JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 45 (2016) no. 2, pp. 109–127
ictp.acad.ro/jnaat

AN IMPROVED SEMILOCAL CONVERGENCE ANALYSIS FOR THE MIDPOINT METHOD

IOANNIS K. ARGYROS* and SANJAY K. KHATTRI†

Abstract. We expand the applicability of the midpoint method for approximating a locally unique solution of nonlinear equations in a Banach space setting. Our majorizing sequences are finer than the known results in scientific literature [1, 3, 4, 10–16, 24–26, 28] and the convergence criteria can be weaker. Finally, numerical work is reported that compares favorably to the existing approaches in the literature [6, 8–16, 24–26,28].

MSC 2010. 65B05, 65G99, 65J15, 65N30, 65N35, 65H10, 47H17, 49M15.
Keywords. Midpoint method, semilocal convergence, majorization sequence, Banach space, Fréchet-derivative.

1. INTRODUCTION

In this study, we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$\mathcal{F}(x) = 0, \tag{1.1}$$

where, $\mathcal{F}$ is a twice Fréchet differentiable operator defined on a convex subset $\mathbf{D}$ of a Banach space $\mathbf{X}$ with values in a Banach space $\mathbf{Y}$. Numerous problems in science and engineering can be reduced to solving the above equation [23, 32]. Consequently, solving these equations is an important scientific field of research. In many situations, finding a closed form solution for the non-linear equation (1.1) is not possible. Therefore, iterative solution techniques are employed for solving these equations.

The study about convergence analysis of iterative methods is usually divided into two categories: semi-local and local convergence analysis. The semilocal convergence analysis is based upon the information around an initial point to give criteria ensuring the convergence of the iterative procedure. While the local convergence analysis is based on the information around a solution to

*Department of Mathematical Sciences, Cameron University, Lawton, Oklahoma 73505-6377, USA, e-mail: iargyros@cameron.edu.
†Department of Engineering, Stord Haugesund University College, Norway, e-mail: sanjay.Khattri@hsh.no.

find estimates of the radii of convergence balls. In this paper, we study the semilocal convergence of the midpoint method defined as

$$(1.2)\qquad \begin{aligned} y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n),\\ x_{n+1} &= x_n - \mathcal{F}'\Big(\tfrac{x_n+y_n}{2}\Big)^{-1}\mathcal{F}(x_n), \quad \text{for each} \quad n = 0, 1, 2, \dots, \end{aligned}$$

where $x_0 \in \mathbf{D}$ is an initial point. Here, $\mathcal{F}'(x)$ denotes the first Fréchet-derivative of the operator $\mathcal{F}$ [23, 32]. It is well-known that the Midpoint method is cubically convergent and it has a long history [see 27–32]. Let $U(w, R)$ and $\overline{U}(w, R)$ stand, respectively, for the open and closed balls in $\mathbf{X}$ with center w and radius $R > 0$. Let the space of bounded linear operators from $\mathbf{X}$ into $\mathbf{Y}$ be denoted by $L(\mathbf{X}, \mathbf{Y})$. The following set of ($\mathbf{C}$) conditions have been used

(1) There exists $x_0 \in \mathbf{D}$ such that $F'(x_0)^{-1} \in L(\mathbf{Y}, \mathbf{X})$.
(2) $\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\| \le \eta$.
(3) $\|\mathcal{F}'(x_0)^{-1}\mathcal{F}''(x)\| \le L$ for all $x \in \mathbf{D}$.
(4) $\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}''(x) - \mathcal{F}''(y))\| \le \mathcal{M}\,\|x - y\|$, for all $x, y \in \mathbf{D}$.

The following sufficient convergence criteria have been given in connection to the ($\mathbf{C}$) conditions

$$(1.3)\qquad \eta \le \frac{4\mathcal{M} + L^2 - L\sqrt{L^2 + 2\mathcal{M}}}{3\mathcal{M}(L + \sqrt{L^2 + 2\mathcal{M}})} \quad [1, 3, 4, 23\text{–}26]$$

or

$$(1.4)\qquad \eta \le \frac{1}{2\mathcal{K}} \quad [12, 14],$$

where

$$\mathcal{K} = L\sqrt{1 + \frac{7\,\mathcal{M}}{6\,L^2}}.$$

However, simple numerical examples can be used to show that criteria (1.3) and (1.4) are unsatisfied but the midpoint method (1.2) still converges to the solution $x^\star$. As an example, let $\mathbf{X} = \mathbf{Y} = \mathbb{R}$, $x_0 = 1$ and $\mathbf{D} = [\zeta, 2 - \zeta]$ for $\zeta \in (0, 1)$. Define function $\mathcal{F}$ on $\mathbf{D}$ by

$$(1.5)\qquad \mathcal{F}(x) = x^5 - \zeta.$$

Then, using conditions ($\mathbf{C}$), we get

$$\eta = \frac{(1-\zeta)}{5}, \quad L = 4(2-\zeta)^3, \quad \mathcal{M} = 12(2-\zeta)^2.$$

Figure 1.1 plots the criteria (1.3) and (1.4) for the problem (1.5). The curve (defined by the right hand side of the inequality (1.3)) intersect the line η (see Figure 1.1) at $\zeta \approx 0.73$ while the curve (defined by the right hand side of the criteria (1.4)) intersect the η line at $\zeta \approx 0.72$. We notice in the Figure 1.1 that for $\zeta < 0.72$ the criteria (1.3) and (1.4) are not satisfied. However, one

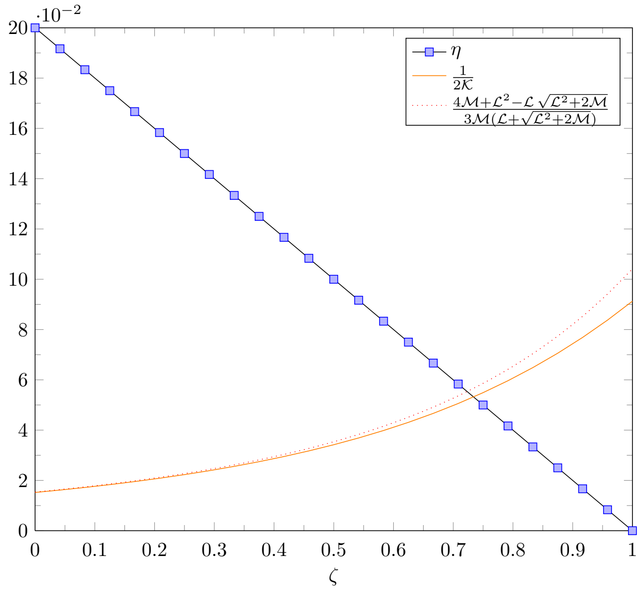

Fig. 1.1. Convergence criteria (1.3) and (1.4) for (1.5).

may see that the method (1.2) is convergent. For additional examples, see the Section 4.

In our work we expand the applicability of the midpoint method (1.2), in cases where (1.3) or (1.4) are not satisfied, using the (**C**) conditions together with the following center Lipschitz condition

(1) $\left\|\mathcal{F}'(x_0)^{-1}\left(\mathcal{F}'(x)-\mathcal{F}'(x_0)\right)\right\| \leq L_0\left\|x-x_0\right\|$ for all $x \in \mathbf{D}$.

We shall refer to ($\mathbf{C}_1$)-($\mathbf{C}_5$) as the (**H**) conditions.

As can be inferred from the studies [1–28], several techniques are usually employed for analyzing the convergence of iterative methods. Among these, the most popular technique is based on the concept of majorizing sequences. In the studies that lead to the convergence conditions (1.3) and (1.4) the computation of the upper bounds on $\left\|\mathcal{F}'(x_n)^{-1}\mathcal{F}'(x_0)\right\|$ was based on ($\mathbf{C}_3$) and the estimate

$$\left\|\mathcal{F}'(x_n)^{-1}\mathcal{F}'(x_0)\right\| \leq \frac{1}{1-L\left\|x_n-x_0\right\|}. \tag{1.6}$$

Instead of ($\mathbf{C}_3$) we use the more precise and less expensive condition ($\mathbf{C}_5$) which leads to

$$\left\|\mathcal{F}'(x_n)^{-1}\mathcal{F}'(x_0)\right\| \leq \frac{1}{1-L_0\left\|x_n-x_0\right\|}. \tag{1.7}$$

Note that

$$L_0 \leq L \tag{1.8}$$

holds in general and L/L_0 can be arbitrarily large [22, 23]. This change in the study of the semilocal convergence of the midpoint method leads to tighter error estimates on the distances $\|y_n - x_n\|$, $\|x_{n+1} - y_n\|$, $\|y_n - x^\star\|$, $\|x_n - x^\star\|$ and weaker convergence criteria.

The rest of the paper is organized as follows. Section 2 develop results on majorizing sequences for the midpoint method (1.2), where as in the Section 3 we present the semilocal convergence of the Midpoint method. Numerical examples are given in the concluding section 4.

2. MAJORIZING SEQUENCES

In this section, we study the convergence of scalar sequences that are majorizing for the Midpoint method (1.2). Let the positive constants be $L_0 > 0$, $L > 0$, $\mathcal{M} \geq 0$ and $\eta > 0$. It is convenient for us to define functions γ, a, α, h_i, $i = 1, 2, 3$ by

$$\gamma(t) = \frac{L\,t}{2\left[1 - \frac{L_0\,t}{2}\right]}, \quad \gamma = \gamma(\eta), \tag{2.1}$$

$$a(t) = \tfrac{1}{24}\left(12L\gamma(t)^2 + 12L\gamma(t) + 7\mathcal{M}\eta\right), \quad a = a(\eta), \tag{2.2}$$

$$\alpha(t) = \frac{a(t)t}{\left[1 - \frac{L_0}{2}(1+\gamma(t))t\right]}, \quad \alpha = \alpha(\eta), \tag{2.3}$$

$$h_1(t) = a(t)t + \tfrac{L_0}{2}(1+\gamma(t))t - 1 \tag{2.4}$$

$$h_2(t) = \tfrac{L}{2}\alpha(t)\,t + \tfrac{\gamma(t)L_0}{2}\left[2(1+\gamma(t)) + \alpha(t)\right]t - \gamma(t) \tag{2.5}$$

and

$$h_3(t) = a(t)t + L_0(1+\gamma(t))(1+\alpha(t))t - 1. \tag{2.6}$$

We denote the minimal positive zeros of the functions h_1, h_2 and h_3 by η_1, η_2 and η_3, respectively. Note that $\alpha(t)$ is well defined on $(0, \eta_1)$ by the choice of η_1. Let us set

$$\eta_0 = \min\{\eta_1, \eta_2, \eta_3\}. \tag{2.7}$$

Then, for all $t \in (0, \eta_0)$ we have

$$\alpha \in (0,1), \tag{2.8}$$

$$h_1(t) < 0, \tag{2.9}$$

$$h_2(t) \leq 0 \tag{2.10}$$

and

$$h_3(t) \leq 0. \tag{2.11}$$

We can show the following result on the convergence of majorizing sequences for the Midpoint method.

Lemma 2.1. *Let the positive constants be* $L_0 > 0$, $L > 0$, $\mathcal{M} \geq 0$ *and* $\eta > 0$. *Suppose that*

$$\eta \begin{cases} \leq \eta_0 & \text{if} \quad \eta_0 \neq \eta_1, \\ < \eta_0 & \text{if} \quad \eta_0 = \eta_1. \end{cases} \tag{2.12}$$

Then, scalar sequence $\{t_n\}$ *generated by*

(2.13)

$$t_0 = 0, \quad s_0 = \eta, \quad t_{n+1} = s_n + \frac{L(s_n - t_n)^2}{2\left[1 - \frac{L_0}{2}(s_n + t_n)\right]},$$

$$s_{n+1} = t_{n+1} + \frac{12L(t_{n+1} - s_n)^2 + \frac{6L^2(s_n - t_n)^3}{1 - \frac{L_0}{2}(t_n + t_n)}(s_n - t_n)^3 + 7\mathcal{M}(s_n - t_n)^3}{24(1 - L_0\, t_{n+1})}$$

is increasing, bounded from above by

$$t^{\star\star} = \left(\frac{1+\gamma}{1-\alpha}\right)\eta \tag{2.14}$$

and converges to its unique least upper bound $t^\star$ *which satisfies*

$$0 \leq t^\star \leq t^{\star\star}. \tag{2.15}$$

Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$

$$0 < t_{n+1} - s_n \leq \gamma(s_n - t_n) \leq \gamma\alpha^n\eta \tag{2.16}$$

and

$$0 < s_{n+1} - t_{n+1} \leq \alpha(s_n - t_n) \leq \alpha^{n+1}\eta. \tag{2.17}$$

Proof. We use mathematical induction to prove (2.16) and (2.17). Estimates (2.16) and (2.17) hold for $n = 0$ by (2.1)-(2.3) and (2.13), since

$$\begin{aligned} s_1 - t_1 &= \frac{12L(t_1 - s_0)^3 + \frac{6L^2(s_0 - t_0)^3}{1 - \frac{L_0}{2}(t_0 + s_0)} + 7\mathcal{M}(s_0 - t_0)^3}{24(1 - L_0\, t_1)} \\ &\leq \frac{12L\gamma^2 + 12L\gamma + 7\mathcal{M}(s_0 - t_0)}{24\left[1 - \frac{L_0}{2}(1+\gamma)\eta\right]}(s_0 - t_0)^2 \\ &\leq \frac{a}{1 - \frac{L_0}{2}(1+\gamma)}(s_0 - t_0)(s_0 - t_0) \leq \alpha(s_0 - t_0) = \alpha\eta. \end{aligned} \tag{2.18}$$

Let us assume that (2.16) and (2.17) hold for all $k \leq \eta$. Then, we have

$$t_{k+1} - s_k \leq \gamma(s_k - t_k) \leq \gamma\alpha^k\eta,$$
$$s_{k+1} - t_{k+1} \leq \alpha(s_k - t_k) \leq \alpha^{k+1}\eta,$$

$$
\begin{aligned}
t_{n+1} &\le s_k + \gamma\alpha^k\eta \le t_k + \alpha^k\eta + \gamma\alpha^k\eta \\
&\le t_{k-1} + \alpha^{k-1}\eta + \alpha^k\eta + \gamma\alpha^{k-1}\eta + \gamma\alpha^k\eta \\
&\le \cdots \le t_2 + (\alpha^2\eta + \alpha^3\eta + \cdots + \alpha^k\eta) + (\gamma\alpha^2\eta + \cdots + \gamma\alpha^k\eta) \\
&\le s_1 + \gamma\alpha\eta + (\alpha^2\eta + \alpha^3\eta + \cdots + \alpha^k\eta) + (\gamma\alpha^2\eta + \cdots + \gamma\alpha^k\eta) \\
&\le t_1 + \alpha\eta + \gamma\alpha\eta + (\alpha^2\eta + \alpha^3\eta + \cdots + \alpha^k\eta) + (\gamma\alpha^2\eta + \cdots + \gamma\alpha^k\eta) \\
&\le \eta + \gamma\eta + \alpha\eta + \gamma\alpha\eta + (\alpha^2\eta + \cdots + \alpha^k\eta) + (\gamma\alpha^2\eta + \cdots + \gamma\alpha^k\eta) \\
&\le \tfrac{1-\alpha^{k+1}}{1-\alpha}(1+\gamma)\eta < \tfrac{1+\gamma}{1-\alpha}\eta = t^{\star\star}, \qquad (2.19)
\end{aligned}
$$

and

$$
\begin{aligned}
s_{k+1} \le t_{k+1} + \alpha^{k+1}\eta &\le \left[\tfrac{1-\alpha^{k+1}}{1-\alpha}(1+\gamma) + \alpha^{k+1}\right]\eta \\
&< \left(\tfrac{1+\gamma}{1-\gamma} + \alpha^{k+1}\right)\eta \le t^{\star\star}.
\end{aligned}
$$

Evidently, estimates (2.16) and (2.17) are true provided that

$$\frac{L(s_k - t_k)}{2\left(1 - \frac{L_0}{2}(s_k + t_k)\right)} \le \gamma \tag{2.20}$$

and

$$\frac{a(s_k - t_k)}{2(1 - L_0 t_{k+1})} \le \alpha. \tag{2.21}$$

Inequality (2.20) can be written as

$$\tfrac{L\alpha^k\eta}{2} + \tfrac{\gamma L_0}{2}\left(2\tfrac{1-\alpha^k}{1-\alpha}(1+\gamma) + \alpha^k\right)\eta - \gamma \le 0. \tag{2.22}$$

Estimate (2.22) motivates us to define recurrent functions f_k on $[0,1)$ for each $k = 1, 2, \ldots$ by

$$f_k(t) = \tfrac{L\,t^k\eta}{2} + \tfrac{\gamma L_0}{2}\left(2\tfrac{1-t^k}{1-t}(1+\gamma) + t^k\right)\eta - \gamma. \tag{2.23}$$

We need a relationship between two consecutive functions f_k. We have by (2.23)

$$
\begin{aligned}
f_{k+1}(t) &= f_k(t) + \tfrac{L\,t^{k+1}\eta}{2} - \tfrac{L\,t^k\eta}{2} + \tfrac{\gamma L_0}{2}\left(2(1+\gamma)(t^k - t^{k-1}) + t^{k+1} - t^k\right)\eta \\
&= f_k(t) + (t-1)\left[\tfrac{L}{2}t + \gamma\,L_0(1+\gamma) + \tfrac{\gamma\alpha L_0}{2}\right]t^{k-1}\eta. \qquad (2.24)
\end{aligned}
$$

It follows from (2.24) that

$$f_{k+1}(t) \le f_k(t) \le \cdots \le f_1(t). \tag{2.25}$$

In view of (2.25), estimate (2.22) holds if

$$f_1(\alpha) \le 0 \tag{2.26}$$

which is true by the choice of η_2. Similarly, estimate (2.21) can be written as

$$a\alpha^k\eta + \alpha L_0(1+\gamma)\left(\tfrac{1-\alpha^{k+1}}{1-\alpha}\right)\eta - \alpha \le 0. \tag{2.27}$$

Define recurrent functions g_k on $[0,1)$ for each $k = 1, 2, \dots$ by

$$g_k(t) = at^{k-1}\eta + L_0(1+\gamma)\left(\frac{1-t^{k+1}}{1-t}\right)\eta - 1. \tag{2.28}$$

Then using (2.28) we get

$$g_{k+1}(t) = g_k(t) + (t-1)\left[a + L_0(1+t)t\right]t^{k-1}\eta. \tag{2.29}$$

Hence, it follows from (2.29) that

$$g_{k+1}(t) \le g_k(t) \le \cdots \le g_1(t). \tag{2.30}$$

In view of (2.30), instead of (2.27), we can show that

$$g_1(\alpha) \le 0, \tag{2.31}$$

which is true by the choice of η_3. The induction for (2.16) and (2.17) is complete. Hence, sequence $\{t_n\}$ is an increasing, bounded from above by $t^{\star\star}$ and as such it converges to its unique least upper bound $t^\star$. The proof of the Lemma is complete. □

We have the following useful and obvious extension of Lemma 2.1

Lemma 2.2. *Suppose there exists $N \ge 0$ such that*

$$t_0 < s_0 < t_1 < \cdots < t_N < s_N < t_{N+1} < \frac{1}{L_0}. \tag{2.32}$$

and

$$s_N - t_N \begin{cases} \le \eta_0 & \text{if} \quad \eta_0 \ne \eta_1 \\ < \eta_0 & \text{if} \quad \eta_0 = \eta_1. \end{cases} \tag{2.33}$$

Then, the conclusions of the Lemma 2.1 hold for sequence $\{t_n\}$. Moreover, the following estimates hold for each $n = 0, 1, 2, 3, \dots$

$$0 < t_{N+1+n} - s_{N+n} \le \gamma_N(s_{N+n} - t_{N+n}) \tag{2.34}$$

and

$$0 < s_{N+1+n} - t_{N+1+n} \le \alpha_N(s_{N+n} - t_{N+n}) \tag{2.35}$$

where $\gamma_N = \gamma(s_N - t_N)$, $\alpha_N = \alpha(s_N - t_N)$ and $t_N^{\star\star} = \frac{1+\gamma_N}{1-\alpha_N}(s_N - t_N)$.

Remark 2.3.

(1) Note that for $N = 0$, the Lemma 2.2 reduces to Lemma 2.1 with $\alpha_0 = \alpha$ and $\gamma_0 = \gamma$.

(2) Let us define sequences $\{r_n\}$ and $\{v_n\}$ by

(2.36)
$$r_0 = 0, \quad q_0 = \eta, \quad r_1 = q_0 + \frac{\mathcal{K}_0 (q_0 - r_0)^2}{2\,(1 - L_3 \eta/2)},$$
$$q_1 = r_1 + \frac{\left(12 L_1 (r_1 - q_0)^2 + \frac{6 L_2 L_2' (q_0 - r_0)^3}{1 - L_0\, r_0/2} + 7 \mathcal{M}_0 (q_0 - r_0)^3\right)}{24 (1 - L_3 r_1)},$$
$$r_{n+1} = q_n + \frac{L (q_n - r_n)^2}{2\left[1 - \frac{L_0}{2}(q_n + r_n)\right]},$$
$$q_{n+1} = r_{n+1} + \frac{\left(12 L (r_{n+1} - q_n)^2 + \frac{6 L^2 (q_n - r_n)^3}{1 - L_0 (r_n + q_n)/2} + 7 \mathcal{M} (r_n - q_n)^3\right)}{24 (1 - L_0 r_{n+1})} \qquad (n \geq 1)$$

for some L_0, L_1, L_2, L_3, $\mathcal{K}_0$, $\mathcal{M}_0$ such that

(2.37) $$L_1 \leq L, \quad L_2 \leq L, \quad L_2' \leq L, \quad \mathcal{K}_0 \leq L, \quad L_3 \leq L_0 \quad \text{and} \quad \mathcal{M}_0 \leq \mathcal{M}$$

and

(2.38)
$$v_0 = 0, \quad u_0 = \eta, \quad v_{n+1} = u_n + \frac{L (u_n - v_n)^2}{2\left[1 - \frac{L_0}{2}(v_n + u_n)\right]},$$
$$u_{n+1} = v_{n+1} + \frac{12 L (v_{n+1} - u_n)^2 + \frac{6 L^2 (u_n - v_n)^3}{1 - \frac{L_0}{2}(v_n + u_n)} + 7 \mathcal{M} (u_n - v_n)^3}{24 (1 - L v_{n+1})}.$$

In view of (1.8), (2.13), (2.36), (2.37)–(2.38) a simple inductive argument shows that

(2.39) $$r_n \leq t_n \leq v_n$$

(2.40) $$q_n \leq s_n \leq u_n,$$

(2.41) $$r_{n+1} - q_n \leq s_{n+1} - t_n \leq u_{n+1} - v_n,$$

(2.42) $$q_{n+1} - r_{n+1} \leq s_{n+1} - t_{n+1} \leq u_{n+1} - v_{n+1}$$

and

(2.43) $$r^\star = \lim_{n \to \infty} r_n \leq t^\star \leq v^\star = \lim_{n \to \infty} v_n.$$

Moreover, (2.39)-(2.42) hold as strict inequalities for $n \geq 1$ if (1.8) and (2.37) hold as strict inequalities. Sequence $\{v_n\}$ was shown to be majorizing for the Midpoint method (1.2) provided that (1.3) or (1.4) hold [1, 3, 4, 10–16, 24–26, 28]. We shall prove in the next section that tighter sequences $\{r_n\}$ and $\{v_n\}$ are also majorizing for the Midpoint method (1.2). Then, certainly these majorizing sequences also converge under (1.3) or (1.4). However, these sequences converge under the new convergence criteria given in the Lemma 2.1 which can be weaker that (1.3) or (1.4) (see Section 4). In the next Section, we shall provide the connection of L_0, L_1, L_2, L_2', L_3, $\mathcal{K}_0$, $\mathcal{M}_0$ to the equation (1.1) and the Midpoint method (1.2) so that estimates (2.37) are satisfied.

3. SEMI-LOCAL CONVERGENCE OF THE MIDPOINT METHOD

We need the following Ostrowski-type representation for the Midpoint method (1.2).

LEMMA 3.1. *Suppose that the Midpoint method* (1.2) *is well defined for each* $n = 0, 1, 2, \ldots$. *Then, the following identities are true for each* $n = 0, 1, 2, \ldots$

(3.1)

$$\begin{aligned}
&\mathcal{F}(x_{n+1}) = \\
&= \int_0^1 \mathcal{F}''(y_n + \theta(x_{n+1} - y_n))(1-\theta)\, d\theta (x_{n+1} - y_n)^2 \\
&\quad + \tfrac{1}{4} \int_0^1 \mathcal{F}''(\tfrac{(x_n + y_n)}{2} + \tfrac{\theta}{2}(y_n - x_n))(y_n - x_n)\, d\theta\, \mathcal{F}'(\tfrac{x_n+y_n}{2})^{-1} \times \\
&\qquad \times \int_0^1 \mathcal{F}''(x_n + \tfrac{\theta}{2}(y_n - x_n))\, d\theta (y_n - x_n)^2 \\
&\quad + \int_0^1 \Big[\mathcal{F}''(x_n + \theta(y_n - x_n))(1-\theta) - \tfrac{1}{2}\mathcal{F}''(x_n + \tfrac{\theta}{2}(y_n - x_n))\Big]\, d\theta (y_n - x_n)^2.
\end{aligned}$$

and

$$x_{n+1} - y_n = -\tfrac{1}{2}\mathcal{F}'(\tfrac{x_n+y_n}{2})^{-1} \int_0^1 \mathcal{F}''(x_n + \tfrac{\theta}{2}(y_n - x_n))\, d\theta (y_n - x_n)^2.$$

Proof. The proof of (3.1) can be found in [1–4]. Using (1.2), we get in turn that

$$\begin{aligned}
&x_{n+1} - y_n = \\
&= \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) - \mathcal{F}'(\tfrac{x_n+y_n}{2})^{-1}\mathcal{F}(x_n) \\
&= \Big(\mathcal{F}'(x_n)^{-1} - \mathcal{F}'(\tfrac{x_n+y_n}{2})^{-1}\Big)\mathcal{F}(x_n) \\
&= \mathcal{F}'(\tfrac{x_n+y_n}{2})^{-1}\Big[\mathcal{F}'(\tfrac{x_n+y_n}{2})\mathcal{F}'(x_n)^{-1} - \mathcal{I}\Big] \\
&= \mathcal{F}'(\tfrac{x_n+y_n}{2})^{-1}\Big[\mathcal{F}'(\tfrac{x_n+y_n}{2}) - \mathcal{F}'(x_n)\Big]\mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\
&= \mathcal{F}'(\tfrac{x_n+y_n}{2})^{-1} \int_0^1 \mathcal{F}''(x_n + \theta(\tfrac{x_n+y_n}{2} - x_n))(\tfrac{x_n+y_n}{2} - x_n)[-(y_n - x_n)]\, d\theta \\
&= -\tfrac{1}{2}\mathcal{F}'(\tfrac{x_n+y_n}{2})^{-1} \int_0^1 \mathcal{F}''(x_n + \tfrac{\theta}{2} y_n - x_n)\, d\theta (y_n - x_n)^2.
\end{aligned}$$

The proof of the Lemma is complete. □

We can show the main semi-local convergence result for the Midpoint method (1.2) under the (**H**) conditions.

Theorem 3.2. *Suppose that the (**H**) conditions and those of the Lemma 2.1 hold. Moreover, suppose that*

$$\overline{U}(x_0, t^\star) \subseteq \mathbf{D}. \tag{3.2}$$

Then, sequence $\{x_n\}$ generated by the Midpoint method (1.2) *is well defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and converges to a solution $x^\star \in \overline{U}(x_0, t^\star)$ of equation $\mathcal{F}(x) = 0$. Moreover, the following estimates hold*

$$\|y_n - x_n\| \leq s_n - t_n, \tag{3.3}$$

$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n, \tag{3.4}$$

$$\|x_n - x^\star\| \leq t^\star - t_n \tag{3.5}$$

and

$$\|y_n - x^\star\| \leq t^\star - s_n. \tag{3.6}$$

Furthermore, if there exists $R \geq t^\star$ such that

$$\overline{U}(x_0, R) \subseteq \mathbf{D} \tag{3.7}$$

and

$$\frac{L_0}{2}(t^\star + R) < 1, \tag{3.8}$$

then, the solution $x^\star$ is unique in $\overline{U}(x_0, R)$

Proof. We shall prove that (3.7) and (3.8) hold using mathematical induction. Using (1.2), ($\mathbf{C}_2$) and (2.13), we get that $\|y_0 - x_0\| = \|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x_0)\| \leq \eta = s_0 - t_0 \leq t^\star$. That is (3.3) holds for $n = 0$ and $y_0 \in \overline{U}(x_0, t^\star)$. We have by ($\mathbf{C}_5$) and the choice of η_1 that

$$\begin{aligned} \|\mathcal{F}'(x_0)^{-1}\left[\mathcal{F}'(\tfrac{x_0+y_0}{2}) - \mathcal{F}'(x_0)\right]\| &\leq \tfrac{L_0}{2}\|y_0 - x_0\| \\ &\leq \tfrac{L_0}{2}(s_0 - t_0) = \tfrac{L_0}{2}\eta < 1. \end{aligned} \tag{3.9}$$

It then follows from (3.9) and the Banach Lemma on invertible operators [23, 32] that

$$\begin{aligned} &\mathcal{F}'(\tfrac{x_0+y_0}{2})^{-1} \in L(\mathbf{Y}, \mathbf{X}), \\ &\left\|\mathcal{F}'(\tfrac{x_0+y_0}{2})^{-1}\mathcal{F}'(x_0)\right\| \leq \tfrac{1}{1-\frac{L_0}{2}\eta}. \end{aligned} \tag{3.10}$$

Using (1.2), (2.13), Lemma 3.1 and (3.10) we obtain

$$\begin{aligned} \|x_1 - y_0\| &\leq \tfrac{1}{2}\tfrac{L\,\|y_0-x_0\|^2}{1-\frac{L_0}{2}\eta} \\ &\leq \tfrac{1}{2}\tfrac{L\,(s_0-t_0)^2}{1-\frac{L_0}{2}\eta} \end{aligned} \tag{3.11}$$

and

$$\|x_1 - y_0\| \leq t_1 - s_0 \leq \gamma(s_0 - t_0). \tag{3.12}$$

Hence, (3.4) holds for $n = 0$. We also get that $\|x_1 - x_0\| \leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq t_1 - s_0 + s_0 - t_0 = t_1 \leq t^\star$, which implies $x_1 \in \overline{U}(x_0, t^\star)$. Let us assume that (3.3), (3.4), $y^\star \in \overline{U}(x_0, t^\star)$ and $x_{k+1} \in \overline{U}(x_0, t^\star)$ hold for all $k \leq n$. It follows from the proof of the Lemma 2.1 and $(\mathbf{C}_5)$ that

$$\begin{aligned} \left\|\mathcal{F}'(x_0)^{-1}\left(\mathcal{F}'(\tfrac{x_k+y_k}{2}) - \mathcal{F}'(x_0)\right)\right\| &\leq \tfrac{L_0}{2}\left(\|x_k - x_0\| + \|y_k - x_0\|\right) \\ &\leq \tfrac{L_0}{2}(t_k + s_k) < 1 \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \left\|\mathcal{F}'(x_0)^{-1}\left(\mathcal{F}'(x_{k+1}) - \mathcal{F}'(x_0)\right)\right\| &\leq L_0\|x_{k+1} - x_0\| \\ &\leq L_0 t_{k+1} < 1. \end{aligned} \tag{3.14}$$

It then follows from (3.13) and (3.14) that

$$\mathcal{F}'(\tfrac{x_k+y_k}{2})^{-1} \in L(\mathbf{Y}, \mathbf{X}),$$

$$\mathcal{F}'(x_{k+1})^{-1} \in L(\mathbf{Y}, \mathbf{X}),$$

$$\left\|\mathcal{F}'(\tfrac{x_k+y_k}{2})^{-1}\mathcal{F}'(x_0)\right\| \leq \frac{1}{1 - \frac{L_0}{2}(t_k + s_k)}, \tag{3.15}$$

$$\left\|\mathcal{F}'(x_{k+1})^{-1}\mathcal{F}'(x_0)\right\| \leq \tfrac{1}{1 - L_0 t_{k+1}}. \tag{3.16}$$

Then, we have by (1.2), $(\mathbf{C}_3)$, Lemma 3.1, (2.13), (3.15) and the induction hypothesis that

$$\begin{aligned} \|x_{k+1} - y_k\| &\leq \tfrac{1}{2}\frac{L\|y_k - x_k\|^2}{1 - \frac{L_0}{2}(s_k + t_k)} \\ &\leq \tfrac{L(s_k - t_k)^2}{2\left[1 - \frac{L_0}{2}(s_k + t_k)\right]} = t_{k+1} - s_k, \end{aligned} \tag{3.17}$$

which shows (3.4). Moreover, using (1.2), $(\mathbf{C}_3)$, $(\mathbf{C}_4)$, (2.13), Lemma 3.1, we obtain in turn

$$\begin{aligned} &\left\|\int_0^1 \mathcal{F}'(x_0)^{-1}\left[\mathcal{F}''(x_k + \theta(y_k - x_k))(1 - \theta) - \tfrac{1}{2}\mathcal{F}''(x_k + \tfrac{\theta}{2}(y_k - x_k))\right] d\theta\right\| \\ &\leq \left\|\int_0^1 \mathcal{F}'(x_0)^{-1}\left[\mathcal{F}''(x_k + \theta(y_k - x_k)) - \mathcal{F}''(x_k)\right] d\theta\right\| \\ &+ \tfrac{1}{2}\left\|\int_0^1 \mathcal{F}'(x_0)^{-1}\left[\mathcal{F}''(x_k) - \mathcal{F}''(x_k + \tfrac{\theta}{2}(y_k - x_k))\right] d\theta\right\| \\ &\leq \mathcal{M}\int_0^1 \theta(1 - \theta)\, d\theta\, \|y_k - x_k\| + \tfrac{\mathcal{M}}{4}\int_0^1 \theta\, d\theta\, \|y_k - x_k\| \\ &= \tfrac{7\mathcal{M}}{24}\|y_k - x_k\|. \end{aligned} \tag{3.18}$$

Thus,

$$\begin{aligned}
(3.19)\qquad & \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k+1})\right\| \\
& \le \tfrac{L}{2}\|x_{k+1}-y_k\|^2 + \tfrac{L^2}{4}\frac{1}{1-\frac{L_0}{2}(s_k+t_k)}\|y_k-x_k\|^3 + \tfrac{7\mathcal{M}}{24}\|y_k-x_k\|^3 \\
& \le \tfrac{L}{2}(t_{k+1}-s_k)^2 + \frac{L^2(s_k-t_k)^3}{4\left[1-\frac{L_0}{2}(s_k+t_k)\right]} + \tfrac{7\mathcal{M}}{24}(s_k-t_k)^3
\end{aligned}$$

and

$$\begin{aligned}
\|y_{k+1}-x_{k+1}\| &\le \left\|\mathcal{F}'(x_{k+1})^{-1}\mathcal{F}'(x_0)\right\|\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k+1})\right\| \\
&\le \frac{\frac{L(t_{k+1}-s_k)^2}{2} + \frac{L^2(s_k-t_k)^3}{4\left[1-\frac{L_0}{2}(t_k+s_k)\right]} + \frac{7\mathcal{M}}{24}(s_k-t_k)^3}{1-L_0\,t_{k+1}} \\
(3.20)\qquad &= s_{k+1}-t_{k+1},
\end{aligned}$$

which shows (3.3). We also have that

$$\begin{aligned}
\|y_{k+1}-x_0\| &\le \|y_{k+1}-x_{k+1}\| + \|x_{k+1}-x_0\| \\
&\le s_{k+1}-t_{k+1}+t_{k+1}-t_0 = s_{k+1} \le t^\star.
\end{aligned}$$

and

$$\begin{aligned}
\|x_{k+2}-x_0\| &\le \|x_{k+2}-y_{k+1}\| + \|y_{k+1}-x_0\| \\
&\le t_{k+2}-s_{k+1}+s_{k+1}-t_0 = t_{k+2} \le t^\star.
\end{aligned}$$

Hence, y_{k+1} and x_{k+2} belong in $\overline{U}(x_0,t^\star)$. It follows from (3.3), (3.4) and the Lemma 2.1 that sequence $\{x_n\}$ is complete in a Banach space $\mathbf{X}$ and as such it converges to some $x^\star \in \overline{U}(x_0,t^\star)$ (since $\overline{U}(x_0,t^\star)$ is a closed set). By letting $k\to\infty$ in (3.19) we obtain $\mathcal{F}(x^\star)=0$. Estimates (3.5) and (3.6) follows from (3.3) by using standard majorization techniques [23, 32]. Finally to show the uniqueness part. Let $y^\star \in \overline{U}(x_0,R)$ be a solution of the equation $\mathcal{F}(x)=0$. Let $Q=\int_0^1 \mathcal{F}'(x^\star+\theta(y^\star-x^\star))\,d\theta$. Using ($\mathbf{C}_5$), (3.7) and (3.8), we get that

$$\begin{aligned}
\left\|\mathcal{F}'(x_0)^{-1}\left[Q-\mathcal{F}'(x_0)\right]\right\| &\le \int_0^1 \|\mathcal{F}'(x_0)^{-1}\left[\mathcal{F}'(x^\star+\theta(y^\star-x^\star))-\mathcal{F}'(x_0)\right]d\theta\| \\
&\le L_0\int_0^1 \left[(1-\theta)\|x^\star-x_0\|+\theta\|y^\star-x_0\|\right]d\theta \\
(3.21)\qquad &\le \tfrac{L_0}{2}(t^\star+R) < 1.
\end{aligned}$$

It follows from (3.21) and the Banach lemma on invertible operators [23, 32] that $Q^{-1}\in L(\mathbf{Y},\mathbf{X})$. Then, using the identity

$$0=\mathcal{F}(y^\star)-\mathcal{F}(x^\star)=Q(y^\star-x^\star)$$

we deduce that $x^\star=y^\star$. The proof of the Theorem is complete. □

Remark 3.3.

(1) The limit point $t^\star$ can be replaced by $t^{\star\star}$ (given in closed from by (2.14)) in Theorem 3.2.

(2) The conclusions of Theorem 3.2 hold if hypotheses of Lemma 2.1 are replaced by those of Lemma 2.2.

(3) It follows from the ($\mathbf{H}$) conditions that there exists constants $\mathcal{K}_0$, L_1, L_2, L_3, $\mathcal{M}_0$ satisfying

$$\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}''(x_0+\tfrac{\theta}{2}(y_0-x_0))\right\| \leq \mathcal{K}_0 \tag{3.22}$$

$$\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}''(y_0+\theta(x_1-y_0))\right\| \leq L_1 \tag{3.23}$$

$$\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}''(x_0+\tfrac{\theta}{2}(y_0-x_0))\right\| \leq L_2 \tag{3.24}$$

$$\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}''(\tfrac{x_0+y_0}{2}+\tfrac{\theta}{2}(y_0-x_0))\right\| \leq L_2' \tag{3.25}$$

$$\left\|\mathcal{F}'(x_0)^{-1}\left[\mathcal{F}'(\tfrac{x_0+y_0}{2})-\mathcal{F}'(x_0)\right]\right\| \leq \tfrac{L_3}{2}\|y_0-x_0\| \tag{3.26}$$

$$\left\|\mathcal{F}'(x_0)^{-1}\left[\mathcal{F}''(x_0+\overline{\theta}(y_0-x_0))-\mathcal{F}''(x_0)\right]\right\| \leq \mathcal{M}_0\overline{\theta}\|y_0-x_0\| \tag{3.27}$$

$\overline{\theta}=\theta$ or $\theta/2$. For all $\theta \in [0,1]$, where, $y_0 = x_0 - \mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)$ and $x_1 = x_0 - \mathcal{F}'(\frac{x_0+y_0}{2})^{-1}\mathcal{F}(x_0)$. Estimates (3.22) -(3.27) are not additional to the ($\mathbf{H}$) conditions, since in practice the verification of ($\mathbf{C}_2$) - ($\mathbf{C}_5$) requires the computation of $\mathcal{K}_0$, L_1, L_2, L_3 and $\mathcal{M}_0$. Note that finding these constants only involves computations at the initial data. Moreover, these constants satisfy (2.37). Furthermore, according to the proof of Theorem 3.2, $\{r_n\}$ is a majorizing sequence for $\{x_n\}$ which is finer than $\{t_n\}$ and $\{v_n\}$ (see also (2.39)-(2.43) and the Tables in the next section).

4. NUMERICAL EXAMPLES

Example 4.1. Let $\mathbf{X} = \mathbf{Y} = \mathbb{R}$ be equipped with the max-norm, $x_0 = 1$, $\mathbf{D} = [\psi, 2-\psi]$. Let us define $\mathcal{F}$ on $\mathbf{D}$ by

$$\mathcal{F}(x) = x^m - \psi. \tag{4.1}$$

Here, $a \in (0, 1.0)$. Through some algebraic manipulations, for the conditions ($\mathbf{H}$), we obtain

$$\eta = \frac{1-\psi}{m}, \quad L = (2-\psi)^{m-2}(m-1), \quad L_0 = \frac{(2-\psi)^{m-1}-1}{1-\psi}$$
$$\text{and} \quad \mathcal{M} = (m-1)(m-2)(2-\psi)^{m-3}.$$

Furthermore, we see that for $m = 8$ and $\psi = 0.79$ the criteria (1.3) and (1.4) yield

$$0.026 \leq 0.021 \quad \text{and} \quad 0.026 \leq 0.020$$

respectively. Thus we observe that the criteria (1.3) and (1.4) are not satisfied. Even though the criteria (1.3) and (1.4) fall short but Midpoint method, starting at $x_0 = 1$, converges for $m = 8$ and $a = 0.79$ as reported in Table 4.1. Moreover from equations (2.4)–(2.6) we obtain

$$\eta_1 = 0.038, \quad \eta_2 = 0.028, \quad \eta_3 = 0.027.$$

n	$\|x_{n+1}-x_n\|$	$\|\mathcal{F}(x_n)\|$	x_n
0	2.879×10^{-02}	2.100×10^{-01}	$1.000\times10^{+00}$
1	2.419×10^{-04}	1.576×10^{-03}	9.712×10^{-01}
2	1.576×10^{-10}	1.026×10^{-09}	9.710×10^{-01}
3	4.357×10^{-29}	2.836×10^{-28}	9.710×10^{-01}
4	9.209×10^{-85}	5.994×10^{-84}	9.710×10^{-01}
5	8.697×10^{-252}	5.661×10^{-251}	9.710×10^{-01}
6	7.327×10^{-753}	4.769×10^{-752}	9.710×10^{-01}
7	$0.000\times10^{+00}$	$1.198\times10^{-2,023}$	9.710×10^{-01}

Table 4.1. Midpoint method applied to (4.1).

From (2.7), we get $\eta_0=\eta_3=0.027$. We notice that the assumption (2.12), of Lemma 2.1, holds. That is $\eta=0.026<\eta_0=0.027$. From (3.22)-(3.26), we obtain

$$\mathcal{K}_0=7,\quad L_1=7\Big(\frac{7+\psi}{8}\Big)^6,\quad L_2=7,\quad L_3=6\frac{\left\|\left(2/3+1/3\,\psi-(-1/3+1/3\,\psi)^2\right)^2-1\right\|}{\left\|-1/3+1/3\,\psi-(-1/3+1/3\,\psi)^2\right\|},$$
$$\mathcal{M}_0=7,\quad L_2'=6\Big(\frac{5+\psi}{3}\Big).$$

We can verify that the conditions (2.37) are fulfilled. Additionally, for the sequences $\{t_n\}$ (given by (2.12)), $\{r_n\}$ (given by (2.36)) and $\{v_n\}$ (given by (2.38)), we produce the Table 4.2. In Table 4.2, we observe that the sequence

n	$t_{n+1}-t_n$	$r_{n+1}-r_n$	$v_{n+1}-v_n$
0	3.084×10^{-02}	2.909×10^{-02}	3.542×10^{-02}
1	6.199×10^{-03}	6.515×10^{-04}	2.818×10^{-02}
2	1.022×10^{-04}	7.473×10^{-08}	-4.567×10^{-03}
3	5.692×10^{-10}	1.162×10^{-19}	6.400×10^{-04}
4	9.876×10^{-26}	4.366×10^{-55}	-4.798×10^{-07}
5	5.158×10^{-73}	2.317×10^{-161}	2.204×10^{-16}
6	7.350×10^{-215}	3.463×10^{-480}	-2.134×10^{-44}
7	2.126×10^{-640}	$1.156\times10^{-1,436}$	1.937×10^{-128}
8	$5.147\times10^{-1,917}$	$0.000\times10^{+00}$	-1.450×10^{-380}
9	$0.000\times10^{+00}$	$0.000\times10^{+00}$	$6.081\times10^{-1,137}$

Table 4.2. Sequences $\{t_n\}$, $\{r_n\}$ and $\{v_n\}$.

$\{r_n\}$ provides tighter error bounds than the sequence $\{t_n\}$. The convergence of the sequence $\{v_n\}$ is not expected, since (1.3) or (1.4) are not satisfied. Note also that $\{v_n\}$ was essentially used as a majorizing sequence for the Midpoint method in [1, 3, 4, 10–16, 24–26, 28]. □

EXAMPLE 4.2. In this example, we provide an application of our results to a special nonlinear Hammerstein integral equation of the second kind. Consider the integral equation

$$x(s) = 1 + \tfrac{4}{5}\int_0^1 G(s,t)x(t)^3\,dt, \quad s \in [0,1], \tag{4.2}$$

where, G is the Green kernel on $[0,1]\times[0,1]$ defined by

$$G(s,t) = \begin{cases} t(1-s), & t \le s; \\ s(1-t), & s \le t. \end{cases} \tag{4.3}$$

Let $\mathbf{X} = \mathbf{Y} = \mathcal{C}[0,1]$ and $\mathbf{D}$ be a suitable open convex subset of $\mathbf{X}_1 := \{x \in \mathbf{X} : x(s) > 0, s \in [0,1]\}$, which will be given below. Define $\mathcal{F} : \mathbf{D} \to \mathbf{Y}$ by

$$[\mathcal{F}(x)](s) = x(s) - 1 - \tfrac{4}{5}\int_0^1 G(s,t)x(t)^3\,dt, \quad s \in [0,1]. \tag{4.4}$$

The first and second derivatives of $\mathcal{F}$ are given by

$$[\mathcal{F}(x)'y](s) = y(s) - \tfrac{12}{5}\int_0^1 G(s,t)x(t)^2 y(t)\,dt, \quad s \in [0,1], \tag{4.5}$$

and

$$[\mathcal{F}(x)''yz](s) = \tfrac{24}{5}\int_0^1 G(s,t)x(t)y(t)z(t)\,dt, \quad s \in [0,1], \tag{4.6}$$

respectively. We use the max-norm. Let $x_0(s) = 1$ for all $s \in [0,1]$. Then, for any $y \in \mathbf{D}$, we have

$$[(I - \mathcal{F}'(x_0))(y)](s) = \tfrac{12}{5}\int_0^1 G(s,t)y(t)\,dt, \quad s \in [0,1], \tag{4.7}$$

which means

$$\|I - \mathcal{F}'(x_0)\| \le \tfrac{12}{5}\max_{s\in[0,1]}\int_0^1 G(s,t)\,dt = \tfrac{12}{5\times 8} = \tfrac{3}{10} < 1. \tag{4.8}$$

It follows from the Banach theorem that $\mathcal{F}'(x_0)^{-1}$ exists and

$$\|\mathcal{F}'(x_0)^{-1}\| \le \frac{1}{1-\frac{3}{10}} = \tfrac{10}{7}. \tag{4.9}$$

On the other hand, we have from (4.4) that

$$\|\mathcal{F}(x_0)\| = \tfrac{4}{5}\max_{s\in[0,1]}\int_0^1 G(s,t)\,dt = \tfrac{1}{10}.$$

Then, we get $\eta = 1/7$. Note that $\mathcal{F}''(x)$ is not bounded in $\mathbf{X}$ or its subset $\mathbf{X}_1$. Take into account that a solution $x^\star$ of equation (1.1) with $\mathcal{F}$ given by (4.3) must satisfy

$$\|x^\star\| - 1 - \tfrac{1}{10}\|x^\star\|^3 \le 0, \tag{4.10}$$

i.e., $\|x^\star\| \le \rho_1 = 1.153467305$ and $\|x^\star\| \ge \rho_2 = 2.423622140$, where ρ_1 and ρ_2 are the positive roots of the real equation $z - 1 - z^3/10 = 0$. Consequently, if we look for a solution such that $x^\star < \rho_1 \in \mathbf{X}_1$, we can consider $\mathbf{D} := \{x : x \in \mathbf{X}_1 \quad \text{and} \quad \|x\| < r\}$, with $r \in (\rho_1, \rho_2)$, as a nonempty open convex subset of $\mathbf{X}$. For example, choose $r = 1.7$. Using (3.7) and (3.8), we have that for any $x, y, z \in \mathbf{D}$

$$
\begin{aligned}
(4.11) \qquad & \|\left[(\mathcal{F}'(x) - \mathcal{F}'(x_0))y\right](s)\| = \\
&= \tfrac{12}{5} \left\| \int_0^1 G(s,t)(x(t)^2 - x_0(t)^2)y(t)\,dt \right\| \\
&\leq \tfrac{12}{5} \int_0^1 G(s,t)\|x(t) - x_0(t)\|\,\|x(t) + x_0(t)\|y(t)\,dt \\
&\leq \tfrac{12}{5} \int_0^1 G(s,t)\,(r+1)\|x(t) - x_0(t)\|y(t)\,dt, \quad s \in [0,1]
\end{aligned}
$$

and

(4.12)

$$\|(F''(x)yz)(s)\| = \tfrac{24}{5} \int_0^1 G(s,t)x(t)y(t)z(t)\,dt, \quad s \in [0,1].$$

Then, we get

$$\|\mathcal{F}'(x) - \mathcal{F}'(x_0)\| \leq \tfrac{12}{5}\tfrac{1}{8}(r+1)\|x - x_0\| = \tfrac{81}{100}\|x - x_0\|, \tag{4.13}$$

$$\|F''(x)\| \leq \tfrac{24}{5} \times \tfrac{r}{8} = \tfrac{51}{50} \tag{4.14}$$

and

$$\|\left[\left[F''(x) - \mathcal{F}''(\overline{x})\right]yz\right](s)\| = \tfrac{24}{5} \left\| \int_0^1 G(s,t)\,(x(t) - \overline{x}(t)))y(t)z(t) \right\| dt \tag{4.15}$$

$$\leq \tfrac{24}{5}\tfrac{1}{8}\|x - \overline{x}\| = \tfrac{3}{5}\|x - \overline{x}\|. \tag{4.16}$$

Now we can choose constants as follows:

$$\mathcal{M} = \tfrac{6}{7}, \quad L = \tfrac{51}{35}, \quad L_0 = \tfrac{81}{70}, \quad \text{and} \quad \eta = \tfrac{1}{7}.$$

From equations (2.4) – (2.6), we obtain

$$\eta_1 = 0.578, \quad \eta_2 = 0.427, \quad \eta_3 = 0.408.$$

Thus from (2.7)

$$\eta_0 = \eta_2 = 0.427.$$

Since $\eta_0 \neq \eta_1$. Thus from (2.12), we get

$$\tfrac{1}{7} \leq 0.3473064574.$$

Thus, the assumption (2.12) holds. Furthermore, it can be checked that the criteria (1.3) ($0.143 < 0.307$) and (1.4) ($0.143 < 0.304$) also hold. Likewise we select the constants

$$\mathcal{K}_0 = \tfrac{51}{50}, \quad \mathcal{M}_0 = \tfrac{4}{9}, \quad L_1 = \tfrac{51}{50}, \quad L_2 = \tfrac{52}{55}, \quad L_2' = \tfrac{50}{45}.$$

We can verify that the conditions (2.37) are fulfilled. Additionally, to verify the criteria (2.33) and check the convergence of the sequences $\{t_n\}$ (given by

(2.12)), $\{r_n\}$ (given by (2.36)) and $\{v_n\}$ (given by (2.38)), we produce the Table 4.3.

n	$t_{n+1} - t_n$	$r_{n+1} - r_n$	$v_{n+1} - v_n$	$s_n - t_n$
0	1.510×10^{-01}	1.548×10^{-01}	1.591×10^{-01}	1.429×10^{-01}
1	2.989×10^{-03}	1.689×10^{-03}	3.404×10^{-03}	2.985×10^{-03}
2	2.897×10^{-08}	2.334×10^{-09}	4.631×10^{-08}	2.897×10^{-08}
3	2.650×10^{-23}	6.183×10^{-27}	1.176×10^{-22}	2.650×10^{-23}
4	2.028×10^{-68}	1.150×10^{-79}	1.925×10^{-66}	2.028×10^{-68}
5	9.088×10^{-204}	7.386×10^{-238}	8.441×10^{-198}	9.088×10^{-204}
6	8.182×10^{-610}	1.959×10^{-712}	7.121×10^{-592}	8.182×10^{-610}
7	$5.971 \times 10^{-1,828}$	$0.000 \times 10^{+00}$	$4.275 \times 10^{-1,774}$	$5.971 \times 10^{-1,828}$
8	$0.000 \times 10^{+00}$	$0.000 \times 10^{+00}$	$0.000 \times 10^{+00}$	$0.000 \times 10^{+00}$
9	$0.000 \times 10^{+00}$	$0.000 \times 10^{+00}$	$0.000 \times 10^{+00}$	$0.000 \times 10^{+00}$

Table 4.3. Sequences $\{t_n\}$, $\{r_n\}$ and $\{v_n\}$.

In the Table 4.3, we observe that the sequence $\{r_n\}$ provides tighter error bounds than sequences $\{t_n\}$ and $\{v_n\}$. This is also true by (2.39). Additionally, we notice in Table 4.3 that the criterion (2.33) holds. That is $s_n - t_n \leq \eta_0$.

Concerning the uniqueness balls, let us denote the radii corresponding to (3.21), (1.4)[see 1, 3, 4, 23–26] and (1.3) [see 12, 14] by γ_1, γ_2 and γ_3, respectively. These are given as the smallest positive roots of the polynomials

$$p_1(t) = L_0\, t - 1 \quad (\text{for} \quad t^\star = R), \tag{4.17}$$

$$p_2(t) = \tfrac{\mathcal{K}}{2} t^2 - t + \eta, \tag{4.18}$$

and

$$p_3(t) = \tfrac{\mathcal{M}}{3} t^3 + \tfrac{L}{2} t^2 - t + \eta \tag{4.19}$$

respectively. Using the values of L_0, L, $\mathcal{M}$ and η we get

$$\gamma_1 = 0.864, \quad \gamma_2 = 0.168, \quad \gamma_3 = 0.164. \tag{4.20}$$

Here, $\mathcal{K} = 1.767$. Note that $\overline{U}(x_0, r-1) \subseteq \mathbf{D}$, $L_0 < L$ and $\gamma_3 < \gamma_2 < \gamma_1$. Therefore, the new approach provides the largest uniqueness ball and since $r - 1 < \gamma_1$, we deduce that $x^\star$ is unique in $\overline{U}(x_0, r-1) = \overline{U}(1, 0.7) \subseteq \mathbf{D}$.

REFERENCES

[1] S. Amat, C. Bermúdez, S. Busquier, M. J. Legaz, S. Plaza, *On a family of high-order iterative methods under Kantorovich conditions and some applications*, Abstract and Applied Analysis, **2012** (2012), Article ID 782170, 14 pp.

[2] S. Amat, C. Bermúdez, S. Busquier, S. Plaza, *On a third-order Newton-type method free of bilinear operators,* Num. Lin. Alg. Appl., **17** (2010) no. 4, pp. 639–653.

[3] S. Amat, S. Busquier, *Third-order iterative methods under Kantorovich conditions,* J. Math. Anal. Appl., **336** (2007) no. 1, 243–261.

[4] S. Amat, S. Busquier, J.M. Gutiérrez, *Third-order iterative methods with applications to Hammerstein equations: A unified approach,* J. Comp. Appl. Math., **235** (2011) no. 9, 2936–2943.

[5] I.K. Argyros, D. Chen, *The midpoint method for solving nonlinear operator equations in Banach space*, Appl. Math. Lett., **5** (1992), pp. 7–9.

[6] I.K. Argyros, D. Chen, *The midpoint method in Banach spaces and the Ptak error estimates,* Appl. Math. Comp., **62** (1994), 1–15.

[7] I.K. Argyros, D. Chen, *On the midpoint iterative method for solving nonlinear operator equations in Banach space and its applications in integral equations*, Rev. Anal. Numér. Théor. Approx. **23** (1994), 139–152.

[8] I.K. Argyros, Y.J. Cho, S. Hilout, *On the midpoint method for solving equations,* Appl. Math. Comput., **216** (2010), 2321–2332.

[9] I.K. Argyros, S. Hilout, *On the semilocal convergence of a Newton-type method of order three,* J. Korea S. M. A. Ser. B.Pure and Appl. Math., **17** (2010), 1–27.

[10] I.K. Argyros, *The convergence of a Halley-Chebysheff-type method under Newton-Kantorovich hypotheses,* Appl. Math. Lett., **6** (1993) no. 5, 71–74.

[11] I.K. Argyros, D. Chen, *Results on the Chebyshev method in Banach spaces*, Proyecciones, **12** (1993), 119–128.

[12] I.K. Argyros, *On the convergence of an Euler-Chebysheff-type method under Newton-Kantorovich hypotheses*, Pure Mathematics and Applications, **4** (1993) no. 3, 369–373.

[13] I.K. Argyros, D. Chen, Q.S. Qian, *A note on the Halley method in Banach spaces*, Appl. Math. Comput., **58** (1993), 215–224.

[14] I.K. Argyros, D. Chen, Q.S. Qian, *Error bound representations of Chebyshev-Halley type methods in Banach spaces*, Rev. Academia de Ciencias Zaragoza, **49** (1994), 57–69.

[15] I.K. Argyros, *On the method of tangent hyperbolas*, Approximation Theory and its Applications, **12** (1996), 78–95.

[16] I.K. Argyros, *On the method of tangent parabolas*, Functiones et Approximatio Commentarii Mathematici, (1996), 3–15.

[17] I.K. Argyros, *The super-Halley method using divided differences*, Appl. Math. Lett., **10** (1997) no. 4, 91–95.

[18] I.K. Argyros, *Chebysheff-Halley-like methods in Banach spaces*, J. Appl. Math. Comput., **4** (1997) no. 1, 83–107.

[19] I.K. Argyros, *On the monotone convergence of an Euler-Chebysheff-type method in partially ordered topological spaces*, Rev. Anal. Numer. Theor. Approx., **27** (1998) no. 1, 23–31.

[20] I.K. Argyros, *Error bounds for the Chebyshev method in Banach spaces,* Bulletin of the Institute of Mathematics Academia Sinica, **26** (1998) no. 4, 269–282.

[21] I.K. Argyros, *Improved error bounds for a Chebysheff-Halley-type method*, Acta Mathematica Hungarica, **84** (1999) no. 3, 209–219.

[22] I.K. Argyros, J.A. Ezquerro, J.M. Gutiérrez, M.A. Hernández, S. Hilout, *On the semilocal convergence of efficient Chebyshev-Secant-type methods*, J. Comput. Appl. Math., **235** (2011) no. 10, 3195–3206.

[23] I.K. Argyros, Y.J. Cho, S. Hilout, *Numerical Methods for Equations and Variational Inclusions*, CRC Press/Taylor & Francis, 2012.

[24] J.A. Ezquerro, *A modification of the Chebyshev method*, IMA J. Numer. Anal., **17** (1997) no. 4, 511–525.

[25] J.A. Ezquerro, J. M. Gutiérrez, M. A. Hernández, M. A. Salanova, *Chebyshev-like methods and quadratic equations,* Rev. Anal. Numér. Théor. Approx., **28** (1999) no. 1, 23–35.

[26] J.A. Ezquerro, M.A. Hernández, *New Kantorovich-type conditions for Halley's method*, Appl. Numer. Anal. Comput. Math., **2** (2005) no. 1, 70–77.

[27] E. Halley, *A new, exact, and easy method of finding roots of any equations generally, and that without any previous reduction*, Philos. Trans. Roy. Soc. London, **18** (1694), 136–145.

[28] M.A. Hernández, *Second-Derivative-Free Variant of the Chebyshev method for nonlinear equations*, J. Optim. Theory Appl., **104** (2000) no. 3, 501–515.

[29] W. Werner, *Some improvements of classical iterative methods for the solution of nonlinear equations,* Numerical Solution of Nonlinear Equations Lecture Notes in Mathematics, 878/1981, (1981), 426–440.

[30] V. Candela, A. Marquina, *Recurrence relations for rational cubic methods II: The Chebyshev method*, Computing, **45** (1990) no. 4, 355–367.

[31] S. Kanno, *Convergence theorems for the method of tangent hyperbolas*, Mathematica Japonica, **87** (1992) no. 4, 711–722.

[32] A.M. Ostrowski, *Solution of Equations in Euclidean and Banach Spaces,* Academic Press, New York, 3rd Ed., 1973.

[33] T. Yamamoto, *On the method of tangent hyperbolas in Banach spaces,* J. Comput. Appl. Math., **21** (1988) no. 1, 75–86

Received by the editors: February 16, 2013.