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BERNSTEIN OPERATORS OF SECOND KIND AND BLENDING SYSTEMS

DANIELA INOAN*, FADEL NASAIREH* and IOAN RAŞA*

Abstract. We consider the fundamental polynomials associated with the Bernstein operators of second kind. They form a blending system for which we study some shape preserving properties. Modified operators are introduced; they have better interpolation properties. The corresponding blending system is also studied.

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1. INTRODUCTION

A system $(f_0, f_1, \ldots, f_m)$ of nonnegative, continuous functions on an interval $[a, b]$ is said to be a *blending system* if

$$\sum_{i=1}^{m} f_i(x) = 1, \text{ for all } x \in [a, b].$$

If $P_0, P_1, \ldots, P_m \in \mathbb{R}^s$ are some given points, using a blending system one can define the curve

$$\gamma(t) = \sum_{i=1}^{m} f_i(t) P_i, \quad t \in [a, b].$$

The points $P_0, P_1, \ldots, P_m$ are called *control points* of the curve γ with respect to the blending system $(f_0, f_1, \ldots, f_m)$.

Blending systems are important instruments in Computer Aided Geometric Design (see [1], [2] and the references therein). Of particular interest are the blending systems for which some shape properties of the curve γ are inherited from the properties of the control polygon $P_0 P_1 \ldots P_m$. Bernstein and B-spline bases are well-known systems of totally positive blending functions that preserve monotonicity and convexity. In [1], [3] were given general results that connect shape preservation with the total positivity of the system of functions.

We will study some properties of two particular blending systems derived from an operator introduced by P. Soardi in [9].

*Technical University of Cluj-Napoca, Department of Mathematics, Memorandumului str. no. 28, Cluj-Napoca, Romania, e-mail: {daniela.inoan, Fadel.Nasaireh, Ioan.Rasa}@math.utcluj.ro.

Let $n \geq 1$, $n \in \mathbb{N}$ and $m = [n/2]$. The positive linear Soardi operator β_n is defined for a function $f \in C[0,1]$ and $x \in [0,1]$ by

$$\beta_n f(x) = \sum_{k=0}^{m} f\left(\tfrac{n-2m+2k}{n}\right) w_{n,k}(x),$$

where $w_{n,k}$ are the fundamental polynomials

$$w_{n,k}(x) =$$
$$= \tfrac{n+1-2m+2k}{(n+1)2^{n+1}x}\tbinom{n+1}{m-k}\Big((1-x)^{m-k}(1+x)^{n+1-m+k} - (1-x)^{n+1-m+k}(1+x)^{m-k}\Big).$$

Monotonicity or convexity preserving properties and a Voronovskaja-type formula for this operator can be found in [8]. In [7] there were given some inequalities for generalized convex functions that involve the operator β_n. A recursive de Casteljau type algorithm for this operator was described in [5].

Theorem 1. *The system* $(w_{n,0}(x), w_{n,1}(x), \ldots, w_{n,m}(x))$, $x \in [0,1]$ *is a blending system.*

Proof. It follows directly from the fact that $\beta_n 1 = 1$, which is proved in [9]. $\square$

2. TOTAL POSITIVITY

We recall (see [6], [4] for instance) that a system of functions $(u_0, \ldots, u_m)$ defined on an interval I is *totally positive (TP)* if for any $t_0, \ldots, t_p \in I$ with $t_0 < t_1 < \cdots < t_p$ the corresponding collocation matrix

$$M\begin{pmatrix} u_0, \ldots, u_m \\ t_0, \ldots, t_p \end{pmatrix} = (u_j(t_i))_{i=0,\ldots,p;j=0,\ldots,m} \tag{1}$$

has only nonnegative minors.

The system is *totally positive of order* r *(*TP_r*)*, $1 \leq r \leq m+1$ if for any collocation matrix (1), all the $k \times k$ minors, $k \in \{1, \ldots, r\}$ are nonnegative.

A system of functions $(u_0, \ldots, u_m)$ is a *Chebyshev system* if all its square collocation matrices $M\begin{pmatrix} u_0, \ldots, u_m \\ t_0, \ldots, t_m \end{pmatrix}$ have positive determinant. If the determinant is nonnegative, the system is *weak Chebyshev.*

Lemma 2. *Let* $0 < a_0 < a_1 < \cdots < a_m$, $m \in \mathbb{N}$. *The system*

$$(\sinh a_0 t, \sinh a_1 t, \ldots, \sinh a_m t)$$

is totally positive on $[0, \infty)$.

Proof. Let $0 \leq t_0 < t_1 < \cdots < t_m$. We study the minors of the corresponding collocation matrix.

For $s \in \{0, \ldots, m\}$ denote

$$M(s,t) = \sinh a_s t = \sum_{i=0}^{\infty} \tfrac{a_s^{2i+1} t^{2i+1}}{(2i+1)!}, \quad K(s,i) = \tfrac{a_s^{2i+1}}{(2i+1)!}, \quad E(i,t) = t^{2i+1}.$$

Let $\{s_1, \dots, s_k\} \subset \{0, \dots, m\}$ be row indices and $\{s'_1, \dots, s'_k\} \subset \{0, \dots, m\}$ column indices in a minor of the collocation matrix. Then, using the basic composition formula (see for instance [6]) we have:

$$\begin{vmatrix} \sinh a_{s_1} t_{s'_1} & \dots & \sinh a_{s_k} t_{s'_1} \\ \dots & \dots & \dots \\ \sinh a_{s_1} t_{s'_k} & \dots & \sinh a_{s_k} t_{s'_k} \end{vmatrix} =$$

$$= M \begin{pmatrix} s_1, \dots, s_k \\ t_{s'_1}, \dots, t_{s'_k} \end{pmatrix}$$

$$= \sum_{0 \le i_1 < \dots < i_k < \infty} K \begin{pmatrix} s_1, \dots, s_k \\ i_1, \dots, i_k \end{pmatrix} E \begin{pmatrix} i_1, \dots, i_k \\ t_{s'_1}, \dots, t_{s'_k} \end{pmatrix}$$

$$= \sum_{0 \le i_1 < \dots < i_k < \infty} \tfrac{1}{(2i_1+1)! \dots (2i_k+1)!} \begin{vmatrix} a_{s_1}^{2i_1+1} & \dots & a_{s_k}^{2i_1+1} \\ \dots & \dots & \dots \\ a_{s_1}^{2i_k+1} & \dots & a_{s_k}^{2i_k+1} \end{vmatrix} \cdot \begin{vmatrix} t_{s'_1}^{2i_1+1} & \dots & t_{s'_k}^{2i_1+1} \\ \dots & \dots & \dots \\ t_{s'_1}^{2i_k+1} & \dots & t_{s'_k}^{2i_k+1} \end{vmatrix}$$

$$\ge 0.$$

□

The next property was mentioned in [8]; we give here a more detailed proof.

THEOREM 3. *The system* $(w_{n,0}, \dots, w_{n,m})$ *is totally positive on* $[0,1]$.

Proof. We have

$$w_{n,k}(x) =$$

$$= \tfrac{n+1-2m+2k}{(n+1)2^{n+1}} \tbinom{n+1}{m-k} \tfrac{1}{x} (1-x^2)^{\frac{n+1}{2}} \left[\left(\tfrac{1+x}{1-x} \right)^{\frac{n+1}{2}-m+k} - \left(\tfrac{1+x}{1-x} \right)^{-\frac{n+1}{2}+m-k} \right]$$

so with the exception of a strictly positive factor, the system is

$$\left((e^t)^{\frac{n+1}{2}-m} - (e^t)^{-\frac{n+1}{2}+m}, \dots, (e^t)^{\frac{n+1}{2}} - (e^t)^{-\frac{n+1}{2}} \right)$$

where we used the notation $\frac{1+x}{1-x} = e^t$. We get $t = \log \frac{1+x}{1-x}$, an increasing function on $[0, \infty)$ and the system

$$(\mathcal{S}) \quad (\sinh a_0 t, \sinh a_1 t, \dots, \sinh a_m t),$$

with $a_j = \frac{n+1}{2} - m + j$.

From Lemma 2 we have that the system $(\mathcal{S})$ is totally positive and according to [4], page 161, this implies the total positivity of $(w_{n,0}, \dots, w_{n,m})$. □

THEOREM 4. *For each* $j = 0, 1, \dots, m$,

$$(S_j) \quad (w_{n,j}(x), w_{n,j+1}(x), \dots, w_{n,m}(x))$$

is a Chebyshev system on $[0,1]$.

Proof. Let $x \in (0,1)$. As in the previous proof, we see that instead of (S_j) is enough to study the system $(\sinh a_j t, \sinh a_{j+1} t, \dots, \sinh a_m t)$, with $0 < a_j < \cdots < a_m$ and $t \in (0,\infty)$. For any $0 < t_j < t_{j+1} < \cdots < t_m$, the determinant $\Delta = \begin{vmatrix} \sinh a_j t_j & \dots & \sinh a_m t_j \\ \dots & \dots & \dots \\ \sinh a_j t_m & \dots & \sinh a_m t_m \end{vmatrix}$ can be written like in the proof of Lemma 2 and we obtain that it is strictly positive.

Let $x \in (0,1]$ and consider $0 < x_j < \cdots < x_m \le 1$. The situation differs from the previous case only if $x_m = 1$. Then

$$\begin{vmatrix} w_{n,j}(x_j) & w_{n,j+1}(x_j) & \dots & w_{n,m}(x_j) \\ \dots & \dots & \dots & \dots \\ w_{n,j}(1) & w_{n,j+1}(1) & \dots & w_{n,m}(1) \end{vmatrix} =$$

$$= \begin{vmatrix} w_{n,j}(x_j) & w_{n,j+1}(x_j) & \dots & w_{n,m}(x_j) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{vmatrix}$$

$$= \begin{vmatrix} w_{n,j}(x_j) & w_{n,j+1}(x_j) & \dots & w_{n,m-1}(x_j) \\ \dots & \dots & \dots & \dots \\ w_{n,j}(x_{m-1}) & w_{n,j+1}(x_{m-1}) & \dots & w_{n,m-1}(x_{m-1}) \end{vmatrix} > 0,$$

like in the previous case.

Let $x \in [0,1)$ and consider $0 \le x_j < \cdots < x_m < 1$. The situation differs from the first case only if $x_1 = 0$. Then, instead of Δ we will have, with $0 = t_j < t_{j+1} < \cdots < t_m$,

$$\begin{vmatrix} \frac{\sinh a_j t_j}{t_j} & \dots & \frac{\sinh a_m t_j}{t_j} \\ \sinh a_j t_{j+1} & \dots & \sinh a_m t_{j+1} \\ \dots & \dots & \dots \\ \sinh a_j t_m & \dots & \sinh a_m t_m \end{vmatrix} = \sum_{0 \le i_j < \cdots < i_m < \infty} \tfrac{1}{(2i_j+1)!} \cdots \tfrac{1}{(2i_m+1)!} \Delta_1 \cdot \Delta_2 > 0,$$

since we have $\Delta_1 = \begin{vmatrix} a_j^{2i_j+1} & \dots & a_m^{2i_j+1} \\ \dots & \dots & \dots \\ a_j^{2i_m+1} & \dots & a_m^{2i_m+1} \end{vmatrix} > 0,$

$\Delta_2 = \begin{vmatrix} t_j^{2i_j} & t_{j+1}^{2i_j+1} & \dots & t_m^{2i_j+1} \\ \dots & \dots & \dots & \dots \\ t_j^{2i_m} & t_{j+1}^{2i_m+1} & \dots & t_m^{2i_m+1} \end{vmatrix} = 0$ for $i_j > 0$ and $\Delta_2 > 0$ for $i_j = 0$.

The case $x \in [0,1]$, with $0 = x_j < x_{j+1} < \cdots < x_m = 1$ can be treated as a combination of the previous three cases. □

3. SHAPE PRESERVING PROPERTIES

A blending system $(f_0, \dots, f_m)$ is said to be *monotonicity preserving* [1] if $\alpha_0 \le \alpha_1 \le \cdots \le \alpha_m$ implies that $\sum_{i=0}^m \alpha_i f_i(t)$ is an increasing function.

The system is said to be *strictly monotonicity preserving* if $\alpha_0 < \alpha_1 < \cdots < \alpha_m$ implies that $\sum_{i=0}^m \alpha_i f_i(t)$ is strictly increasing.

In the case of a monotonicity preserving blending system, if the projections of the control points $P_0, \ldots, P_m$ onto a line are ordered, then the projections of the points of the curve γ on the same line are also ordered.

LEMMA 5. [1, Corollary 2.4] *A totally positive blending system is monotonicity preserving.*

This allows us to state, following directly from Theorem 3:

THEOREM 6. *Soardi's blending system $(w_{n,0}, \ldots, w_{n,m})$ is monotonicity preserving.*

LEMMA 7. [1, Proposition 2.1] *Let $(f_0, \ldots, f_m)$ be a blending system on $[a, b]$ and let*

$$g_i = \sum_{j=i}^{m} f_j, \quad i = 0, \ldots, m.$$

Then:

a) $(f_0, \ldots, f_m)$ is monotonicity preserving if and only if $g_1, \ldots, g_m$ are increasing functions.

b) $(f_0, \ldots, f_m)$ is strictly monotonicity preserving if and only if $g_1, \ldots, g_m$ are increasing functions and $\sum_{i=1}^{m} g_i$ is strictly increasing.

THEOREM 8. *Soardi's blending system $(w_{n,0}, \ldots, w_{n,m})$ is strictly monotonicity preserving.*

Proof. According to Theorem 6 and Lemma 7 we have to prove only that the function $\sum_{i=1}^{m} g_i = \sum_{i=1}^{m} i w_{n,i}$ is strictly increasing.

It was proved in [8, Theorem 2.1], that if a function $f \in C[0, 1]$ is strictly increasing then $\beta_n f$ is also strictly increasing.

Let $f \in C[0, 1]$ be such that $f\left(\frac{n-2m+2k}{n}\right) = k$, for $k = 0, \ldots, m$ and such that it is strictly increasing. Then also

$$\sum_{i=1}^{m} i w_{n,i} = \sum_{i=1}^{m} f\left(\frac{n-2m+2i}{n}\right) w_{n,i} = \beta_n f$$

is strictly increasing. □

As a consequence of the property of monotonicity preserving, the blending system $(w_{n,0}, \ldots, w_{n,m})$ is also length diminishing (see [1]), that is the length of the curve $\sum_{i=0}^{m} P_i w_{n,i}$ is smaller then the length of the control polygon $P_0 P_1 \ldots P_m$.

The fact that Soardi's blending system is totally positive and a Chebyshev system implies also some other properties: geometrically (strictly) convexity preserving (see [1, Theorems 3.9 and 3.10]) or geometrically (strictly) k-convexity preserving (see [1, Proposition 5.7 and 5.8]).

4. A MODIFIED SOARDI OPERATOR

Starting from Soardi's operator we define $\Gamma_n : C[0,1] \to C[0,1]$ by

$$\Gamma_n f = \frac{e_1}{\beta_n e_1}\beta_n f + \left(1 - \frac{e_1}{\beta_n e_1}\right) f(0).$$

This is a positive linear operator. It easy to prove, by direct computation, that $\Gamma_n e_0 = e_0$, $\Gamma_n e_1 = e_1$ and $\Gamma_n f(0) = f(0)$, $\Gamma_n f(1) = f(1)$ for every $f \in C[0,1]$. To obtain a blending system we consider two situations: n being even or odd.

For $n = 2m$, we have

$$\Gamma_n f(x) = \sum_{k=0}^{m} f\left(\frac{n-2m+2k}{n}\right)\gamma_{n,k}(x),$$

where

$$\begin{aligned}\gamma_{n,0}(x) &= \frac{x}{\beta_n e_1(x)}\left(w_{n,0}(x) - 1\right) + 1,\\ \gamma_{n,k}(x) &= \frac{x}{\beta_n e_1(x)} w_{n,k}(x), \quad \text{for } k = 1, 2, \ldots, m.\end{aligned}$$

For $n = 2m+1$, we have

$$\Gamma_n f(x) = f(0)\theta_{n,0}(x) + \sum_{k=0}^{m} f\left(\frac{n-2m+2k}{n}\right)\theta_{n,k+1}(x),$$

where

$$\begin{aligned}\theta_{n,0}(x) &= -\frac{x}{\beta_n e_1(x)} + 1,\\ \theta_{n,k+1}(x) &= \frac{x}{\beta_n e_1(x)} w_{n,k}(x), \quad \text{for } k = 0, 1, \ldots, m.\end{aligned}$$

THEOREM 9. *For $n = 2m$ the system $(\gamma_{n,0}(x), \gamma_{n,1}(x), \ldots, \gamma_{n,m}(x))$, $x \in [0,1]$ is a blending system.*

For $n = 2m+1$ the system $(\theta_{n,0}(x), \theta_{n,1}(x), \ldots, \theta_{n,m+1}(x))$, $x \in [0,1]$ is a blending system.

Proof. It follows directly from $\Gamma_n e_0 = e_0$. □

LEMMA 10. *For any $n \in \mathbb{N}$, the function $g : [0,1] \to \mathbb{R}$, $g(x) = \frac{x}{\beta_n e_1(x)}$ is strictly increasing.*

Proof. Denote $h(x) = \beta_n e_1(x) - x\beta_n' e_1(x)$. We have $h'(x) = -x\beta_n'' e_1(x) \leq 0$, since $\beta_n e_1(x)$ is a convex function (see [8, Theorem 2.1]). Thus h is decreasing on $[0,1]$. By direct computation we get $h(1) = \frac{1}{n} > 0$, which implies $h(x) > 0$ for any $x \in [0,1]$. Finally the conclusion follows from $g'(x) = \frac{h(x)}{(\beta_n e_1(x))^2} > 0$. □

THEOREM 11. *a) For any even number $n = 2m$, the blending system $(\gamma_{n,0}(x), \gamma_{n,1}(x), \ldots, \gamma_{n,m}(x))$ is strictly monotonicity preserving.*

b) For any odd number $n = 2m+1$, the blending system

$$(\theta_{n,0}(x), \theta_{n,1}(x), \ldots, \theta_{n,m+1}(x))$$

is strictly monotonicity preserving.

Proof. We will use Lemma 7. For $n = 2m$ and $i = 1, \dots, m$ we have

$$g_i = \sum_{j=i}^{m} \gamma_{n,j}(x) = \frac{x}{\beta_n e_1(x)} \sum_{j=i}^{m} w_{n,j}(x)$$

By the previous lemma, the function $\frac{x}{\beta_n e_1(x)}$ is strictly increasing. Also the function $\sum_{j=i}^{m} w_{n,j}(x)$ is strictly increasing since $(w_{n,0}, \dots, w_{n,m})$ is monotonicity preserving. So g_i is increasing and Lemma 7 gives the monotonicity preserving property of $(\gamma_{n,0}(x), \gamma_{n,1}(x), \dots, \gamma_{n,m}(x))$.

Moreover, the function

$$\sum_{i=1}^{m} g_i = \frac{x}{\beta_n e_1(x)} \sum_{i=1}^{m} i w_{n,i}(x)$$

is strictly increasing, so the blending system is strictly monotonicity preserving.

The case $n = 2m + 1$ can be treated in the same way. $\square$

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