x) \equiv y_{5,t}(x)$ in [6], [7] does not include the determination of the critical points, a goal which we are now going to address for $Z_{5,t}$. To begin with, it is readily verified that there holds

$$(12) \qquad Z_{5,t}(-1) = Z_{5,t}(1) = 1.$$

We first turn to the determination of the three equioscillation points of $Z_{5,t}$ in the interior of $\mathbf{I}$. The equation $Z_{5,t}^{(1)}(x) = 0$ in the variable x, which is equivalent to

$$\begin{aligned}(13) \quad & -\sqrt{2}t^5(1+2t+5t^2)(-1-2t+11t^2)v_1^3 - (-1+5t^2)\times \\ & \times(-2-17t-21t^2+43t^3+83t^4+237t^5+625t^6+825t^7+275t^8)x + \\ & +3\sqrt{2}t^5v_1^3(-3-10t+6t^2+40t^3+85t^4+10t^5)x^2 + \\ & +4(1+t)^3(1-5t^2)^2(1+5t-5t^2+15t^3)x^3 - \\ & -10\sqrt{2}(1+t)^4(t-5t^3)^2v_1x^4 = 0,\end{aligned}$$

(with v_1 from (10)) we have solved with a symbolic mathematical computation program (*Mathematica*™, version 10, symbol *Solve*). It renders in particular the following three solutions x_1, x_2, x_3, as can be verified by inserting them backwards into the left hand side of (13):

$$x_1 := x_1(t) = \frac{v_2 - v_3 - \sqrt{v_4 + v_5}}{10\sqrt{2}}, \tag{14}$$

$$x_2 := x_2(t) = \frac{v_2 + v_3 - \sqrt{v_4 - v_5}}{10\sqrt{2}}, \tag{15}$$

$$x_3 := x_3(t) = \frac{v_2 - v_3 + \sqrt{v_4 + v_5}}{10\sqrt{2}}, \tag{16}$$

where

$$v_2 := v_2(t) = \frac{3-2t-5t^2}{t^3 v_1} + \frac{3(1+t^2)v_1}{(1+t)^2}, \tag{17}$$

with v_1 according to (10),

$$v_3 := v_3(t) = \sqrt{\frac{(1+5t^2(1+2t))^2}{t(1+t)^3(-1+5t^2)}}, \tag{18}$$

$$v_4 := v_4(t) = 175 - \frac{2}{t} - \frac{68}{(1+t)^3} + \frac{82}{(1+t)^2} - \frac{148}{1+t} + \frac{25(-1+5t)}{-1+5t^2}, \tag{19}$$

$$v_5 := v_5(t) = \frac{2t(1+5t(2+t))^2 v_1 v_3}{(1+t)^2(1+5t^2(1+2t))}. \tag{20}$$

Plugging these zeros of $Z^{(1)}_{5,t}$ into the initial function $Z_{5,t}$ gives $Z_{5,t}(x_1) = -1, Z_{5,t}(x_2) = 1$ and $Z_{5,t}(x_3) = -1$. Thus x_1, x_2, x_3 behave like the three ordered equioscillation points $z_i(t), i = 1, 2, 3$, of $Z_{5,t}$ in the interior of $\mathbf{I}$. And in fact they coincide with the $z_i(t)$'s since there is no other choice left: From the requirements which $Z_{5,t}$ must fulfill we know that we have

$$Z_{5,t}(x) = -1 \quad \text{only if } x \in \{z_1(t), z_3(t), C_5(t)\}, \tag{21}$$

$$Z_{5,t}(x) = 1 \quad \text{only if } x \in \{-1, z_2(t), 1, B_5(t)\}, \tag{22}$$

$$Z^{(1)}_{5,t}(x) = 0 \quad \text{only if } x \in \{z_1(t), z_2(t), z_3(t), A_5(t)\}. \tag{23}$$

Consider first x_1 with property $Z_{5,t}(x_1) = -1$ and $Z^{(1)}_{5,t}(x_1) = 0$. Consequently, in view of (21) and (23), $x_1 \in \{z_1(t), z_3(t), C_5(t)\} \cap \{z_1(t), z_2(t), z_3(t), A_5(t)\}$, which implies $x_1 \in \{z_1(t), z_3(t)\}$. Analogously we get $x_3 \in \{z_1(t), z_3(t)\}$ and obviously $x_1 \neq x_3$ holds as can be seen by evaluating $x_1 = x_1(t)$ and $x_3 = x_3(t)$ at $t = -0.1 \in J_5$, for example. Hence either $x_1 = z_1(t)$ and $x_3 = z_3(t)$ with $x_1 < x_3$ or $x_3 = z_1(t)$ and $x_1 = z_3(t)$ with $x_3 < x_1$. But the latter inequality cannot occur as can be seen by evaluating $x_1 = x_1(t)$ and $x_3 = x_3(t)$ at $t = -0.1$, for example. Hence we have $x_1 = z_1(t) < x_3 = z_3(t)$. Consider next $x_2 = x_2(t)$ with property $Z_{5,t}(x_2) = 1$ and $Z^{(1)}_{5,t}(x_2) = 0$. Consequently, in view of (22) and (23), $x_2 \in \{-1, z_2(t), 1, B_5(t)\} \cap \{z_1(t), z_2(t), z_3(t), A_5(t)\}$, which implies $x_2 = z_2(t)$. Hence we have as claimed

$$x_1 = z_1(t) < x_2 = z_2(t) < x_3 = z_3(t). \tag{24}$$

We now turn to the determination of the three Zolotarev points $A_5(t) < B_5(t) < C_5(t)$ of $Z_{5,t}$, where $1 < A_5(t)$. It is tempting to determine $B_5(t)$ and $C_5(t)$ as solutions of the polynomial equations $Z_{5,t}(x) \pm 1 = 0$ with the aid of a symbolic computation program. This approach, however, leads to complex-valued solutions. We therefore proceed as follows:

According to [16, Formula (5.11)], the numbers $B_5(t)$ and $C_5(t)$ satisfy a set of four equations of which the first two of these read (using the shorthand $z_i = z_i(t), i = 1, 2, 3$)

$$2(-z_1 + z_2 - z_3) + B_5(t) = C_5(t) \tag{25}$$

$$2 + 2(-z_1^2 + z_2^2 - z_3^2) + B_5(t)^2 = C_5(t)^2, \tag{26}$$

from which $B_5(t)$ and $C_5(t)$ can be recovered by substitution:

$$\begin{aligned} B_5(t) &= \tfrac{1}{2}\left(z_1 - z_2 + 3z_3 + \tfrac{-1+2z_1(z_1-z_2)}{z_1-z_2+z_3}\right) = \\ &= \tfrac{1}{10\sqrt{2}}(3v_2 - 2v_3 + 2\sqrt{v_4 - v_5}) \end{aligned} \tag{27}$$

and

$$\begin{aligned} C_5(t) &= -\tfrac{1+z_1^2+3z_2^2-4z_2z_3+z_3^2+4z_1(-z_2+z_3)}{2(z_1-z_2+z_3)} = \\ &= \tfrac{1}{10\sqrt{2}}\left(\tfrac{1+t(5+t(-5+4v_1v_3+t(15+4v_1v_3)))}{t^2(1+t)v_1}\right). \end{aligned} \tag{28}$$

The value of $A_5(t)$ we deduce from Formula (5.21) in [16] where $A_5(t)$ is expressed with the aid of $B_5(t)$ and $C_5(t)$:

$$\begin{aligned} A_5(t) &= \tfrac{1}{5}(2B_5(t) + 2C_5(t) - z_1 - z_2 - z_3) = \tfrac{-2-z_2^2+(z_1-z_3)^2}{5(z_1-z_2+z_3)} = \\ &= \tfrac{1}{10\sqrt{2}}(v_2 + v_3 + \sqrt{v_4 - v_5}). \end{aligned} \tag{29}$$

The employed terms v_1, v_2, v_3, v_4, v_5 above have been defined in (10), (17)-(20). Virtually for any parameter $t \in J_5$ one is now able to determine $Z_{5,t}$ (by calculating its six coefficients (4)-(9)) as well as its six critical points given in (14)-(16) (identifying there $x_i = z_i(t), i = 1, 2, 3$) and in (27)-(29). The knowledge of these points allows one to calculate $Z_{5,t}$ in alternative fashions, for example, by means of interpolation formulas since the values of $Z_{5,t}$ at those points (and at z_0, z_4) are known. A particular alternative form to represent $Z_{5,t}$ ($t \in J_5$) can be deduced from [20, Lemma 1]. It is a concrete implementation of the expression as given in [20] since we know the critical points which enter into this expression:

$$Z_{5,t}(x) = 1 - \frac{2(x^2-1)(x-B_5(t))(x-z_2(t))^2}{(d^2-1)(d-B_5(t))(d-z_2(t))^2}, \quad \text{with } d := C_5(t). \tag{30}$$

The knowledge of the Zolotarev points allows to provide a concrete implementation of the famous Pell's equation (aka: Abel's equation) for hard-core Zolotarev polynomials (see [20, p. 149], [24, p. 2486]), stated here for $n = 5$:

$$(Z_{5,t}(x))^2 - (x^2-1)(x-B_5(t))(x-C_5(t))\left(\frac{Z_{5,t}^{(1)}(x)}{5(x-A_5(t))}\right)^2 = 1. \tag{31}$$

Summarizing we get

Proposition 2. *At the four points* $-1 = z_0 < z_2(t) < z_4 = 1 < B_5(t)$ *the polynomial* $Z_{5,t}$ *attains the value* 1, *whereas at the three points* $z_1(t) < z_3(t) < C_5(t)$ *it attains the value* -1. *The first derivative of* $Z_{5,t}$ *with respect to* x *vanishes at the four points* $z_1(t) < z_2(t) < z_3(t) < A_5(t)$. *There holds* $1 < A_5(t) < B_5(t) < C_5(t)$, *so that* $|Z_{5,t}(x)| \le 1$ *only for* $x \in \mathbf{I} \cup [B_5(t), C_5(t)]$. *The six critical points of* $Z_{5,t}$ *are explicitly determined by* (14)-(16) *(identifying there* $x_i = z_i(t), i = 1, 2, 3$*) and by* (27)–(29). *Pell's (aka: Abel's) equation for the quintic hard-core Zolotarev polynomial is given in* (31), *with* $x \in \mathbf{I}, t \in J_5$.

An alternative proof, compared to the one given in [7], for the maximal range J_5 in (11) of the parameter t of $Z_{5,t}$ can now be had as follows: We let $A_5(t) \in (1, \infty)$ tend first towards 1 and then towards infinity and study the limiting behavior of t. Solving $A_5(t) = 1$ numerically, we obtain the solutions $t = -0.1055728090...$ and $t = 1$, of which the latter drops out because $Z_{5,1}$ is not defined. The former one, evaluated to high precision, is readily seen (*e.g.*, by applying *Mathematica*™'s *RootApproximant* - symbol) to be identical to the irrational number t°. And indeed $A_5(t^\circ) = 1$ holds as is verified by insertion. Thus, $t \to t^\circ$ from the right. Employing *Mathematica*™'s *Limit* - symbol would have produced the same finding. If $A_5(t) \to \infty$, then also $B_5(t) \to \infty$ as well as $C_5(t) \to \infty$ since $1 < A_5(t) < B_5(t) < C_5(t)$. To guess for which parameter t the expression $C_5(t)$ becomes infinite, we numerically solve the equation $C_5(t) - 10^n = 0$ for t and various large values of n and get, approximately, $t = -\frac{5}{4}10^{-2n-1}$. This indicates that for $t \to 0$ from the left the value $C_5(t)$ will tend to infinity. And this is indeed the case as can be seen from the power series expansion of $C_5(t)$ about the point $t = 0$:

$$C_5(t) = \frac{3i}{10\sqrt{2}\sqrt{t}} - \frac{19i\sqrt{t}}{20\sqrt{2}} + \frac{73it^{3/2}}{16\sqrt{2}} - \frac{167it^{5/2}}{32\sqrt{2}} + \frac{3869it^{7/2}}{256\sqrt{2}} + \mathcal{O}(t)^{9/2}. \tag{32}$$

Employing *Mathematica*™'s *Limit* - symbol would have produced the same finding. This completes our alternative proof for the maximal range J_5 of the parameter t of $Z_{5,t}$. We leave it to the reader to verify that when t is approaching the limits of J_5, then $Z_{5,t}(x)$ will transform into $-T_5(\frac{x+t^\circ}{1-t^\circ})$ respectively into $T_4(x)$, see also [4, p. 456] and Section 4 below.

Subsequently we shall need the values of the first four derivatives of $Z_{5,t}$ evaluated at the point $x = z_4 = 1$. We provide them here for the reader's convenience:

$$\begin{aligned} Z^{(1)}_{5,t}(1) = &- 8\kappa(-1+t)^3(-1+5t^2)(-2-21t+2t^2(-34+\sqrt{2}v_1)+ \\ &+ 6t^3(-15+2\sqrt{2}v_1) + 10t^4(-5+3\sqrt{2}v_1)+ \\ &+ 5t^5(-5+4\sqrt{2}v_1)), \end{aligned} \tag{33}$$

$$\begin{aligned} Z^{(2)}_{5,t}(1) = &8\kappa(-40\sqrt{2}(1+t)^4(t-5t^3)^2 v_1 + \\ &+ 12(1+t)^3(1-5t^2)^2(1+5t-5t^2+15t^3)+ \\ &+ 6\sqrt{2}t^5 v_1^3(-3-10t+6t^2+40t^3+85t^4+10t^5)- \end{aligned} \tag{34}$$

$$-(-1+5t^2)(-2-17t-21t^2+43t^3+83t^4+237t^5+ \\ +625t^6+825t^7+275t^8)),$$

$$Z^{(3)}_{5,t}(1) = 48\kappa(-20\sqrt{2}(1+t)^4(t-5t^3)^2v_1+ \\ +4(1+t)^3(1-5t^2)^2(1+5t-5t^2+15t^3)+ \\ +\sqrt{2}t^5v_1^3(-3-10t+6t^2+40t^3+85t^4+10t^5)), \tag{35}$$

$$Z^{(4)}_{5,t}(1) = 192\kappa(1+t)^3(-10\sqrt{2}(1+t)(t-5t^3)^2v_1+ \\ +(1-5t^2)^2(1+5t-5t^2+15t^3)), \tag{36}$$

with κ and v_1 according to (10).

3. THE QUINTIC SCHUR POLYNOMIAL

A. A. Markov's inequality [12] asserts an estimate on $|P^{(1)}_n(x)|$ and can be restated as

$$\sup_{x\in\mathbf{I}} \sup_{P_n\in\mathbf{B}_n} \frac{|P^{(1)}_n(x)|}{n^2} = 1. \tag{37}$$

This maximum will be attained if, up to the sign, $x = 1$ and $P_n = T_n \in \mathbf{B}_n$. Schur [21, § 2], considered the related extremal problem under the additional condition $P^{(2)}_n(\xi) = 0$, where $\xi \in \mathbf{I}$ is given:

Determine $\xi \in \mathbf{I}$ and $P_n \in \mathbf{B}_n$ for which

$$M_n := \sup_{\xi\in\mathbf{I}} \sup_{P_n\in\mathbf{B}_{n,\xi,2}} \frac{|P^{(1)}_n(\xi)|}{n^2}, \tag{38}$$

is attained, where

$$\mathbf{B}_{n,\xi,2} = \{P_n \in \mathbf{B}_n : P^{(2)}_n(\xi) = 0\}. \tag{39}$$

Erdös and Szegö provided the following solution in terms of the hard-core Zolotarev polynomial $Z_{n,t}$ [4, Th. 2]:

Let $n \geq 4$. The maximum (38) will be attained only if $\xi = 1$ and $P_n = \pm Z_{n,t}$ or if $\xi = -1$ and $P_n = \pm Q_{n,t}$ (where $Q_{n,t}(x) = Z_{n,t}(-x)$). For $n = 3$ the maximum (38) will be attained only if $\xi = 0$ and $P_3 = \pm T_3$.

For a general polynomial degree n the coefficients $a_i(t)$ of $Z_{n,t}$ and the optimal parameter $t = t^*$ for which the corresponding Z_{n,t^*} attains the maximum in (38), as well as the value M_n itself, remain arcane. However, as for the first Zolotarev case, $n = 4$, we have shed some new light on the above Erdös-Szegö solution by providing explicit analytical expressions for the value M_4 as well as for the optimal parameter $t = t^*$, and hence for the extremal coefficients $a_i(t^*), i = 0, 1, 2, 3, 4$, of the extremizer polynomial Z_{4,t^*}, see [18].

We now proceed to provide an analogous amendment for the second Zolotarev case in the above Erdös-Szegö solution, but with the reservation that for $n = 5$ the optimal parameter $t = t^*$ of the sought-for Schur polynomial cannot be expressed by radicals. Rather, t^* will be derived numerically (to any precision, and in three different fashions). Therefore, also the coefficients $a_i(t^*)$ of Z_{5,t^*} as well as the value M_5 cannot be determined in a closed analytic form so that we resort to numerical approximations. In the presentation of our results we will chop numerical results after the tenth valid decimal place.

Let now $n = 5$. According to [4, Th. 2], it suffices to consider the polynomials $Z_{5,t} \in \mathbf{B}_{5,1,2}$. The equation $Z^{(2)}_{5,t}(1) = 0$ in the variable t (see (34)) renders, when solved with *Mathematica*™'s *NSolve* - symbol, six real (approximate) solutions:

$$\begin{aligned} t_1 &= -3.1614415379..., t_2 = -1.3939833463..., \\ t_3 &= -\tfrac{1}{\sqrt{5}} = -0.4472135954..., \\ t_4 &= -0.0582703679..., t_5 = \tfrac{1}{\sqrt{5}}, t_6 = 0.4591395093... . \end{aligned} \tag{40}$$

Of these only $t^* := t_4$ is applicable since it satisfies $t^* \in J_5$. Hence the largest value of $|Z^{(1)}_{5,t}(1)|$ subject to $Z^{(2)}_{5,t}(1) = 0$ is attained if

$$t = t^* = -0.0582703679..., \tag{41}$$

and we get, after inserting t^* into $|Z^{(1)}_{5,t}(1)|$,

$$|Z^{(1)}_{5,t^*}(1)| = 7.5924835389..., \tag{42}$$

which implies

$$M_5 = \sup_{\xi \in \mathbf{I}} \sup_{P_5 \in \mathbf{B}_{5,\xi,2}} \frac{|P^{(1)}_5(\xi)|}{5^2} = 0.3036993415... . \tag{43}$$

An alternative way to deduce the optimal parameter $t = t^*$ of $Z_{5,t}$ with regard to (38) is, utilizing our knowledge of the Zolotarev points $A_5(t) < B_5(t) < C_5(t)$, to solve an equation which necessarily must be satisfied by $t = t^*$, see [4, formula (2.17)] and [16, formula (5.20)]:

$$\tfrac{25(A_5(t)-1)^2}{(B_5(t)-1)(C_5(t)-1)} - 2\Big(\tfrac{2}{A_5(t)-1} - \tfrac{1}{B_5(t)-1} - \tfrac{1}{C_5(t)-1}\Big) - 1 = 0. \tag{44}$$

Solving (44) with *Mathematica*™'s *NSolve* - symbol produces (after an excessive runtime) the identical root $t = t^*$ as given in (41). A third way to compute $t = t^*$ is to construct a polynomial, say P_m, with smallest possible degree m and smallest integer coefficients which has t^* among its real roots, and then to solve the polynomial equation $P_m(s) = 0$, either by radicals (if possible) or numerically. A desired such minimal polynomial P_m of degree $m = 10$ can be obtained by means of *Mathematica*™'s *Solve* - symbol (applied to $Z^{(2)}_{5,t}(1) = 0$) or *RootApproximant* - symbol (applied to sufficiently many (> 70) decimal places of t^* when computation is done with high precision in

(40)). In this way we get

$$(45)\quad P_{10}(s) = 50 + 949s + 1269s^2 - 5772s^3 - 13600s^4 - 5802s^5 + +19518s^6 + 49380s^7 + 54230s^8 + 26525s^9 + 4325s^{10}.$$

First we are going to check if the equation $P_{10}(s) = 0$ can be solved by radicals. To this end we employ the open source symbolic mathematical computation program *GAP* (package *Radiroot*, function *IsSolvablePolynomial*) to find out that the answer is in the negative: The Galois group of P_{10} is not solvable so that the zeros of P_{10} cannot be expressed by radicals. Solving the equation $P_{10}(s) = 0$ numerically (to a desired precision), *e.g.*, with *Mathematica*™'s *NSolve* - symbol, yields the six real solutions

$$(46)\quad \begin{aligned} s_1 &= -3.1614415379..., \quad s_2 = -1.3939833463...,\\ s_3 &= -0.4385675589..., \quad s_4 = -0.0582703679...,\\ s_5 &= 0.4591395093..., \quad s_6 = 0.4627324263..., \end{aligned}$$

of which s_4 coincides with t^* as given in (41). It is obvious from this set of solutions that $s_4 = t^*$ is that negative zero of P_{10} which has smallest modulus. It is not unusual to describe a sought-for constant (here: t^*) as a certain zero of a minimal algebraic polynomial with integer coefficients: Consider, for example, the definition of J. H. Conway's constant as the unique positive zero of some polynomial P_{71}, see ([5], p. 453).

Having determined (numerically) the optimal parameter t^* which selects, according to [4, Th. 2], the quintic Schur polynomial Z_{5,t^*} among the infinitely many quintic hard-core Zolotarev polynomials $Z_{5,t}$, we obtain, by insertion, the numerical approximations for the coefficients of Z_{5,t^*} (and hence for the coefficients of its first and second derivative) as well as the numerical approximations for its critical points:

$$(47)\quad \begin{aligned} Z_{5,t^*}(x) &= \sum_{i=0}^{5} a_i(t^*)x^i = 0.7437050451... - 2.8454432113...x -\\ &\quad -6.5707799509...x^2 + 8.9780145139...x^3 + 6.8270749058...x^4 -\\ &\quad -6.1325713026...x^5, \end{aligned}$$

and it is readily checked that there holds $Z^{(1)}_{5,t^*}(1) = 7.5924835389...$ and $Z^{(2)}_{5,t^*}(1) = 0$.

The equioscillation points of Z_{5,t^*} in the interior of $\mathbf{I}$ are, approximately, see (14)–(16),

$$(48)\quad \begin{aligned} z_1(t^*) &= -0.7699336349... < z_2(t^*) = -0.1696253638... <\\ &< z_3(t^*) = 0.5589586326..., \end{aligned}$$

and it is readily checked that there holds $Z_{5,t^*}(z_1(t^*)) = -1, Z_{5,t^*}(z_2(t^*)) = 1, Z_{5,t^*}(z_3(t^*)) = -1$ and, furthermore, $Z^{(1)}_{5,t^*}(z_i(t^*)) = 0$ for $i = 1, 2, 3$. The

Zolotarev points of Z_{5,t^*} (to the right of $\mathbf{I}$) are, approximately, see (27)–(29),

(49)

$A_5(t^*) = 1.2711990490... < B_5(t^*) = 1.4524990812... < C_5(t^*) = 1.5351983581...,$

and it is readily checked that there holds $Z^{(1)}_{5,t^*}(A_5(t^*)) = 0, Z_{5,t^*}(B_5(t^*)) = 1$ and $Z_{5,t^*}(C_5(t^*)) = -1$, and furthermore, that equation (31) holds for $t = t^*$. According to [16, formula (5.26)], the first term in (44), evaluated at $t = t^*$, coincides with $Z^{(1)}_{5,t^*}(1)$, and also this auxiliary equation can now be readily cross-checked. Summarizing we thus obtain the following amendment to [4, Th. 2] for the second Zolotarev case, $n = 5$:

Proposition 3. *Let t^* denote the negative zero with smallest modulus of the polynomial of degree $n = 10$ as given in* (45)*, where the numerical value of t^* is given in* (41)*. Let $Z_{5,t}$ denote the quintic hard-core Zolotarev polynomial with parameterized coefficients as given in* (4)-(9) *and with critical points as given in* (14)-(16) *and* (27)-(29)*. Then, Z_{5,t^*} is a Schur polynomial which solves Schur's Markov-type extremal problem* (38) *for $n = 5$. The numerical values of its coefficients and of its critical points are given in* (47)-(49)*, and the numerical value of the sought-for maximum M_5 is given in* (43)*.*

4. THE QUINTIC SHADRIN POLYNOMIALS

A. A. Markov's inequality (37) for the first derivative of $P_n \in \mathbf{B}_n$ was generalized to the k-th derivatives by his half-brother V. A. Markov [13, p. 93] in 1892. It can be restated as follows, see also [15, p. 545], [19, Th. 2.24]:

$$\sup_{x\in\mathbf{I}} \sup_{P_n\in\mathbf{B}_n} \frac{|P_n^{(k)}(x)|}{\prod\limits_{j=0}^{k-1} \frac{n^2-j^2}{2j+1}} = 1 \quad (1 \le k \le n). \tag{50}$$

For each k this maximum will be attained (up to the sign) if $x = 1$ and $P_n = T_n$. Shadrin [23] has analogously generalized Schur's problem, *i.e.*, extending (38) to the k-th derivatives, and it can be stated as follows: Determine $\xi \in \mathbf{I}$ and $P_n \in \mathbf{B}_n$ for which

$$M_{n,k} := \sup_{\xi\in\mathbf{I}} \sup_{P_n\in\mathbf{B}_{n,\xi,k+1}} \frac{|P_n^{(k)}(\xi)|}{\prod\limits_{j=0}^{k-1} \frac{n^2-j^2}{2j+1}}, \tag{51}$$

is attained, where $\mathbf{B}_{n,\xi,k+1} = \{P_n \in \mathbf{B}_n : P_n^{(k+1)}(\xi) = 0\}, 2 \le k \le n - 2$, and $n \ge 4$.

Shadrin [23, Prop. 4.4], also provided the following solution:

Let $n \ge 4$ and $2 \le k \le n - 2$. The maximum (51) will be attained (up to the sign) if $\xi = 1$ and P_n is a (proper or improper) Zolotarev polynomial, Z_n, or if $\xi = \omega_{k,n}$ = the rightmost zero of $T_n^{(k+1)}$ and $P_n = T_n$, so that altogether *there holds (under the assumptions* $Z_n \in \mathbf{B}_{n,1,k+1}$ *and* $T_n^{(k+1)}(\omega_{k,n}) = 0$*)*

$$M_{n,k} := \frac{\max\{|Z_n^{(k)}(1)|, |T_n^{(k)}(\omega_{k,n})|\}}{\prod_{j=0}^{k-1} \frac{n^2-j^2}{2j+1}}. \tag{52}$$

The extremal polynomials for $2 \le k \le n-2$ we therefore term *Shadrin polynomials.* The proper Zolotarev polynomial $Z_n := Z_{n,t}$ has been introduced in Section 2. Apart from sign and reflection, the improper Zolotarev polynomial relative to $\mathbf{I}$ is either the distorted Chebyshev polynomial $Z_n := T_{n,\sigma}$, with $T_{n,\sigma}(x) := T_n(\frac{x-\sigma}{1+\sigma})$, where $0 < \sigma \le \tan^2(\frac{\pi}{2n})$, or the familiar Chebyshev polynomial of degree n or $n-1$, $Z_n = T_n$ respectively $Z_n = T_{n-1}$, see [1], [2], [15, p. 406]. Let now $n = 5$ and choose $k = 2$ (the case $n = 4$ and $k = 2$ is treated in [18]). In view of (52) the goal is to evaluate $\max\{|Z_5^{(2)}(1)|, |T_5^{(2)}(\omega_{2,5})|\}$, given that $Z_5^{(3)}(1) = 0 = T_5^{(3)}(\omega_{2,5})$. It turns out that the improper Zolotarev polynomial $Z_5 \in \{T_4, T_5, T_{5,\sigma}\}$ cannot be extremal due to $T_4^{(3)}(1) = 192 \neq 0$, resp. $T_5^{(3)}(1) = 840 \neq 0$, resp. $T_{5,\sigma}^{(3)}(1) = \frac{120(7-18\sigma+7\sigma^2)}{(1+\sigma)^5} \neq 0$ (for $0 < \sigma \le \tan^2(\frac{\pi}{10}) = -t^\circ = 1 - \frac{2}{\sqrt{5}}$).

For the proper Zolotarev polynomial $Z_5 = Z_{5,t}$ we find, again employing *Mathematica*™, that the condition $Z_{5,t}^{(3)}(1) = 0$ (see (35)) renders seven real (approximate) solutions for t:

$$\begin{aligned} t_1 &= -1.8058692666..., t_2 = -1, t_3 = -\tfrac{1}{\sqrt{5}} = -0.4472135954..., \\ t_4 &= -0.0230782942..., t_5 = \tfrac{1}{\sqrt{5}}, \\ t_6 &= 0.5194288192..., t_7 = 23.4433908091... . \end{aligned} \tag{53}$$

Of these only $t^{**} := t_4$ is applicable since it satisfies $t^{**} \in J_5$. Hence the largest value of $|Z_{5,t}^{(2)}(1)|$ subject to $Z_{5,t}^{(3)}(1) = 0$ is attained for

$$t = t^{**} = -0.0230782942... \tag{54}$$

which leads, after insertion, to

$$|Z_{5,t^{**}}^{(2)}(1)| = 36.6462826529... . \tag{55}$$

It is tempting to express $t_4 = t^{**}$ as a closed algebraic form in terms of radicals of some polynomial equation. A desired such minimal polynomial can again be deduced with the aid of *Mathematica*™, see Section 3. We so likewise obtain here a minimal integer polynomial $\overline{P}_m$ of degree $m = 10$ which has t^{**} among its real roots:

$$\begin{aligned} \overline{P}_m(s) \;=\; & -8 - 369s - 937s^2 + 1539s^3 + 7503s^4 + 7245s^5 - \\ & -8935s^6 - 26415s^7 - 23075s^8 - 6000s^9 + 300s^{10}. \end{aligned} \tag{56}$$

But similar to the case $k = 1$ and polynomial (45), t^{**} cannot be expressed in terms of radicals since the Galois group of (56) is not solvable, as we have checked with the aid of *GAP*. Among the three negative roots of the equation $\overline{P}_m(s) = 0$, *i.e.*, $s_1 = -1.8058692666...$, $s_2 = -0.4119616991...$, $s_3 = t^{**}$, obviously t^{**} is the one with smallest modulus. Comparing $|Z^{(2)}_{5,t^{**}}(1)|$ to $|T_5^{(2)}(\omega_{2,5})|$, where $\omega_{2,5} =$ the rightmost zero of $T_5^{(3)} = \frac{1}{2\sqrt{2}}$, we get

$$\left|T_5^{(2)}\left(\tfrac{1}{2\sqrt{2}}\right)\right| = 20\sqrt{2} = 28.2842712474... < |Z^{(2)}_{5,t^{**}}(1)|. \tag{57}$$

Thus we have, subject to $Z_5^{(3)}(1) = 0 = T_5^{(3)}(\omega_{2,5})$,

$$\max\left\{|Z_5^{(2)}(1)|, \left|T_5^{(2)}\left(\tfrac{1}{2\sqrt{2}}\right)\right|\right\} = |Z^{(2)}_{5,t^{**}}(1)| = 36.6462826529... \tag{58}$$

and obviously

$$\prod_{j=0}^{1} \tfrac{5^2-j^2}{2j+1} = 200, \tag{59}$$

holds, so that finally we get

$$M_{5,2} = 0.1832314132... . \tag{60}$$

Having determined (numerically) the optimal parameter t^{**}, we are in a position to provide, by insertion, the numerical approximations for the coefficients of the Shadrin polynomial $Z_{5,t^{**}}$ (for $k = 2$) as well as for its critical points:

$$\begin{aligned} Z_{5,t^{**}}(x) \quad = \quad & \sum_{i=0}^{5} a_i(t^{**})x^i = 0.9050563187... - 1.7415460912...x - \\ & -7.5064008470...x^2 + 5.3134265584...x^3 + \\ & +7.6013445283...x^4 - 3.5718804671...x^5, \end{aligned} \tag{61}$$

and it is readily checked that (55) holds and that $Z^{(3)}_{5,t^{**}}(1)$ vanishes. Furthermore, we get

$$\begin{aligned} z_1(t^{**}) = -0.7501496712... < z_2(t^{**}) = -0.1065526272... < \\ < z_3(t^{**}) = 0.6335508926... \end{aligned} \tag{62}$$

and

$$\begin{aligned} A_5(t^{**}) = 1.9256371632... < B_5(t^{**}) = 2.3412124512... < \\ < C_5(t^{**}) = 2.3613047539... . \end{aligned} \tag{63}$$

Summarizing we have thus established:

PROPOSITION 4. *Let t^{**} denote the negative zero with smallest modulus of the polynomial of degree $n = 10$ as given in* (56)*, where the numerical value of t^{**} is given in* (54)*. Let $Z_{5,t}$ denote the quintic hard-core Zolotarev polynomial. Then, $Z_{5,t^{**}}$ is a Shadrin polynomial which solves Shadrin's Markov-type extremal problem to determine* (51) *for $n = 5$ and $k = 2$. The numerical values of its coefficients and of its critical points are given in* (61)–(63) *and the numerical value of the sought-for maximum $M_{5,2}$ is given in* (60)*.*

Let now $n = 5$ and $k = 3$, so that the goal is to evaluate $\max\{|Z_5^{(3)}(1)|, |T_5^{(3)}(\omega_{3,5})|\}$, subject to $Z_5^{(4)}(1) = 0 = T_5^{(4)}(\omega_{3,5})$. It is left to the reader to check that, likewise as for $k = 2$, $Z_5 \in \{T_4, T_5, T_{5,\sigma}\}$ cannot be extremal due to $Z_5^{(4)}(1) \neq 0$. For $Z_5 = Z_{5,t}$ the condition $Z_{5,t}^{(4)}(1) = 0$ (see (36)) renders, likewise as for $k = 2$, six real (approximate) solutions for t:

$$t_1 = -3.0314515138..., t_2 = -1, t_3 = -\tfrac{1}{\sqrt{5}}, \tag{64}$$
$$t_4 = -0.0048304566..., t_5 = \tfrac{1}{\sqrt{5}}, t_6 = 0.4577656892... .$$

Of these only $t^{***} := t_4$ is applicable since it satisfies $t^{***} \in J_5$. Hence the largest value of $|Z_{5,t}^{(3)}(1)|$ subject to $Z_{5,t}^{(4)}(1) = 0$ is attained for

$$t = t^{***} = -0.0048304566... \tag{65}$$

which yields, after insertion,

$$|Z_{5,t^{***}}^{(3)}(1)| = 109.2942452670... . \tag{66}$$

Again one might ask whether $t_4 = t^{***}$ can be expressed by radicals of some polynomial equation. In this case the answer is in the positive, since the following minimal polynomial P_6, which contains t^{***} as a zero, has a Galois group which is solvable (as can be checked with *GAP*):

$$P_6(t) = -1 - 210t - 615t^2 + 420t^3 + 2625t^4 + 3150t^5 + 775t^6. \tag{67}$$

But we shall not dwell on this since it will turn out that $Z_{5,t^{***}}$, which is approximately given by

$$\begin{aligned} Z_{5,t^{***}}(x) &= \sum_{i=0}^{5} a_i(t^{***})x^i = 0.9805806750... - 0.7882729973...x - \\ &\quad -7.9021418328...x^2 + 2.3725852288...x^3 + \\ &\quad +7.9215611578...x^4 - 1.5843122315...x^5, \end{aligned} \tag{68}$$

is not a Shadrin polynomial for $k = 3$. Indeed, since $\prod_{j=0}^{2} \frac{5^2-j^2}{2j+1} = 840$ and $\omega_{3,5} =$ the rightmost zero of $T_5^{(4)} = 0$, we get

$$|T_5^{(3)}(0)| = 120 > |Z_{5,t^{***}}^{(3)}(1)|, \tag{69}$$

and hence, subject to $Z_{5,t}^{(4)}(1) = 0 = T_5^{(4)}(\omega_{3,5})$,

$$\max\{|Z_5^{(3)}(1)|, |T_5^{(3)}(0)|\} = |T_5^{(3)}(0)| = 120, \tag{70}$$

$$M_{5,3} = \tfrac{1}{7} = 0.1428571428... . \tag{71}$$

Summarizing we have thus established:

PROPOSITION 5. *The quintic Chebyshev polynomial of the first kind,* $T_5 \in \mathbf{B}_5$ *with* $T_5(x) = 16x^5 - 20x^3 + 5x$*, is a Shadrin polynomial which solves Shadrin's Markov-type extremal problem to determine* (51) *for* $n = 5$ *and* $k = 3$*. The sought-for maximum* $M_{5,3}$ *has the value* $\frac{1}{7}$*.*

In concluding this Section, we are going to compare our deduced maximum values (43), (60), (71) to Shadrin's estimates, $\lambda_{n,k}$, for $M_{n,k}$, see [23, Th. 7.1 and Rem. 5.5]):

$$M_{n,k} \leq \lambda_{n,k} = \tfrac{n-1}{(k+1)(n-1+k)}. \tag{72}$$

For $n = 5$ and $k \in \{1, 2, 3\}$ we thus obtain

$$M_{5,1} := M_5 = 0.3036993415... < 0.4 = \tfrac{2}{5} = \lambda_{5,1} \tag{73}$$

$$M_{5,2} = 0.1832314132... < 0.\overline{2} = \tfrac{2}{9} = \lambda_{5,2} \tag{74}$$

$$M_{5,3} = 0.1428571428... = \tfrac{1}{7} = \lambda_{5,3}. \tag{75}$$

5. TWO NEW ALGEBRAIC APPROACHES TO ZOLOTAREV'S FIRST PROBLEM FOR QUINTIC POLYNOMIALS

Zolotarev's first problem (out of four) calls for a best approximation by polynomials of degree $\leq n-2$ to the function f_σ, with $f_\sigma(x) = x^n - n\sigma x^{n-1}$ and $x \in \mathbf{I}$, or equivalently, calls for a polynomial $\check{P}_{n,\sigma}$ of degree n with fixed first and second leading coefficient, given by $\check{P}_{n,\sigma}(x) = x^n - n\sigma x^{n-1} + b_{n-2}x^{n-2} + ... + b_1 x + b_0$, which deviates least from the zero function in $\mathbf{I}$, see [2], [25]. Here, σ is a given real number and the deviation is measured in the uniform norm. Equipped with the previous results we are able to solve Zolotarev's first problem, for $n = 5$, algebraically and even in two fashions, thus avoiding the use of elliptic functions. Our solutions complement and simplify existing algebraic approaches to solve Zolotarev's first problem for $n = 5$. Note that Zolotarev's first problem extends P. L. Chebyshev's classical approximation problem [19, p. 67 and p. 87], to determine a monic polynomial of degree n which deviates least from the zero function in $\mathbf{I}$, measured in the uniform norm (a solution is $2^{1-n}T_n$, which corresponds to $\sigma = 0$).

Consider now $\check{P}_{5,\sigma}$ with some $\sigma \in \mathbb{R}\setminus\{0\}$. The goal is to specify, at least numerically, its four variable coefficients $b_i (i = 0, 1, 2, 3)$ such that $\sup_{x\in\mathbf{I}} |\check{P}_{5,\sigma}(x)|$ becomes least. The following is well known: For $0 < |\sigma| \leq \tan^2(\frac{\pi}{10}) = -t^\circ = 1 - 2/\sqrt{5} = 0.1055728090...$ the desired least-deviating quintic polynomial, which is related to an improper quintic Zolotarev polynomial, can be deduced by elementary means, see [2, Th. 1], [15, Th. 1.2.20], where a solution for an arbitrary degree n is displayed. However, for $n = 5$ and for $|\sigma| > \tan^2(\frac{\pi}{10})$ the desired solution, which is related to a quintic hard-core Zolotarev polynomial, is usually expressed by means of elliptic functions, see [2, Th. 2], [15, Th. 1.2.21], where a solution for an arbitrary degree n is displayed. Schiefermayr [20, p. 156], has established, for arbitrary n and $|\sigma| > \tan^2(\frac{\pi}{2n})$, an algebraic solution formula which can be applied immediately provided a subset of the critical points of $Z_{n,t}$ is known, a premise that holds for the case $n=5$ under consideration (see Section 2). If the critical points are not known in advance, then an algorithm is advised how to compute that subset. This algorithm, however, requires polynomial equations to be solved which for $n \geq 5$ get *very bulky* [20]. Schiefermayr's solution formula reads for $n=5$:

A least-deviating polynomial $\check{P}_{5,\sigma} = \check{P}_{5,\sigma,\check{t}}$ *with* $\sigma > \tan^2(\frac{\pi}{10}) = -t^\circ$ *is given by*

$$\begin{aligned} (76) \qquad \check{P}_{5,\sigma,\check{t}}(x) &= (x - B_5(\check{t}))(x^2-1)(x - z_2(\check{t}))^2 - \\ &-\tfrac{1}{2}(C_5(\check{t}) - B_5(\check{t}))(C_5(\check{t})^2 - 1)(C_5(\check{t}) - z_2(\check{t}))^2 \end{aligned}$$

subject to

$$(77) \qquad 2z_2(t) + B_5(t) = 5\sigma,$$

meaning that one has first to determine some $t = \check{t} \in J_5$ which solves equation (77) and then to compute $\check{P}_{5,\sigma,\check{t}}(x)$ with the aid of this value. To apply the formula in this way, the three critical points $B_5(t), C_5(t)$ and $z_2(t)$ of $Z_{5,t}$ need to be known.

Example 6. We choose $\sigma = 2$, say, so that our goal is to determine $\check{P}_{5,2,\check{t}}$. (77) yields

$$(78) \qquad 2z_2(t) + B_5(t) = \tfrac{v_6}{2\sqrt{2}} = 10$$

with

$$(79) \qquad v_6 := v_6(t) = \tfrac{1+5t-5t^2+15t^3}{t^2(1+t)v_1} = \frac{1+5t-5t^2+15t^3}{t^2(1+t)\sqrt{\frac{(1+t)(-1+5t^2)}{t^3}}}.$$

Applying *Mathematica*™'s *NSolve* - symbol we get

$$(80) \qquad t = \check{t} = -0.0012391497...$$

as the unique approximate solution to (78) from J_5, which is a zero of the minimal polynomial $P_6(u) = -1-810u-2415u^2+1620u^3+11025u^4+12150u^5+3775u^6$, and the zeros of P_6 cannot be expressed by radicals, as we have checked with *GAP*. In an intermediate step we then rearrange terms, see (15), (27), (28):

$$\begin{aligned} (81) \qquad & (x - B_5(t))(x^2-1)(x - z_2(t))^2 - \\ & -\tfrac{1}{2}(C_5(t) - B_5(t))(C_5(t)^2-1)(C_5(t) - z_2(t))^2 \\ & = (3v_2 - 6v_3 + 2\sqrt{v_4 - v_5} - v_6) \times \\ & \times \tfrac{(-v_2+3v_3+\sqrt{v_4-v_5}+v_6)^2(-1+\frac{1}{200}(4v_3+v_6)^2)}{4000\sqrt{2}} + \\ & + \left(\tfrac{-v_2-v_3+\sqrt{v_4-v_5}}{10\sqrt{2}} + x\right)^2 \left(\tfrac{-3v_2+2v_3-2\sqrt{v_4-v_5}}{10\sqrt{2}} + x\right)(-1+x^2). \end{aligned}$$

Inserting $t = \check{t}$ into v_2, v_3, v_4, v_5, v_6 (which have been defined in (17)-(20), (79)) and expanding, finally gives

$$\begin{aligned}(82)\qquad &\check{P}_{5,2,\check{t}}(x) = -1.2468982438... + 0.4993781094...x + \\ &+9.9937810175...x^2 - 1.4993781094...x^3 - 10x^4 + x^5.\end{aligned}$$

This polynomial has the required two leading coefficients $b_5 = 1, b_4 = -10$ and the resulting coefficients $b_i, i = 0,1,2,3$ (with $b_3 = -1 - b_1$, due to $\check{P}_{5,\sigma}(1) = \check{P}_{5,\sigma}(-1)$) are optimal. Dividing $\check{P}_{5,2,\check{t}}$ by its least deviation

$$\begin{aligned}(83)\qquad |\check{P}_{5,2,\check{t}}(\pm 1)| &= \tfrac{1}{2}(C_5(\check{t}) - B_5(\check{t}))(C_5(\check{t})^2 - 1)(C_5(\check{t}) - z_2(\check{t}))^2 \\ &= 1.2531172262...\end{aligned}$$

renders the related hard-core Zolotarev polynomial $-Z_{5,\check{t}} \in \mathbf{B}_5$ with

$$\begin{aligned}&-Z_{5,\check{t}}(x) = -0.9950371902... + 0.3985086941...x + 7.9751365698...x^2 - \\ (84)\qquad &-1.1965186321...x^3 - 7.9800993796...x^4 + 0.7980099379...x^5.\end{aligned}$$

$\square$

Consider now $\check{P}_{5,\sigma}(x)$ assuming $\sigma < 0$ but $|\sigma| > \tan^2(\frac{\pi}{10})$. Then the least-deviating polynomial (76) changes to $-\check{P}_{5,\sigma(x),\check{t}}(-x)$ and the right hand side in (77) to $5|\sigma|$.

Example 7. We reconsider an example from literature where a least-deviating quintic polynomial with prescribed leading coefficients has been computed algebraically, with different approaches, by G. E. Collins ([3], p. 186) and independently by V. A. Malyshev [11, p. 937]: The goal is to find among all polynomials of the form $x^5 + x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$, where both leading coefficients b_5 and b_4 are equated to 1, one that deviates least from the zero-function in $\mathbf{I}$ (measured in the uniform norm). In the representation of $\check{P}_{5,\sigma}(x)$ we now have $1 = -5\sigma$, *i.e.*, $\sigma = -0.2$ with $|\sigma| > \tan^2(\frac{\pi}{10}) = -t^\circ$. Solving the equation $2z_2(t) + B_5(t) = \frac{v_6}{2\sqrt{2}} = 5|\sigma| = 1$ (*e.g.*, with *Mathematica*™'s *Solve*-symbol) renders the unique solution $t = \check{t}$ from J_5 as an expression in radicals:

$$\begin{aligned}(85)\qquad \check{t} &:= \frac{\sqrt{10}}{185}\left(\frac{-25}{\sqrt{10}} + \sqrt{-356 + \frac{220\sqrt{10}}{\sqrt{-178+a}} - a} - \sqrt{-178 + a})\right), \\ \text{with } a &:= (74\sqrt{37})^{2/3}\left((7 - 3\sqrt{5})^{1/3} + (7 + 3\sqrt{5})^{1/3}\right), \\ &= -0.0654947997...,\end{aligned}$$

which is a zero of the minimal polynomial $P_4(u) = 1 + 20u + 78u^2 + 100u^3 + 185u^4$. Inserting this $\check{t}$ into $-\check{P}_{5,\sigma,\check{t}}(-x)$, see (76), yields a least deviating polynomial whose optimal numerical coefficients are:

$$\begin{aligned}
&(86) \qquad & b_0 &= 0.1065834340... \\
&(87) \qquad & b_1 &= 0.4581775889... \\
&(88) \qquad & b_2 &= -0.9557598788... \\
&(89) \qquad & b_3 &= -1.4581775889... = -1 - b_1 \\
&(90) \qquad & b_4 &= b_5 = 1.
\end{aligned}$$

They indeed coincide with those numerical coefficients as derived in [3] (with opposite signs) and [11]. We observe that the numerical coefficients (86), (87), (89) as given in [11] are biased after the 34th decimal place, and (88) as given in [11] is biased after the 33rd one. Dividing this polynomial by its least deviation

$$(91) \qquad \tfrac{1}{2}(C_5(\check{t}) - B_5(\check{t}))(C_5(\check{t})^2 - 1)(C_5(\check{t}) - z_2(\check{t}))^2 = 0.1508235551...$$

renders the following related hard-core Zolotarev polynomial $Q_{5,\check{t}} \in \mathbf{B}_5$:

$$\begin{aligned}
(92) \qquad Q_{5,\check{t}}(x) &= Z_{5,\check{t}}(-x) = 0.7066763140... + 3.0378384100...x - \\
&\quad -6.3369403942...x^2 - 9.6681024902...x^3 + \\
&\quad +6.6302640801...x^4 + 6.6302640801...x^5.
\end{aligned}$$

We can even provide explicit analytic expressions for the coefficients of the least deviating polynomial $-\check{P}_{5,\sigma,\check{t}}(-x) = \sum_{i=0}^{5} b_i x^i$, where the b_i's are given numerically in (86) - (90). Our procedure is to expand the right hand side of (81) in powers of x and then to replace t by $\check{t}$, according to (85), in those terms which represent the b_i's. Let us proceed so exemplarily for the optimal b_3: From (81) we deduce that the coefficient of x^3 in $-\check{P}_{5,\sigma,\check{t}}(-x)$ is

(93)

$$-1+\tfrac{1}{100}(-3v_2+2v_3-2\sqrt{v_4-v_5})(-v_2-v_3+\sqrt{v_4-v_5})+\tfrac{1}{200}(-v_2-v_3+\sqrt{v_4-v_5})^2.$$

Inserting now $t = \check{t}$ from (85) into the v_i's $= v_i(t)$'s yields, after some simplifications,

$$\begin{aligned}
(94) \qquad b_3 &= \tfrac{1}{15}\left(-18 - \sqrt{-36 + 40\sqrt{\tfrac{2}{b}} - 2b} + \sqrt{2b}\right), \\
\text{with } b &:= -6 - 22\left(\tfrac{2}{c}\right)^{1/3} + 2^{2/3}c^{1/3} \text{ and } c := 151 + 75\sqrt{5}, \\
&= -1.4581775889...,
\end{aligned}$$

which is a zero of the minimal polynomial $P_4(u) = 1600 + 5120u + 6048u^2 + 3240u^3 + 675u^4$.

The coefficients b_2, b_1, b_0 can be deduced in a similar vein. Collins [3], using a different method, provides the quartic minimal polynomials for $-b_3, -b_2, -b_0$ (and utilizes $-b_1 = 1 + b_3$) from which the explicit analytic expressions for

$-b_3, -b_2, -b_0$ (and hence for $-b_1$) can be deduced with the aid of *Mathematica*™'s *Solve* - symbol. We note that the minimal polynomial in [3] for $-b_0$ contains a misprint: The coefficient of x^0 should read -1178141 (not -117814). See also Remark 10 below. □

An alternative path to obtaining an algebraic solution to Zolotarev's first problem for $n = 5$ is via the novel power form representation (3) - (11): Dividing (3) by the leading coefficient $a_5(t)$ results in a monic power form representation of type

$$x^5 + \frac{(-v_6)}{2\sqrt{2}} x^4 + \text{ lower degree terms.} \tag{95}$$

Identifying (95) with $\check{P}_{5,\sigma}(x)$ for $\sigma = 2$ means that the (parameterized) coefficient of x^4 will be equated with -10. Solving the resulting equation $\frac{(-v_6)}{2\sqrt{2}} = -10$ for t (again with *Mathematica*™'s *NSolve* - symbol) yields, consistently with Example 6, $t = \check{t} = -0.0012391497...$ as the only solution from J_5. The hard-core Zolotarev polynomial $Z_{5,\check{t}}$ has the leading coefficient $a_5(\check{t}) = -0.7980099379...$, see (9). Dividing now $Z_{5,\check{t}}$ by $a_5(\check{t})$ renders a polynomial of type $x^5 - 10x^4 +$ lower degree terms, which is identical to $\check{P}_{5,2,\check{t}}(x)$, see (82). For $\sigma = -0.2$ we consider the representation $-Y_{5,t}(-x)$, see (3), and we proceed analogously, that is, we divide it by its leading coefficient, set the residual coefficient of x^4 equal to 1 and solve the resulting equation in the variable t. This will yield, consistently with Example 7, $t = \check{t} = -0.0654947997...$ as the unique solution from J_5. The hard-core Zolotarev polynomial $Z_{5,\check{t}}$ has the leading coefficient $a_5(\check{t}) = -6.6302640801...$. Replacing $Z_{5,\check{t}}$ by $Q_{5,\check{t}}$ with $Q_{5,\check{t}}(x) = Z_{5,\check{t}}(-x)$ recovers the polynomial (92). Dividing it by its leading coefficient yields the polynomial with the optimal coefficients (86) - (90). Its graph is sketched in [11, p. 937]. □

In concluding this Section, we summarize, to the best our knowledge, the currently available constructive approaches to solve Zolotarev's first problem algebraically for $n = 5$ and for a given σ with $|\sigma| > \tan^2(\frac{\pi}{10})$:

(1) M. L. Sodin and P. M. Yuditskii [24] derive the least deviating $\check{P}_{n,\sigma}$ by representing it by means of involved determinants. No explicit power form representation of the optimal $\check{P}_{n,\sigma}$ and also no explicit example is given.

(2) Malyshev [11, pp. 934], too derives the coefficients of the optimal $\check{P}_{n,\sigma}$ by means of determinants, but no explicit power form representation of the optimal $\check{P}_{n,\sigma}$ is given. For $n = 5$ two auxiliary polynomials $U_{6,-5\sigma}$ (of degree 6 in the variable x) and $V_{6,-5\sigma}$ (of degree 6 in the variable y) are provided which depend on the parameter -5σ, and hence on σ. For $\sigma = -0.2$ the zeros of $U_{6,1}$ and $V_{6,1}$ are computed and are then employed to determine, by computing certain determinants, the explicit least deviating polynomial with coefficients (86) - (90), see Example 7 above. The reference [24] is not given.

(3) Schiefermayr [20] derives the least deviating $\check{P}_{n,\sigma}$ by representing it in a modified power form which entails the Zolotarev points $B_n(t)$ and $C_n(t)$ as well as a subset of the equioscillation points of $Z_{n,t}$. This subset of the critical points of $Z_{n,t}$ has to be computed by means of a given algorithm which involves determinants. For $n = 5$ this subset consists of $\{B_5(t), C_5(t), z_2(t)\}$, see (76), (77). References [11] and [24] are given, but no explicit example for $n = 5$.
(4) Our modification of Schiefermayr's approach [20] for $n = 5$ which uses the prior knowledge of the set $\{B_5(t), C_5(t), z_2(t)\}$ from Section 2. The computation of this set by the algorithm stipulated in [20] is thus dispensable, see (76), (77) and Examples 6 and 7.
(5) Our alternative approach as indicated above (after Example 7) which is justified by [4, Th. 3]. It builds on the novel explicit algebraic power form representation of $Z_{5,t}$ (see Section 2), that is, identifying $Z_{5,t}(x)/a_5(t)$ with $\check{P}_{n,\sigma}$ and equating the coefficients of the respective power x^4 (and with obvious modifications if $\sigma < 0$ but $|\sigma| > \tan^2\left(\frac{\pi}{10}\right)$).

We note that Collins' algebraic approach [3] for $n = 5$ solves Zolotarev's first problem only for the single dedicated parameter $\sigma = -0.2$, see [3, p. 185]. E. Kaltofen [8, p. 8], with reference to [3], but not mentioning that incompleteness for $n = 5$ and also not mentioning the solution in [24], poses Zolotarev's first problem as an open problem for $n \geq 6$. D. Lazard [9], in response to this challenge, does notice (on p. 197) the incompleteness of Collins' solution for $n = 5$, but does not reference either to the solution in [24] and also not to the then available solution in [11]. He claims to have solved Zolotarev's first problem algebraically, by symbolic computation, up to $n = 12$; however, no constructive representation of the least deviating $\check{P}_{n,\sigma}$ and also no explicit example is given.

6. CONCLUDING REMARKS

Remark 8. An iterative numerical method to compute, in particular, (42), (48) and (49) is advised in [16, Section 5d].

Remark 9. V. I. Lebedev [10] considers a generalized proper Zolotarev polynomial which depends on two parameters. The second parameter, μ, satisfies $1 \leq \mu < n - 1$ and the choice $\mu = 1$ takes us back to the classical proper Zolotarev polynomial (with only one parameter) as described in Section 2 above. In particular, the sextic polynomial $\widetilde{Z}_6 := T_3(Z_{2,t})$, with $t > 1$, is such a generalized Zolotarev polynomial with $\mu = 3$, see [10, Formula (2.50)]. According to [10, Lemma 2.1], $\widetilde{Z}_6$ has only four (rather than six) equioscillation points in $\mathbf{I}$ so that $\widetilde{Z}_6$ does not represent a classical proper Zolotarev polynomial as described in Section 2 above, contrary to what is indicated in

[6, p. 15], and [7, pp. 3]. Therefore, the problem to determine an explicit algebraic power form representation of a sextic hard-core Zolotarev polynomial (with six equioscillation points in $\mathbf{I}$) is still open.

REMARK 10. The quintic polynomial P_5^* given by $P_5^*(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + x^4 + x^5$ (with two fixed leading coefficients $b_4 = b_5 = 1$, see [3], [11], and Example 7), which deviates least from the zero function in $\mathbf{I}$, is worth the effort to write down its optimal coefficients b_0, b_1, b_2, b_3 and its least deviation in explicit form in terms of radicals. The corresponding numerical values are given in (86)–(89) and (91). We already know the expression for b_3, see (94), and hence we know $b_1 = -1 - b_3$. For b_2 we get, by the methods described,

$$b_2 = \tfrac{1}{675}\left(-690 + \sqrt{30}\left(\sqrt{-38 + \tfrac{556}{3}\sqrt{\tfrac{10}{3d}} - d} + \sqrt{d}\right)\right) \tag{96}$$

with

$$d = -\tfrac{38}{3} + 2^{2/3}\alpha^{1/3} + 2^{2/3}\beta^{1/3}, \tag{97}$$

$$\alpha = (3053 + 1345\sqrt{5}),\ \beta = (3053 - 1345\sqrt{5}). \tag{98}$$

For b_0 we get likewise the expression

$$b_0 = \tfrac{1}{84375}\left(11211 + \sqrt{6}\left(\sqrt{-9837934 + \tfrac{210863861500}{3}\sqrt{\tfrac{2}{3D}} - D} - \sqrt{D}\right)\right) \tag{99}$$

with

$$D = -\tfrac{9837934}{3} + 2^{2/3}\gamma^{1/3} + 2^{2/3}\delta^{1/3}, \tag{100}$$

$$\gamma = 84447248882562640537 + 36683761704646421875\sqrt{5}, \tag{101}$$

$$\delta = 84447248882562640537 - 36683761704646421875\sqrt{5}. \tag{102}$$

The least deviation of P_5^* from the zero function is given by

$$P_5^*(\pm 1) = b_0 + b_2 + 1, \tag{103}$$

which is the unique positive root of the minimal polynomial defined by

$$\begin{aligned} P_4(z) = {} & -11943936 - 693026816z + 13578720768z^2 - \\ & -85074300000z^3 + 192216796875z^4. \end{aligned} \tag{104}$$

REFERENCES

[1] N. I. ACHIESER, *Theory of Approximation*, Dover Publications, Mineola N.Y., 2003 (Russian 1947).

[2] B. C. CARLSON, J. TODD, *Zolotarev's first problem - the best approximation by polynomials of degree $\leq n-2$ to $x^n - n\sigma x^{n-1}$ in* $[-1,1]$, Aeq. Math., **26** (1983), pp. 1–33.

[3] G. E. COLLINS, *Application of quantifier elimination to Solotareff's approximation problem*, in: Stability Theory: Hurwitz Centenary Conference, (R. Jeltsch *et al.*, eds.), Ascona 1995, Birkhäuser Verlag, Basel, 1996 (ISNM **121**), 181–190.

[4] P. Erdös, G. Szegö, *On a problem of I. Schur*, Ann. Math. **43** (1942), pp. 451–470.

[5] S. R. Finch, *Mathematical Constants*, Cambridge University Press, Cambridge (UK), 2003 (EMA **94**).

[6] G. Grasegger, N. Th. Vo, *An algebraic-geometric method for computing Zolotarev polynomials*, Technical report no. 16-02, RISC Report Series, Johannes Kepler University, Linz, Austria, 2016, pp. 1–17. URL `http://www.risc.jku.at/publications/download/risc_5271/RISCReport1602.pdf`

[7] G. Grasegger, *An algebraic-geometric method for computing Zolotarev polynomials - Additional information*, Technical report no. 16-07, RISC Report Series, Johannes Kepler University, Linz, Austria, 2016, pp. 1–12. URL `http://www.risc.jku.at/publications/download/risc_5340/RISC1607.pdf`

[8] E. Kaltofen, *Challenges of symbolic computation: My favorite open problems. With an additional open problem by Robert M. Corless and David J. Jeffrey*, J. Symb. Comput. **29** (2000), pp. 891–919.

[9] D. Lazard, *Solving Kaltofen's challenge on Zolotarev's approximation problem*, in: Proceedings International Symposium on Symbolic and Algebraic Computation (ISSAC, Genoa, Italy, 2006), ACM, New York, 2006, pp. 196–203.

[10] V. I. Lebedev, *Zolotarev polynomials and extremum problems*, Russ. J. Numer. Anal. Math. Model. **9** (1994), pp. 231–263.

[11] V. A. Malyshev, *The Abel equation*, St. Petersburg Math. J. **13** (2002), pp. 893–938 (Russian 2001).

[12] A. A. Markov, *On a question of D. I. Mendeleev*, Zapiski. Imper. Akad. Nauk., St. Petersburg, **62** (1889), pp. 1–24 (Russian). URL `www.math.technion.ac.il/hat/fpapers/mar1.pdf`

[13] V. A. Markov, *On functions deviating least from zero in a given interval*, Izdat. Akad. Nauk., St. Petersburg, 1892, iv + 111 (Russian). URL `www.math.technion.ac.il/hat/fpapers/vmar.pdf`

[14] W. Markoff, *Über Polynome, die in einem gegebenen Intervalle möglichst wenig von Null abweichen*, Math. Ann. **77** (1916), pp. 213–258. (Abridged German translation of [13])

[15] G. V. Milovanović, D. S. Mitrinović, Th. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore, 1994.

[16] F. Peherstorfer, K. Schiefermayr, *Description of extremal polynomials on several intervals and their computation. II*, Acta Math. Hungar. **83** (1999), pp. 59–83.

[17] H.-J. Rack, *On polynomials with largest coefficient sums*, J. Approx. Theory **56** (1989), pp. 348–359.

[18] H.-J. Rack, *The first Zolotarev case in the Erdös-Szegö solution to a Markov-type extremal problem of Schur*, Stud. Univ. Babeş-Bolyai Math. **62** (2017), pp. 151–162.

[19] Th. J. Rivlin, *Chebyshev Polynomials*, 2nd ed., Wiley, New York, 1990.

[20] K. Schiefermayr, *Inverse polynomial images which consists of two Jordan arcs - An algebraic solution*, J. Approx Theory **148** (2007), pp. 148–157.

[21] I. Schur, *Über das Maximum des absoluten Betrages eines Polynoms in einem gegebenen Intervall*, Math. Z., **4** (1919), pp. 271–287.

[22] A. Shadrin, *Twelve proofs of the Markov inequality*, in: Approximation Theory: A volume dedicated to Borislav Bojanov (D. K. Dimitrov *et al.*, eds.), M. Drinov Acad. Publ. House, Sofia, 2004, pp. 233–298.

[23] A. Shadrin, *The Landau-Kolmogorov inequality revisited*, Discrete Contin. Dyn. Syst., **34** (2014), pp. 1183–1210.

[24] M. L. Sodin, P. M. Yuditskii, *Algebraic solution of a problem of E. I. Zolotarev and N. I. Akhiezer on polynomials with smallest deviation from zero*, J. Math. Sci. **76** (1995), pp. 2486–2492 (Russian 1991).

[25] E. I. Zolotarev, *Applications of elliptic functions to problems of functions deviating least and most from zero*, Zapiski Imper. Akad. Nauk, St.Petersburg, **30** (1877), Oeuvres vol. 2, pp. 1–59 (Russian). URL `http://www.math.technion.ac.il/hat/fpapers/zolo1.pdf`

[26] G. Graesegger, N. Th. Vo, *An algebraic-geometric method for computing Zolotarev polynomials*, in: Proceedings International Symposium on Symbolic and Algebraic Computation (ISSAC, Kaiserslautern, Germany, 2017), ACM, New York, 2017, pp. 173–180.

Note added in the proof. The conference paper [26], published in August 2017, is based on the technical reports [6], [7].

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