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SOME APPLICATIONS OF QUADRATURE RULES FOR MAPPINGS ON $L_p[u, v]$ SPACE *VIA* OSTROWSKI-TYPE INEQUALITY

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Abstract. Some Ostrowski-type inequalities are stated for $L_p[u, v]$ space and for mappings of bounded variations. Applications are also given for obtaining error bounds of some composite quadrature formulae.

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1. INTRODUCTION

In 1938, Ostrowski introduced a bound for the absolute value of the difference of a function to its average over a finite interval. His well known result named as Ostrowski's inequality [10].

PROPOSITION 1. *Let $g : [u, v] \to \mathbb{R}$ be a differentiable function, also*

$$|g'(z)| \leq M,$$

for some positive real constant M, for all $z \in (u, v)$. Then the following inequality holds for every $z \in [u, v]$

$$\left| g(z) - \tfrac{1}{v-u} \int_u^v g(s)ds \right| \leq \left[\tfrac{1}{4} + \tfrac{\left(z - \frac{u+v}{2}\right)^2}{(v-u)^2} \right] (v-u)M, \tag{1}$$

where the constant $\frac{1}{4}$ is sharp.

Let $g, h : [u, v] \to \mathbb{R}$ be two absolutely continuous functions such that functions and their product are integrable, the Čebyšev functional [2] is defined by

$$\begin{aligned} T(g,h) &= \tfrac{1}{v-u} \int_u^v \left(g(z) - \tfrac{1}{v-u} \int_u^v g(z)dz \right) \left(h(z) - \tfrac{1}{v-u} \int_u^v h(z)dz \right) dz \\ &= \tfrac{1}{v-u} \int_u^v g(z)h(z)dz - \tfrac{1}{(v-u)^2} \left(\int_u^v g(z)dz \right) \left(\int_u^v h(z)dz \right). \end{aligned}$$

In 1934, the following result proved by Grüss [5] (see also [6]):

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Proposition 2. *Let $T(g,h)$ be as defined above, we have*

$$|T(g,h)| \leq \tfrac{1}{4}(M_1 - m_1)(M_2 - m_2), \tag{2}$$

where $m_1, m_2, M_1, M_2 \in \mathbb{R}$ and satisfy the conditions

$$m_1 \leq g(z) \leq M_1 \quad \text{and} \quad m_2 \leq h(z) \leq M_2, \quad \forall\, z \in [u,v].$$

By G.V. Milovanović in [8], an application of classical Ostrowski inequality in quadrature formula was given for the very first time, also its generalization to functions in several variables was given in this article.

A generalization of Ostrowski inequality developed by Milovanović and Pečarić [9], which is stated as:

Proposition 3. *Let $g : \mathbb{R} \to \mathbb{R}$ be $n(>1)$ times differentiable function such that $|g^n(z)| \leq M$ $(\forall z \in (u,v))$. Then for every $z \in [u,v]$*

$$\left|\frac{1}{n}\Big(g(z) + \sum_{k-1}^{n-1} F_k\Big) - \frac{1}{v-u}\int_u^v g(s)ds\right| \leq \frac{M}{n(n+1)!}\frac{(z-u)^{n+1}+(v-z)^{n+1}}{v-u} \tag{3}$$

where F_k is defined by

$$F_k \equiv F_k(g;n;\xi,u,v) \equiv \frac{n-k}{k!}\frac{g^{k-1}(u)(z-u)^k - g^{k-1}(v)(z-v)^k}{v-u}.$$

In 1997, Dragomir and Wang [3] proved the following proposition by using (2) which is known as Ostrowski-Grüss inequality.

Proposition 4. *If $g : [u,v] \to \mathbb{R}$ such that g' is bounded and*

$$m \leq g'(z) \leq M,$$

for all $z \in [u,v]$ and for real constants m and M, then

$$\left| g(z) - \frac{g(v)-g(u)}{v-u}\left(z - \frac{u+v}{2}\right) - \frac{1}{v-u}\int_u^v g(s)ds \right| \leq \frac{1}{4}(v-u)(M-m). \tag{4}$$

In this paper, we first derive an integral identity for differentiable functions by using the kernel (5). Then, we apply this equality to get our results for functions whose first derivative is bounded. First section is based on introduction and preliminaries. In the second and in the third section we prove inequalities for absolutely continuous mappings in which $g' \in L_p[u,v]$ for $p \geq 1$ and mappings of bounded variation, respectively. In the last section, we will give some applications for composite quadrature rules.

2. THE CASE WHERE $g' \in L_p[u,v], p \geq 1$

In order to prove our main results, we need the following lemma from [7]:

Lemma 5. *Let $g : [u,v] \to \mathbb{R}$ be a function. Then for the kernel $P(z,s)$ on $[u,v]$ given as*

$$P(z,s) = \begin{cases} s - z + \frac{v-u}{2}, & \text{if} \quad s \in [u,z], \\ s - z - \frac{v-u}{2}, & \text{if} \quad s \in (z,v], \end{cases} \tag{5}$$

the following identity holds

$$\tag{6} \tfrac{1}{v-u}\int_u^v P(z,s)g'(s)ds \;=\; g(z) - \tfrac{g(v)-g(u)}{v-u}\left(z-\tfrac{u+v}{2}\right) - \tfrac{1}{v-u}\int_u^v g(s)ds.$$

In this section, we are going to present Ostrowski-type integral inequality for $g' \in L_p[u,v]$.

Theorem 6. *Let $g : I \to \mathbb{R}$ be an absolutely continuous mapping on I^o, the interior of the interval I, where $u, v \in I$ with $u < v$. If $g' \in L_p[u,v]$, for $p \geq 1$, then we get the following inequality*

$$\tag{7} \begin{aligned} &\left| g(z) - \tfrac{g(v)-g(u)}{v-u}\left(z-\tfrac{u+v}{2}\right) - \tfrac{1}{v-u}\int_u^v g(s)ds \right| \leq \\ &\leq \tfrac{\|g'\|_p}{(v-u)(q+1)^{\frac{1}{q}}}\left[\left(\tfrac{u+v}{2}-z\right)^{q+1} + \left(z-\tfrac{u+v}{2}\right)^{q+1} + 2\left(\tfrac{v-u}{2}\right)^{q+1}\right]^{\frac{1}{q}}. \end{aligned}$$

Proof. Using the Hölder inequality in (6), for any $z \in [u,v]$, we get

$$\begin{aligned} &\left| g(z) - \tfrac{g(v)-g(u)}{v-u}\left(z-\tfrac{u+v}{2}\right) - \tfrac{1}{v-u}\int_u^v g(s)ds \right| = \\ &= \tfrac{1}{v-u}\left|\int_u^v P(z,s)g'(s)ds\right| \\ &\leq \tfrac{1}{v-u}\left(\int_u^v |P(z,s)|^q ds\right)^{\frac{1}{q}}\left(\int_u^v |g'(s)|^p ds\right)^{\frac{1}{p}} \\ &= \tfrac{1}{v-u}\left[\int_u^z \left|s-\left(z-\tfrac{v-u}{2}\right)\right|^q ds + \int_z^v \left|s-\left(z+\tfrac{v-u}{2}\right)\right|^q ds\right]^{\frac{1}{q}} \|g'\|_p \\ &= \tfrac{\|g'\|_p}{(v-u)(q+1)^{\frac{1}{q}}}\left[\left(\tfrac{u+v}{2}-z\right)^{q+1} + \left(z-\tfrac{u+v}{2}\right)^{q+1} + 2\left(\tfrac{v-u}{2}\right)^{q+1}\right]^{\frac{1}{q}}. \end{aligned}$$

□

Remark 7. If we substitute $q = 1$ (and $p = \infty$) in (7), then we get the following Corollary. □

Corollary 8. *Let $g : I \to \mathbb{R}$ be an absolutely continuously mapping on I^o, the interior of the interval I, where $u, v \in I$ with $u < v$. If g' is bounded on $[u,v]$, then the following inequality holds for any $z \in [u,v]$*

$$\tag{8} \left| g(z) - \tfrac{g(v)-g(u)}{v-u}\left(z-\tfrac{u+v}{2}\right) - \tfrac{1}{v-u}\int_u^v g(s)ds \right| \leq \left[\tfrac{1}{4} + \tfrac{\left(z-\frac{u+v}{2}\right)^2}{(v-u)^2}\right](v-u)\|g'\|_\infty.$$

Remark 9. The inequality (8) is the generalization of Ostrowski inequality which is presented in Proposition 5, *i.e.*, by replacing $g(u) = g(v)$ in (8), we get (3) and also by choosing $\|g'\|_\infty = M$ we get (1). □

Remark 10. If we replace $z = \frac{u+v}{2}$ in (8), then we get the following midpoint inequality

$$\tag{9} \left| g\left(\tfrac{u+v}{2}\right) - \tfrac{1}{v-u}\int_u^v g(s)ds \right| \leq \tfrac{1}{4}(v-u)\|g'\|_\infty,$$

where the constant $\frac{1}{4}$ is sharp. □

REMARK 11. By replacing $z = u$ or $z = v$ in (8), we get the trapezoidal inequality

$$\left|\frac{g(u)+g(v)}{2} - \frac{1}{v-u}\int_u^v g(s)ds\right| \leq \frac{1}{2}(v-u)\|g'\|_\infty. \tag{10}$$ □

COROLLARY 12. *Let g be a function as defined in Theorem 7.*
1) *If we replace $z = \frac{u+v}{2}$ in (7), then we get the midpoint inequality $\forall\ p \geq 1$:*

$$\left|g\left(\frac{u+v}{2}\right) - \frac{1}{v-u}\int_u^b g(s)ds\right| \leq \frac{1}{2}\left[\frac{(v-u)}{(q+1)}\right]^{\frac{1}{q}} \|g'\|_p \tag{11}$$

where the constant $\frac{1}{2}\left[\frac{(v-u)}{(q+1)}\right]^{\frac{1}{q}}$ is sharp.
2) *If we replace $z = u$ or $z = v$ in (7), we get the trapezoidal inequality $\forall$ $p \geq 1$:*

(12)
$$\left|\frac{g(u)+g(v)}{2} - \frac{1}{v-u}\int_u^v g(s)ds\right| \leq \frac{1}{(v-u)(q+1)^{\frac{1}{q}}}\left[\left(\frac{v-u}{2}\right)^{q+1} + 3\left(\frac{v-u}{2}\right)^{q+1}\right]^{\frac{1}{q}} \|g'\|_p.$$

a) *If q is odd, then*

$$\left|\frac{g(u)+g(v)}{2} - \frac{1}{v-u}\int_u^v g(s)ds\right| \leq \frac{1}{2}\left(\frac{2(v-u)}{q+1}\right)^{\frac{1}{q}} \|g'\|_p. \tag{13}$$

b) *If q is even, then*

$$\left|\frac{g(u)+g(v)}{2} - \frac{1}{v-u}\int_u^v g(s)ds\right| \leq \frac{1}{2}\left(\frac{v-u}{q+1}\right)^{\frac{1}{q}} \|g'\|_p. \tag{14}$$ □

REMARK 13. By the inequality (9) we retrieve the result of Corollary 5 and the inequality (11) gives us the result of Corollary 8 of M. W. Alomari paper [1], respectively. □

3. THE CASE WHERE g IS OF BOUNDED VARIATION

THEOREM 14. *Let $g : [u, v] \to \mathbb{R}$ be a function of bounded variation. Then the following inequality holds for any $z \in [u, v]$:*

$$\left|g(z) - \frac{g(v)-g(u)}{v-u}\left(z - \frac{u+v}{2}\right) - \frac{1}{v-u}\int_u^v g(s)ds\right| \leq \frac{1}{2}\max\left\{\left|\frac{u+v-2z}{v-u}\right|, 1\right\}\bigvee_u^v g \tag{15}$$

where $\bigvee_u^v g$ is the total variation of g over $[u, v]$ and the constant $\frac{1}{2}$ is sharp.

Proof. Recalling the definition from [1], for a continuous function $p : [c, d] \to \mathbb{R}$ and a function $\nu : [c, d] \to \mathbb{R}$ of bounded variation, the following inequality holds:

$$\left| \int_c^d p(s) d\nu(s) \right| \leq \sup_{s \in [c,d]} |p(s)| \bigvee_c^d \nu. \tag{16}$$

Now using Lemma 6 with the inequality (16) for $p(s) = P(z, s)$, and $\nu(s) = g(s)$, $s \in [u, v]$, we get

$$\begin{aligned}
&\left| \tfrac{1}{v-u} \int_u^v P(z, s) dg(s) \right| \leq \\
&\leq \tfrac{1}{v-u} \left| \int_u^z P(z, s) dg(s) \right| + \tfrac{1}{v-u} \left| \int_z^v P(z, s) dg(s) \right| \\
&\leq \tfrac{1}{v-u} \sup_{s \in [u,z]} |P(z, s)| \bigvee_u^z g + \tfrac{1}{v-u} \sup_{s \in (z,v]} |P(z, s)| \bigvee_z^v g \\
&= \tfrac{1}{v-u} \max\left\{ \left| \tfrac{u+v}{2} - z \right|, \tfrac{v-u}{2} \right\} \bigvee_u^z g + \tfrac{1}{v-u} \max\left\{ \left| \tfrac{u+v}{2} - z \right| \right\} \bigvee_z^v g := M(z).
\end{aligned}$$

We notice that

$$M(z) \leq \tfrac{1}{v-u} \max\left\{ \left| \tfrac{u+v}{2} - z \right|, \tfrac{v-u}{2} \right\} \left[\bigvee_u^z g + \bigvee_z^v g \right] = \tfrac{1}{2} \max\left\{ \left| \tfrac{u+v-2z}{v-u} \right|, 1 \right\} \bigvee_u^v g$$

which proves the inequality (15).

To prove that the constant $\frac{1}{2}$ in inequality (15) is sharp, we suppose that the inequality (15) is valid for a constant $K > 0$, *i.e.*,

$$\left| g(z) - \tfrac{g(v)-g(u)}{v-u} \left(z - \tfrac{u+v}{2} \right) - \tfrac{1}{v-u} \int_u^v g(s) ds \right| \leq K \max\left\{ \left| \tfrac{u+v-2z}{v-u} \right|, 1 \right\} \bigvee_u^v g \tag{17}$$

for any $z \in [u, v]$.

Consider the mapping $g : [u, v] \to \{0, 1\}$ is defined as

$$g(s) = \begin{cases} 0, & s \in (u, v) \\ 1, & s \in \{u, v\}. \end{cases}$$

For $z = u$, we have

$$\int_u^v g(s) ds = 0 \quad \text{and} \quad \bigvee_u^v g = 2.$$

By using (17), we obtain,

$$1 \leq 2K \text{ or } \tfrac{1}{2} \leq K,$$

and thus it is proved that the constant $\frac{1}{2}$ is sharp. □

Corollary 15. *Let g be a function as defined in Theorem 15.*
1) *If we replace $z = \frac{u+v}{2}$ in (15), then we get the midpoint inequality*

$$\left| g\left(\tfrac{u+v}{2}\right) - \tfrac{1}{v-u} \int_u^v g(s)ds \right| \leq \tfrac{1}{2} \bigvee_u^v g \tag{18}$$

where the constant $\frac{1}{2}$ is sharp.
2) *If we replace $z = u$ or $z = v$ in (15), then we get the trapezoidal inequality*

$$\left| \tfrac{g(u)+g(v)}{2} - \tfrac{1}{u-v} \int_u^v g(s)ds \right| \leq \tfrac{1}{2} \bigvee_u^v g \tag{19}$$

where the constant $\frac{1}{2}$ is sharp.

Remark 16. The inequalities (18) and (19) are the results of Corollary 2 of M.W. Alomari paper [1] and the Corollaries 2.6 and 2.4 of S.S. Dragomir paper [4], respectively. □

4. APPLICATIONS TO NUMERICAL QUADRATURE RULES

Now, we are going to discuss some applications in numerical quadrature rules, which can be used to get some sharp bounds.

Let $I_n : u = z_0 < z_1 < \cdots < z_n = v$ be a partition of the interval $[u, v]$ and let $\Delta z_k = z_{k+1} - z_k, k \in \{0, 1, 2, \cdots, n-1\}$. Then

$$\sum_{k=0}^{n-1} \tfrac{1}{\Delta z_k} \int_{z_k}^{z_{k+1}} g(s)ds = Q_n(I_n, g) + R_n(I_n, g) \tag{20}$$

Consider a general quadrature formula

$$Q_n(I_n, g) := \sum_{k=0}^{n-1} \left[g(\xi_k) - \tfrac{g(z_{k+1})-g(z_k)}{\Delta z_k} \left(\xi_k - \tfrac{z_{k+1}+z_k}{2} \right) \right] \tag{21}$$

for all $\xi_k \in [a, b]$.

Theorem 17. *Let g be defined as in Theorem 7. Then (20) holds where $Q_n(I_n, g)$ is given by formula (21) and the remainder $R_n(I_n, g)$ satisfies the estimates*

(22)

$$|R_n(I_n, g)| \leq$$

$$\leq \tfrac{1}{\Delta z_k} \left(\tfrac{1}{q+1} \right)^{\frac{1}{q}} \sum_{k=0}^{n-1} \left[\left(\tfrac{z_k+z_{k+1}}{2} - \xi_k \right)^{q+1} + \left(\xi_k - \tfrac{z_k+z_{k+1}}{2} \right)^{q+1} + 2 \left(\tfrac{\Delta z_k}{2} \right)^{q+1} \right]^{\frac{1}{q}} \|g'\|_p$$

for all $\xi_k \in [z_k, z_{k+1}]$.

Proof. Applying inequality (7) on the intervals, $[z_k, z_{k+1}]$, we can state that

$$R_k(I_k, g) = \tfrac{1}{\Delta z_k} \int_{z_k}^{z_{k+1}} g(s)ds - g(\xi_k) - \tfrac{g(z_{k+1})-g(z_k)}{\Delta z_k} \left(\xi_k - \tfrac{z_k+z_{k+1}}{2} \right)$$

we sum the inequalities presented above over k from 0 to $n-1$. This gives

$$R_n(I_n,g)=\sum_{k=0}^{n-1}\tfrac{1}{\Delta z_k}\int_{z_k}^{z_{k+1}}g(t)dt-\sum_{k=0}^{n-1}\left[g(\xi_k)-\tfrac{g(z_{k+1})-g(z_k)}{\Delta z_k}\left(\xi_k-\tfrac{z_k+z_{k+1}}{2}\right)\right].$$

It follows from (7) that

$$|R_n(I_n,g)|=$$

$$=\left|\sum_{k=0}^{n-1}\tfrac{1}{\Delta z_k}\int_{z_k}^{z_{k+1}}g(s)ds-\sum_{k=0}^{n-1}\left[g(\xi_k)-\tfrac{g(z_{k+1})-g(z_k)}{\Delta z_k}\left(\xi_k-\tfrac{z_k+z_{k+1}}{2}\right)\right]\right|$$

$$\le\tfrac{1}{\Delta z_k}\left(\tfrac{1}{q+1}\right)^{\frac{1}{q}}\sum_{k=0}^{n-1}\left[\left(\tfrac{z_k+z_{k+1}}{2}-\xi_k\right)^{q+1}+\left(\xi_k-\tfrac{z_k+z_{k+1}}{2}\right)^{q+1}+2\left(\tfrac{\Delta z_k}{2}\right)^{q+1}\right]^{\frac{1}{q}}\|g'\|_p.$$

Corollary 18. *Let $q=1$ (and $p=\infty$) in (22). Then (20) holds, where $Q_n(I_n,g)$ is given by formula (21) and the remainder $R_n(I_n,g)$ satisfies the estimate*

$$|R_n(I_n,g)|\le\sum_{k=0}^{n-1}\Delta z_k\left[\tfrac{1}{4}+\tfrac{\left(\xi_k-\frac{z_k+z_{k+1}}{2}\right)^2}{(\Delta z_k)^2}\right]\|g'\|_\infty \tag{23}$$

for all $\xi_k\in[z_k,z_{k+1}]$.

Theorem 19. *Let g be a function as defined in Theorem 15. Then (20) holds, where $Q_n(I_n,g)$ is given by formula (21) and the remainder satisfies the estimate*

$$|R_n(I_n,g)|\le\sum_{k=0}^{n-1}\tfrac{1}{2}\max\left\{\left|\tfrac{z_k+z_{k+1}-2\xi_k}{z_{k+1}-z_k}\right|,1\right\}\bigvee_{z_k}^{z_{k+1}}g \tag{24}$$

for all $\xi_k\in[z_k,z_{k+1}]$.

Proof. Applying inequality (15) on the intervals, $[z_k,z_{k+1}]$, we can state that

$$R_k(I_k,g)=\tfrac{1}{\Delta z_k}\int_{z_k}^{z_{k+1}}g(s)ds-g(\xi_k)-\tfrac{g(z_{k+1})-g(z_k)}{\Delta z_k}(\xi_k-\tfrac{z_k+z_{k+1}}{2}).$$

We sum the inequalities presented above over k from 0 to $n-1$. This gives

$$R_n(I_n,g)=\sum_{k=0}^{n-1}\tfrac{1}{\Delta z_k}\int_{z_k}^{z_{k+1}}g(s)ds-\sum_{k=0}^{n-1}\left[g(\xi_k)+\tfrac{g(z_{k+1})-g(z_k)}{\Delta z_k}\left(\xi_k-\tfrac{z_k+z_{k+1}}{2}\right)\right].$$

It follows from (15) that

$$|R_n(I_n,g)|=\left|\sum_{k=0}^{n-1}\tfrac{1}{\Delta z_k}\int_{z_k}^{z_{k+1}}g(s)ds-\sum_{k=0}^{n-1}\left[g(\xi_k)+\tfrac{g(z_{k+1})-g(z_k)}{\Delta z_k}\left(\xi_k-\tfrac{z_k+z_{k+1}}{2}\right)\right]\right|$$

$$\le\sum_{k=0}^{n-1}\tfrac{1}{2}\max\left\{\left|\tfrac{z_k+z_{k+1}-2\xi_k}{\Delta z_k}\right|,1\right\}\bigvee_{z_k}^{z_{k+1}}g.\qquad\square$$

If we choose,

$$\xi_k = \frac{z_k+z_{k+1}}{2}$$

in (21), then quadrature formula becomes:

$$Q_n(I_n, g) := \sum_{k=0}^{n-1} \left[g\left(\frac{z_k+z_{k+1}}{2}\right)\right]. \tag{25}$$

REMARK 20. If (20) holds and $Q_n(I_n, g)$ is given by formula (25).

1) Let g be as in Theorem 7 where the remainder $R_n(I_n, g)$ from (11) and (9) becomes respectively

$$|R_n(I_n, g)| \leq \frac{1}{2}\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \sum_{k=0}^{n-1} [\Delta z_k]^{\frac{1}{q}} \|g'\|_p$$

and

$$|R_n(I_n, g)| \leq \frac{1}{4} \sum_{k=0}^{n-1} [\Delta z_k] \|g'\|_\infty.$$

2) Let g be as in Theorem 15 where the remainder $R_n(I_n, g)$ from (18) becomes

$$|R_n(I_n, g)| \leq \frac{1}{2} \sum_{k=0}^{n-1} \bigvee_{z_k}^{z_{k+1}} g. \qquad \square$$

If we choose, $\xi_k = z_k$ or $\xi_k = z_{k+1}$ in (21), then quadrature formula becomes:

$$Q_n(I_n, g) := \sum_{k=0}^{n-1} \left[\frac{g(z_k)+g(z_{k+1})}{2}\right]. \tag{26}$$

REMARK 21. If (20) holds and $Q_n(I_n, g)$ is given by formula (26).

1) Let g be as in Theorem 7 where the remainder $R_n(I_n, g)$ from (12), (13), (14) and (10) becomes respectively

$$|R_n(I_n, g)| \leq \frac{1}{(q+1)^{\frac{1}{q}}} \sum_{k=0}^{n-1} \frac{1}{\Delta z_k} \left[\left(\frac{-\Delta z_k}{2}\right)^{q+1} + 3\left(\frac{\Delta z_k}{2}\right)^{q+1}\right]^{\frac{1}{q}} \|g'\|_p,$$

$$|R_n(I_n, g)| \leq \frac{1}{2}\left(\frac{2}{q+1}\right)^{\frac{1}{q}} \sum_{k=0}^{n-1} (\Delta z_k)^{\frac{1}{q}} \|g'\|_p,$$

$$|R_n(I_n, g)| \leq \frac{1}{2}\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \sum_{k=0}^{n-1} (\Delta z_k)^{\frac{1}{q}} \|g'\|_p$$

and

$$|R_n(I_n, g)| \leq \frac{1}{2} \sum_{k=0}^{n-1} [\Delta z_k] \|g'\|_\infty.$$

2) Let g be as in Theorem 15 where the remainder $R_n(I_n, g)$ from (19) becomes
$$|R_n(I_n, g)| \leq \frac{1}{2} \sum_{k=0}^{n-1} \bigvee_{z_k}^{z_{k+1}} g. \qquad \square$$

5. CONCLUSION

We have given some remarks on Ostrowski type inequalities for absolutely continuous functions in which $g' \in L_p$ space. Using the results of L_p space, we have also given some special results for L_∞ space. Our Corollary 9 of Theorem 7 is the generalization of Ostrowski inequality[10] which is presented in 1938 by A. M Ostrowski. Furthermore, by putting suitable substitutions we get midpoint and trapezoidal rules which are presented in [1, 4]. At the end we have also given some applications for numerical integration.

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