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APPROXIMATION THEOREMS FOR KANTOROVICH TYPE LUPAŞ-STANCU OPERATORS BASED ON q-INTEGERS

SEVILAY KIRCI SERENBAY[†] and ÖZGE DALMANOĞLU[*]

Abstract. In this paper, we introduce a Kantorovich generalization of q-Stancu-Lupaş operators and investigate their approximation properties. The rate of convergence of these operators are obtained by means of modulus of continuity, functions of Lipschitz class and Peetre's K-functional. We also investigate the convergency of the operators in the statistical sense and give a numerical example in order to estimate the error in the approximation.

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1. INTRODUCTION

For a function $f(x)$ defined on the interval $[0,1]$, the linear operator $R_{n,q} : C[0,1] \to C[0,1]$ defined by

$$R_{n,q}(f) = R_n(f,q;x) = \sum_{k=0}^{n} f\left(\tfrac{[k]}{[n]}\right) b_{nk}(q;x) \tag{1}$$

where

$$b_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k-1)/2} x^k (1-x)^{n-k}}{(1-x+qx)\dots(1-x+q^{n-1}x)} \tag{2}$$

is called Lupaş operators [12]. For $q>0$, $R_n(f,q;x)$ are linear positive operators on $C[0,1]$ and for $q=1$ they turn into the well known Bernstein operators. The following identities hold for the $R_n(f,q;x)$ operators:

$$\begin{aligned} R_n(e_0,q;x) &= 1 \\ R_n(e_1,q;x) &= x \\ R_n(e_2,q;x) &= x^2 + \tfrac{x(1-x)}{[n]}\left(\tfrac{1-x+q^n x}{1-x+xq}\right). \end{aligned} \tag{3}$$

Lupaş investigated the approximation properties of the operators on $C[0,1]$ and estimated the rate of convergence in terms of modulus of continuity. In [14]

[†]Baskent University, Department of Mathematics Education, Ankara, Turkey, e-mail: sevilaykirci@gmail.com.
[*]Baskent University, Department of Mathematics Education, Ankara, Turkey, e-mail: ozgedalmanoglu@gmail.com.

the authors studied Voronovskaja type theorems for the q-Lupaş operators for fixed $q > 0$. In [16], Ostrovska presented new results for the convergence of the sequence $R_n(f, q_n; x)$ in $C[0,1]$. She established approximation theorems for the cases $q \in (0,1)$ and $q \in (1,\infty)$, respectively, and studied the convergence of $\{R_n(f, q_n; x)\}$, $q \neq 1$ is fixed, obtaining the limit operator of the Lupaş q-analogue of the Bernstein operator. In [5], Doğru and Kanat considered a King type modification of Lupaş operators and investigated the statistical approximation properties of the operators. Very recently, Doğru *et al.* [7] introduced a Stancu type generalization of q-Lupaş operators as

$$R_n^{\alpha,\beta}(f; q, x) = [n+1] \sum_{k=0}^{n} f\left(\tfrac{[k]+[\alpha]}{[n]+[\beta]}\right) b_{n,k}(q; x) \tag{4}$$

where $b_{n,k}(q;x)$ is given in (2). They studied the approximation properties and also introduced the r-th generalization of these operators.

Since q-Bernstein operators has attracted a lot of interest, many generalizations of them have been discovered and studied by several authors. Here we will mention some of them related to our study. For example in [13] an integral modification, called Kantorovich type generalization of q-Bernstein operators, have been studied. The authors constructed the operators as

$$B_{n,q}^{*}(f; x) := \sum_{k=0}^{n} p_{n,k}(q; x) \int_0^1 f\left(\tfrac{[k]+q^k t}{[n+1]}\right) d_q t$$

where $f \in C[0,1], 0 < q < 1$ and

$$p_{n,k}(q; x) = \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)^{n-k} \tag{5}$$

and studied some approximation properties of them. Özarslan and Vedi [17] introduced q-Bernstein-Schurer-Kantorovich operators as

$$K_n^{p}(f; q, x) := \sum_{k=0}^{n+p} p_{n+p,k}(q; x) \int_0^1 f\left(\tfrac{[k]+q^k t}{[n+1]}\right) d_q t.$$

Acu *et al.* [1] introduced a new q-Stancu-Kantorovich operators as

$$S_{n,q}^{*(\alpha,\beta)}(f; x) := \sum_{k=0}^{n} p_{n,k}(q; x) \int_0^1 f\left(\tfrac{[k]+q^k t+\alpha}{[n+1]+\beta}\right) d_q t$$

where $0 \le \alpha \le \beta$, $f \in C[0,1]$ and $p_{n,k}(q;x)$ is given in (5). She also established a q-analogue of Stancu-Schurer-Kantorovich operators in [2] where she gave the convergence theorems both in classical and statistical sense and obtained a Voronovskaya type result.

For every $n \in N$ and $q \in (0,1)$, Doğru and Kanat [6] defined the Kantorovich type modification of Lupaş operators as

$$(6) \quad R_n(f,q;x) = [n+1] \sum_{k=0}^{n} \left(\int\limits_{[k]/[n+1]}^{[k+1]/[n+1]} f(t) d_q t \right) \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \tfrac{q^{-k} q^{k(k-1)/2} x^k (1-x)^{n-k}}{(1-x+qx)...(1-x+q^{n-1}x)}.$$

Recently, Agrawal *et al.* [3] studied the approximation properties of Lupaş-Kantorovich operators based on Pólya distribution.

In this paper we present a Kantorovich generalization of the Lupaş-Stancu operators based on the q-integers. Our purpose is to study the local and global approximation results for these operators. We also investigate statistical approximation properties using Korovkin type statistical approximation theorem.

2. CONSTRUCTION OF THE OPERATORS

Before proceeding further we recall some basic notations from q-calculus (see [4] and [10]).

Let $q > 0$. For each nonnegative integer r, the q-integer $[r]$, the q-factorial $[r]!$ and the q-binomial coefficient $\left[\begin{smallmatrix} r \\ k \end{smallmatrix} \right]$, $(r \geq k \geq 0)$ are defined by

$$[r] := [r]_q := \begin{cases} \frac{1-q^r}{1-q}, & q \neq 1, \\ r, & q = 1, \end{cases}$$

$$[r]! := \begin{cases} [r][r-1]...[1], & q \geq 1, \\ 1, & q = 1, \end{cases}$$

and

$$\left[\begin{matrix} r \\ k \end{matrix} \right] := \tfrac{[r]!}{[r-k]![k]!}, \quad 0 \leq k \leq r,$$

respectively. The q-Jackson integral on the interval $[0,b]$ is defined as

$$(7) \quad \int\limits_0^b f(t) d_q t = (1-q) b \sum_{j=0}^{\infty} f(q^j b) q^j, \quad 0 < q < 1,$$

provided that the series is convergent. The Newton's binomial formula is given by

$$(8) \quad (1+x)(1+qx)...(1+q^{n-1}x) = \sum_{k=0}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] q^{k(k-1)/2} x^k.$$

The Euler's formula is

$$(9) \quad \sum_{k=0}^{\infty} \tfrac{q^{k(k-1)/2} x^k}{(1-q)^k [k]!} = \prod_{k=0}^{\infty} (1 + q^k x)$$

which can be derived from Newton's binomial formula. Let $0 < q < 1$. We introduce the Kantorovich type q-Lupaş-Stancu operators as

$$R_{n,q}^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k-1)/2} x^k (1-x)^{n-k}}{\prod_{s=0}^{n-1}(1-x+q^s x)} \left(\int_0^1 f\left(\frac{[k]+q^k t+\alpha}{[n+1]+\beta}\right) d_q t \right) \tag{10}$$

where $0 \leq \alpha \leq \beta$ and $f \in C[0,1]$.

LEMMA 1. *For all $n \in \mathbb{N}$, $x \in [0,1]$ and $0 < q < 1$, we have the following equalities:*

$$R_{n,q}^{(\alpha,\beta)}(1;x) = 1$$

$$\begin{aligned} R_{n,q}^{(\alpha,\beta)}(t;x) &= \frac{[n]}{[n+1]+\beta}\left\{x + \frac{\alpha}{[n]} + \frac{1-x+q^n x}{[2][n]}\right\} \\ &= \frac{1}{[n+1]+\beta}\left(\frac{2q}{[2]}[n]x + \alpha + \frac{1}{[2]}\right) \end{aligned} \tag{11}$$

$$\begin{aligned} R_{n,q}^{(\alpha,\beta)}(t^2;x) &= \frac{q}{1-x+qx}\frac{[n][n-1]}{([n+1]+\beta)^2}x^2 + \left(1 + \frac{2q}{[2]}\frac{1-x+q^n x}{1-x+qx} + 2\alpha\right)\frac{[n]}{([n+1]+\beta)^2}x \\ &+ \left(\alpha^2 + \frac{2\alpha}{[2]}(1-x+q^n x) + \frac{1}{[3]}\frac{(1-x+q^n x)\left(1-x+q^{n+1}x\right)}{1-x+qx}\right)\frac{1}{([n+1]+\beta)^2} \end{aligned}$$

Proof. Taking $\frac{x}{1-x}$ instead of x in (8) one gets the first equality of (11). Taking $\frac{qx}{1-x}$ and $\frac{q^2 x}{1-x}$ instead of x in (8) we have

$$\prod_{s=1}^{n}(1-x+q^s x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2}(qx)^k (1-x)^{n-k} \tag{12}$$

$$\prod_{s=2}^{n+1}(1-x+q^s x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2}\left(q^2 x\right)^k (1-x)^{n-k} \tag{13}$$

respectively. Using the definition of q-Jackson integral given in (7) and the first equality of (3), we can write

$$\begin{aligned} R_{n,q}^{(\alpha,\beta)}(t;x) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k-1)/2} x^k (1-x)^{n-k}}{\prod_{s=0}^{n-1}(1-x+q^s x)} \left(\int_0^1 \frac{[k]+q^k t+\alpha}{[n+1]+\beta} d_q t \right) \\ &= \frac{1}{[n+1]+\beta} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k-1)/2} x^k (1-x)^{n-k}}{\prod_{s=0}^{n-1}(1-x+q^s x)} \left([k] + \frac{q^k}{[2]} + \alpha\right) \\ &= \frac{1}{[n+1]+\beta} \left\{ [n]x \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} \frac{q^{k(k-1)/2}(qx)^k (1-x)^{n-k-1}}{\prod_{s=0}^{n-1}(1-x+q^s x)} \right. \\ &\left. + \frac{1}{[2]} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k-1)/2}(qx)^k (1-x)^{n-k}}{\prod_{s=0}^{n-1}(1-x+q^s x)} + \alpha \right\} \end{aligned}$$

From the identity (12) we get the desired identity for $R_{n,q}^{(\alpha,\beta)}(t;x)$.

$$
\begin{aligned}
&R_{n,q}^{(\alpha,\beta)}(t^2;x) = \\
&= \frac{1}{([n+1]+\beta)^2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{k(k-1)/2}x^k(1-x)^{n-k}}{\prod_{s=0}^{n-1}(1-x+q^s x)}\left(\int_0^1\left([k]+q^k t+\alpha\right)^2 d_q t\right) \\
&= \frac{1}{([n+1]+\beta)^2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{k(k-1)/2}x^k(1-x)^{n-k}}{\prod_{s=0}^{n-1}(1-x+q^s x)}\left\{([k]+\alpha)^2+\frac{2q^k}{[2]}(\alpha+[k])+\frac{q^{2k}}{[3]}\right\} \\
&= \frac{1}{([n+1]+\beta)^2}\Bigg\{qx^2[n][n-1]\sum_{k=0}^{n-2}\begin{bmatrix} n-2 \\ k \end{bmatrix}\frac{q^{k(k-1)/2}\left(q^2x\right)^k(1-x)^{n-k-2}}{\prod_{s=0}^{n-1}(1-x+q^s x)} \\
&\quad+(2\alpha+1)\,[n]x\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}\frac{q^{k(k-1)/2}(qx)^k(1-x)^{n-k-1}}{\prod_{s=0}^{n-1}(1-x+q^s x)} \\
&\quad+\alpha^2+\frac{2\alpha}{[2]}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{k(k-1)/2}(qx)^k(1-x)^{n-k}}{\prod_{s=0}^{n-1}(1-x+q^s x)} \\
&\quad+\frac{2q}{[2]}[n]x\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}\frac{q^{k(k-1)/2}\left(q^2x\right)^k(1-x)^{n-k-1}}{\prod_{s=0}^{n-1}(1-x+q^s x)} \\
&\quad+\frac{1}{[3]}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{q^{k(k-1)/2}\left(q^2x\right)^k(1-x)^{n-k}}{\prod_{s=0}^{n-1}(1-x+q^s x)}\Bigg\}
\end{aligned}
$$

Using the identities given in (12) and (13) we get

$$
\begin{aligned}
R_{n,q}^{(\alpha,\beta)}(t^2;x) = \frac{1}{([n+1]+\beta)^2}&\left\{\frac{qx^2[n][n-1]}{1-x+qx}\,(2\alpha+1)\,[n]x+\alpha^2+\frac{2\alpha}{[2]}\,(1-x+q^n x)\right. \\
&\left.+\frac{2q}{[2]}[n]x\frac{1-x+q^n x}{1-x+qx}+\frac{1}{[3]}\frac{(1-x+q^n x)\left(1-x+q^{n+1}x\right)}{(1-x+qx)}\right\}.
\end{aligned}
$$

Arranging the terms we have the desired result. □

REMARK 2. From Lemma 1 we have,

$$
R_{n,q}^{(\alpha,\beta)}(t-x;x) = \frac{-(q^n+\beta)}{[n+1]+\beta}x+\frac{1}{[n+1]+\beta}\left\{\alpha+\frac{1-x+q^n x}{[2]}\right\} \tag{14}
$$

$$
\begin{aligned}
R_{n,q}^{(\alpha,\beta)}((t-x)^2;x) \le &\left(\frac{[n]}{([n+1]+\beta)}-1\right)^2+(3+2\alpha)\,\frac{[n]}{([n+1]+\beta)^2} \\
&+\frac{(\alpha+1)^2}{([n+1]+\beta)^2}+\frac{2(\alpha+1)}{([n+1]+\beta)}
\end{aligned} \tag{15}
$$

Proof. The identity (14) is obvious. For the inequality (15) we use the following second central moment of the operator $R_{n,q}^{(\alpha,\beta)}(f;x)$.

$$
\begin{aligned}
&R_{n,q}^{(\alpha,\beta)}((t-x)^2;x) = \\
&= \frac{q}{1-x+qx}\frac{[n][n-1]}{([n+1]+\beta)^2}x^2 \\
&+ \left(1+\frac{2q}{[2]}\frac{1-x+q^n x}{1-x+qx}+2\alpha\right)\frac{[n]}{([n+1]+\beta)^2}x \\
&+ \left\{\left(\alpha^2+\frac{2\alpha}{[2]}(1-x+q^n x)\right)+\frac{1}{[3]}\frac{(1-x+q^n x)(1-x+q^{n+1}x)}{(1-x+qx)}\right\}\frac{1}{([n+1]+\beta)^2} \qquad (16)\\
&- 2x\frac{[n]}{[n+1]+\beta}\left\{x+\frac{\alpha}{[n]}+\frac{1-x+q^n x}{[2][n]}\right\}+x^2
\end{aligned}
$$

For $0<q<1$ and $0\le x\le 1$, we have $\frac{q}{1-x+qx}\le 1$. Also using the inequality $[n-1]<[n]$ we can write

$$\left(\frac{q}{1-x+qx}\frac{[n][n-1]}{([n+1]+\beta)^2}-2\frac{[n]}{[n+1]+\beta}+1\right)x^2\le\left(\frac{[n]}{[n+1]+\beta}-1\right)^2x^2.$$

Since $\max\limits_{0\le x\le 1}\frac{(1-x+q^n x)}{(1-x+qx)}=1$ and $1-x+q^n x\le 1$, we have,

$$\left(1+\frac{2q}{[2]}\frac{1-x+q^n x}{1-x+qx}+2\alpha\right)\le 3+2\alpha$$

and

$$
\begin{aligned}
\left(\alpha^2+\frac{2\alpha}{[2]}(1-x+q^n x)\right)+\frac{1}{[3]}\frac{(1-x+q^n x)(1-x+q^{n+1}x)}{(1-x+qx)} &\le \alpha^2+2\alpha+1\\
&=(\alpha+1)^2.
\end{aligned}
$$

Using the above inequalities in (16) and keeping in mind that $0\le x\le 1$, we finally get the desired result. □

3. DIRECT ESTIMATES

In this section, we give some direct theorems for the operators $R_{n,q}^{(\alpha,\beta)}(f;x)$. In what follows we denote by $\|.\|=\|.\|_{C[0,1]}$ the uniform norm on $C[0,1]$.

THEOREM 3. *Let $f\in C[0,1]$ and $q:=(q_n), 0<q_n<1$ be a sequence satisfying the condition*

$$\lim_{n\to\infty} q_n = 1. \qquad (17)$$

Then we have

$$\lim_{n\to\infty}\left\|R_{n,q_n}^{(\alpha,\beta)}(f;.)-f(.)\right\|=0.$$

Proof. From Lemma 1 and Korovkin's theorem, the proof is obvious because $[n]_{q_n}\to\infty$ as $n\to\infty$. □

Let $f\in C[0,1]$. The modulus of continuity of f is defined by

$$w(f;\delta)=\sup_{\substack{t,x\in[0,1]\\|t-x|\le\delta}}|f(t)-f(x)|.$$

It is well known that for any $\delta > 0$ and each $t \in [0,1]$

$$|f(t) - f(x)| \leq w(f;\delta)\left(1 + \frac{|t-x|}{\delta}\right). \tag{18}$$

The next theorem gives us the rate of convergence of the operators $R_{n,q}^{(\alpha,\beta)}(f;x)$ in terms of modulus of continuity.

Theorem 4. *If $0 < q < 1$, then for any $f \in C[0,1]$, we have*

$$\left\| R_{n,q}^{(\alpha,\beta)}(f;x) - f(x) \right\| \leq 2w(f;\sqrt{\delta_{n,q}})$$

where $\delta_{n,q} = \left(\frac{[n]}{([n+1]+\beta)} - 1\right)^2 + (3+2\alpha)\frac{[n]}{([n+1]+\beta)^2} + \frac{(\alpha+1)^2}{([n+1]+\beta)^2} + \frac{2(\alpha+1)}{([n+1]+\beta)}$.

Proof. We have

$$\begin{aligned} |R_{n,q}^{(\alpha,\beta)}(f;x) - f(x)| &= \left| \sum_{k=0}^{n} b_{n,k}(q;x) \int_0^1 \left(f\left(\frac{[k]+q^k t+\alpha}{[n+1]+\beta}\right) - f(x)\right) d_q t \right| \\ &\leq \sum_{k=0}^{n} b_{n,k}(q;x) \int_0^1 \left(\frac{\left|\frac{[k]+q^k t+\alpha}{[n+1]+\beta} - x\right|}{\delta} + 1 \right) w(f;\delta) d_q t \end{aligned}$$

Using Cauchy-Schwarz inequality we have,

$$\int_0^1 \left| \frac{[k]+q^k t+\alpha}{[n+1]+\beta} - x \right| d_q t \leq \left\{ \int_0^1 \left(\frac{[k]+q^k t+\alpha}{[n+1]+\beta} - x \right)^2 d_q t \right\}^{1/2}$$

from which we can write

$$\sum_{k=0}^{n} b_{n,k}(q;x) \int_0^1 \left(\left| \frac{[k]+q^k t+\alpha}{[n+1]+\beta} - x \right| d_q t \right) \leq \sum_{k=0}^{n} b_{n,k}(q;x) \left\{ \int_0^1 \left(\frac{[k]+q^k t+\alpha}{[n+1]+\beta} - x \right)^2 d_q t \right\}^{1/2}$$

Applying Cauchy-Schwarz inequality once more, the right hand side of the above inequality becomes

$$\begin{aligned} &\sum_{k=0}^{n} b_{n,k}(q;x) \left\{ \int_0^1 \left(\frac{[k]+q^k t+\alpha}{[n+1]+\beta} - x \right)^2 d_q t \right\}^{1/2} \\ &\leq \left\{ \sum_{k=0}^{n} b_{n,k}(q;x) \int_0^1 \left(\frac{[k]+q^k t+\alpha}{[n+1]+\beta} - x \right)^2 d_q t \right\}^{1/2} \left\{ \sum_{k=0}^{n} b_{n,k}(q;x) \right\}^{1/2}. \end{aligned}$$

Hence, by the first equality of (3) , we have

$$\begin{aligned} |R_{n,q}^{(\alpha,\beta)}(f;x) - f(x)| &\leq w(f;\delta) \left\{ 1 + \frac{1}{\delta} \left[\sum_{k=0}^{n} b_{n,k}(q;x) \int_0^1 \left(\frac{[k]+q^k t+\alpha}{[n+1]+\beta} - x \right)^2 d_q t \right]^{1/2} \right\} \\ &= w(f;\delta) \left\{ 1 + \frac{1}{\delta} \left(R_{n,q}^{(\alpha,\beta)}((t-x)^2;x) \right)^{1/2} \right\} \end{aligned}$$

Taking maximum of both sides over the interval $[0,1]$, we have

$$\left\|R_{n,q}^{(\alpha,\beta)}(f;.)-f(.)\right\|\leq w(f;\delta)\left\{1+\tfrac{1}{\delta}\left(\delta_{n,q}\right)^{1/2}\right\}$$

Choosing $\delta=\left(\delta_{n,q}\right)^{1/2}$ we get the result. □

For $0<\alpha\leq 1$, a function $f\in C[0,1]$ belongs to $\mathrm{Lip}_M\left(\alpha\right)$ if

$$\left|f(t)-f(x)\right|\leq M\left|t-x\right|^{\alpha}$$

is satisfied for some $M>0$ and for all $t,x\in[0,1]$. The following theorem gives us the rate of convergence of the operators in terms of the functions of Lipschitz class.

THEOREM 5. *Let $f\in\mathrm{Lip}_M\left(\alpha\right)$ and $q:=(q_n),0<q_n<1$, be a sequence satisfying the conditions given in* (17)*. Then*

$$\left\|R_{n,q}^{(\alpha,\beta)}(f;.)-f(.)\right\|\leq M\left(\delta_{n,q}\right)^{\alpha/2}$$

where $(\delta_{n,q})$ is given in Theorem 4.

Proof. By linearity and positivity of the operator and using the condition $f\in\mathrm{Lip}_M\left(\alpha\right)$, we have

$$|R_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|\leq M\sum_{k=0}^{n}b_{n,k}\left(q;x\right)\int_0^1\left|\tfrac{[k]+q^kt+\alpha}{[n+1]+\beta}-x\right|^{\alpha}d_qt. \tag{19}$$

The Hölder's inequality with $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$ gives us

$$|R_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|\leq M\sum_{k=0}^{n}b_{n,k}\left(q;x\right)\left(\int_0^1\left(\tfrac{[k]+q^kt+\alpha}{[n+1]+\beta}-x\right)^2d_qt\right)^{\alpha/2}.$$

Applying the Hölder's inequality once more for the sum term, we obtain

$$\begin{aligned}&|R_{n,q}^{(\alpha,\beta)}(f;x)-f(x)|\leq\\ &\leq M\left(\sum_{k=0}^{n}b_{n,k}\left(q;x\right)\int_0^1\left(\tfrac{[k]+q^kt+\alpha}{[n+1]+\beta}-x\right)^2d_qt\right)^{\alpha/2}\left(\sum_{k=0}^{n}b_{n,k}\left(q;x\right)\right)^{(2-\alpha)/2}\\ &=M\left(R_{n,q}^{(\alpha,\beta)}((t-x)^2;x)\right)^{\alpha/2}\end{aligned}$$

Taking maximum of both sides of the above inequality over $[0,1]$, we get the desired result. □

Lastly, we will study the rate of convergence of the operators $R_{n,q}^{(\alpha,\beta)}(f;x)$ by means of Peetre's K-functionals. Remember that the Peetre's K-functional is defined by

$$K_2(f;\delta)=\inf_{g\in C^2[0,1]}\left\{\|f-g\|+\delta\left\|g''\right\|\right\}. \tag{20}$$

Recall that the second modulus of a function is defined by

$$w_2(f;\delta) = \sup_{0\leq h\leq \delta} \sup_{x\in[0,1]} \left|f(x+2h) - 2f(x+h) + f(x)\right|.$$

It is known [11, p. 177, Th. 2.4] that there exists a positive constant $C > 0$ such that

$$K_2(f;\delta) \leq Cw_2(f;\sqrt{\delta}). \tag{21}$$

We need the following Lemma for the proof of the theorem on Peetre's K-functional.

Lemma 6. *For $f \in C[0,1]$ and $x \in [0,1]$ one has*

$$\left|R_{n;q}^{(\alpha,\beta)}(f,x)\right| \leq \|f\|$$

Proof. The proof follows from the linearity of the operator $R_{n,q}^{(\alpha,\beta)}(f,x)$ and from the first identity of Lemma 1. □

Theorem 7. *Let $f \in C[0,1], x \in [0,1]$ and $0 < q < 1$. Then there exist a positive constant C such that*

$$\left|R_{n;q}^{(\alpha,\beta)}(f,x) - f(x)\right| \leq Cw_2(f;\sqrt{\alpha_{n,q}}) + w(f;\beta_{n,q}(x))$$

where $\alpha_{n,q} = \delta_{n,q} + \frac{2}{([n+1]+\beta)^2}\left\{3\left([2]\beta+1\right)^2 + 2q^{2n+2}\right\}$ *and*
$\beta_{n,q}(x) = \left|\frac{1}{[n+1]+\beta}\left(\frac{2q}{[2]}[n]x + \alpha + \frac{1}{[2]}\right) - x\right|.$

Proof. Consider the following auxiliary operators $\widetilde{R}_{n;q}^{(\alpha,\beta)}(f,x)$ defined by

$$\widetilde{R}_{n;q}^{(\alpha,\beta)}(f,x) = R_{n;q}^{(\alpha,\beta)}(f,x) + f(x) - f\left(\frac{1}{[n+1]+\beta}\left(\frac{2q}{[2]}[n]x + \alpha + \frac{1}{[2]}\right)\right) \tag{22}$$

Since $R_{n;q}^{(\alpha,\beta)}$ is linear, from Lemma 1,

$$\widetilde{R}_{n;q}^{(\alpha,\beta)}(t-x;q,x) = 0.$$

By Taylor's theorem we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

Applying $\widetilde{R}_{n;q}^{(\alpha,\beta)}$ to the both side of the above equality, we get

$$\begin{aligned}
&\widetilde{R}_{n;q}^{(\alpha,\beta)}(g;x) - g(x) = \\
&= \widetilde{R}_{n;q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du;q,x\right) \\
&= R_{n;q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du\right) -
\end{aligned}$$

$$-\int_x^{\frac{1}{[n+1]+\beta}\left(\frac{2q}{[2]}[n]x+\alpha+\frac{1}{[2]}\right)}\left(\frac{1}{[n+1]+\beta}\left(\frac{2q}{[2]}[n]x+\alpha+\frac{1}{[2]}\right)-u\right)g''(u)du$$

Hence we have

$$\left|\widetilde{R}_{n;q}^{(\alpha,\beta)}(g;x)-g(x)\right|\leq\|g''\|\left\{R_{n;q}^{(\alpha,\beta)}\left(\left|\int_x^t(t-u)du\right|;x\right)\right.$$

$$\left.+\left|\int_x^{\frac{1}{[n+1]+\beta}\left(\frac{2q}{[2]}[n]x+\alpha+\frac{1}{[2]}\right)}\left(\frac{1}{[n+1]+\beta}\left(\frac{2q}{[2]}[n]x+\alpha+\frac{1}{[2]}\right)-u\right)du\right|\right\}$$

$$\leq\|g''\|\left\{R_{n;q}^{(\alpha,\beta)}\left((t-x)^2;x\right)+\left(\frac{1}{[n+1]+\beta}\left(\frac{2q}{[2]}[n]x+\alpha+\frac{1}{[2]}\right)-x\right)^2\right\} \tag{23}$$

For the last term of the above inequality we can write

$$\left(\frac{1}{[n+1]+\beta}\left(\frac{2q}{[2]}[n]x+\alpha+\frac{1}{[2]}\right)-x\right)^2\leq$$
$$\leq 2\left\{\left(\frac{2q[n]}{[2]([n+1]+\beta)}-1\right)^2x^2+\left(\frac{\alpha+\frac{1}{[2]}}{[n+1]+\beta}\right)^2\right\}$$
$$=\frac{2}{[2]^2([n+1]+\beta)^2}\left\{\left(1+q^{n+1}+[2]\beta\right)^2x^2+([2]\alpha+1)^2\right\}$$

Since $0\leq x\leq 1$ and $\alpha\leq\beta$, we have

$$\left(\frac{1}{[n+1]+\beta}\left(\frac{2q}{[2]}[n]x+\alpha+\frac{1}{[2]}\right)-x\right)^2\leq$$
$$\leq\frac{2}{[2]^2([n+1]+\beta)^2}\left\{2\left(([2]\beta+1)^2+q^{2n+2}\right)+([2]\beta+1)^2\right\}$$

from which we get

$$\left|\widetilde{R}_{n;q}^{(\alpha,\beta)}(g;x)-g(x)\right|\leq\|g''\|\left\{\delta_{n,q}+\frac{2}{([n+1]+\beta)^2}\left\{3\left([2]\beta+1\right)^2+2q^{2n+2}\right\}\right\} \tag{24}$$

by (23). On the other hand from (22) and Lemma 6, we have

$$\begin{aligned}\left|\widetilde{R}_{n;q}^{(\alpha,\beta)}(f;x)\right| &\leq \left|R_{n;q}^{(\alpha,\beta)}(f;x)\right|+2\|f\|\\ &\leq 3\|f\|.\end{aligned}$$

Thus, from (22) and (24) we can write

$$\begin{aligned}
&\left|R_{n;q}^{(\alpha,\beta)}(f;x)-f(x)\right| \leq \\
&\leq \left|\widetilde{R}_{n;q}^{(\alpha,\beta)}(f;x)-f(x)\right| + \left|f(x)-f\left(\frac{1}{[n+1]+\beta}\left(\frac{2q}{[2]}[n]x+\alpha+\frac{1}{[2]}\right)\right)\right| \\
&\leq \left|\widetilde{R}_{n;q}^{(\alpha,\beta)}(f-g;x)\right| + |(f-g)(x)| + \left|\widetilde{R}_{n;q}^{(\alpha,\beta)}(g;x)-g(x)\right| \\
&\quad + \left|f(x)-f\left(\frac{1}{[n+1]+\beta}\left(\frac{2q}{[2]}[n]x+\alpha+\frac{1}{[2]}\right)\right)\right| \\
&\leq 4\|f-g\| + \|g''\|\,\alpha_{n,q} + \left|f(x)-f\left(\frac{1}{[n+1]+\beta}\left(\frac{2q}{[2]}[n]x+\alpha+\frac{1}{[2]}\right)\right)\right|
\end{aligned}$$

Taking the infimum on the right hand side over all $g \in C^2[0,1]$ and using (20) and (21) we get

$$\begin{aligned}
\left|R_{n;q}^{(\alpha,\beta)}(f;x)-f(x)\right| &\leq 4K_2(f;\alpha_{n,q}) + w(f;\beta_{n,q}(x)) \\
&\leq Cw_2(f;\sqrt{\alpha_{n,q}}) + w(f;\beta_{n,q}(x)).
\end{aligned}$$

which completes the proof. □

Remark 8. For $q_n \to 1$ as $n \to \infty$, we have $\alpha_{n,q_n} \to 0$ and $\beta_{n,q_n}(x) \to 0$.

4. STATISTICAL CONVERGENCE PROPERTIES

Before proceeding further let us recall the concept of statistical convergence, which was first introduced by H. Fast [8] in 1951 and has been studied frequently in approximation theory for the last two decades.

The natural density of a set $K \subseteq N$ is defined by

$$\delta(K) = \lim_n \frac{1}{n} \left|\{k : \ k \leq n, \ k \in K\}\right|$$

provided the limit exists (see [15]); here $|A|$ denotes the cardinality of the set A. A sequence $x = (x_k)$ is called statistically convergent to a number L if, for every $\epsilon > 0$

$$\delta\{k : \ |x_k - L| \geq \epsilon\} = 0$$

and it is denoted as $st - \lim_k x_k = L$.

A.D. Gadjiev and C. Orhan [9] proved the following Bohman-Korovkin type approximation theorem using the concept of the statistical convergence.

Theorem A. [9] *If the sequence of linear positive operators $A_n : C[a,b] \to C[a,b]$ satisfies the conditions,*

$$st - \lim_n \|A_n(e_\nu;.) - e_\nu(.)\|_{C[a,b]} = 0 \quad , \quad e_\nu(t) = t^\nu$$

for $\nu = 0,1,2$, then for any function $f \in C[a,b]$,

$$st - \lim_n \|A_n(f;.) - f(.)\|_{C[a,b]} = 0.$$

Theorem 9. *Let* (q_n)*,* $0 < q_n < 1$*, be a sequence satisfying*

$$st - \lim_n q_n = 1 \quad \text{and} \quad st - \lim_n q_n^n = c \in (0,1). \tag{25}$$

Then for all $f \in C[0,1]$*, the operator* $R_{n;q_n}^{(\alpha,\beta)}$ *satisfies*

$$st - \lim_n \|R_{n;q_n}^{(\alpha,\beta)}(f,.) - f(.)\| = 0.$$

Proof. It is enough to prove that

$$st - \lim_n \|R_{n;q_n}^{(\alpha,\beta)}(e_i;.) - e_i(.)\| = 0,$$

for $e_i(t) = t^i, i = 0,1,2$, then the proof follows from Theorem A.

For $i = 0$, it is clear from the first identity of Lemma 1 that

$$st - \lim_n \|R_{n;q_n}^{(\alpha,\beta)}(e_0;.) - e_0(.)\| = 0. \tag{26}$$

For $i = 1$, again Lemma 1 implies,

$$R_{n,q}^{(\alpha,\beta)}(e_1,x) - e_1(x) = \left(\tfrac{2q}{[2]}\tfrac{[n]}{[n+1]+\beta} - 1\right)x + \left(\tfrac{\alpha+\frac{1}{[2]}}{[n+1]+\beta}\right)$$

from which we can write

$$|R_{n,q}^{(\alpha,\beta)}(e_1,x) - e_1(x)| \leq \tfrac{1+q^{n+1}+[2]\beta}{[2]([n+1]+\beta)}x + \tfrac{\alpha+\frac{1}{[2]}}{[n+1]+\beta} \tag{27}$$

Now for a given $\epsilon > 0$, let us define the following sets:

$$T := \{n : \|R_{n,q_n}^{(\alpha,\beta)}(e_1,.) - e_1(.)\| \geq \epsilon\},$$

$$T_1 := \{n : \tfrac{1+q^{n+1}+[2]\beta}{[2]([n+1]+\beta)} \geq \tfrac{\epsilon}{2}\}.$$

$$T_2 := \{n : \tfrac{\alpha+\frac{1}{[2]}}{[n+1]+\beta} \geq \tfrac{\epsilon}{2}\}.$$

From (27) it is clear that $T \subseteq T_1 \cup T_2$. So we can write,

$$\delta(T) \leq \delta(T_1) + \delta(T_2) \tag{28}$$

From the conditions (25), we have

$$st - \lim_n \tfrac{1+q^{n+1}+[2]\beta}{[2]([n+1]+\beta)} = 0 \text{ and } st - \lim_n \tfrac{\alpha+\frac{1}{[2]}}{[n+1]+\beta} = 0$$

which implies that the right hand side of the inequality (28) is zero. Therefore we have,

$$\delta\{n : \|R_{n,q_n}^{(\alpha,\beta)}(e_1,.) - e_1(.)\| \geq \epsilon\} = 0$$

which implies

$$st - \lim_n \|R_{n,q_n}^{(\alpha,\beta)}(e_1,.) - e_1(.)\| = 0. \tag{29}$$

Lastly for $i=2$ we can write

$$
\begin{aligned}
|R_{n,q}^{(\alpha,\beta)}(e_2,x)-e_2(x)| &= \left(1-\tfrac{q}{1-x+qx}\tfrac{[n][n-1]}{([n+1]+\beta)^2}\right)x^2 \\
&+\left(1+\tfrac{2q}{[2]}\tfrac{1-x+q^nx}{1-x+qx}+2\alpha\right)\tfrac{[n]}{([n+1]+\beta)^2}x \\
&+\left(\alpha^2+\tfrac{2\alpha}{[2]}(1-x+q^nx)+\tfrac{1}{[3]}\tfrac{(1-x+q^nx)\left(1-x+q^{n+1}x\right)}{1-x+qx}\right)\tfrac{1}{([n+1]+\beta)^2}
\end{aligned}
$$

from which we have

$$
\begin{aligned}
\|R_{n,q}^{(\alpha,\beta)}(e_2,.)-e_2(.)\| &\leq \left(1-q\tfrac{[n][n-1]}{([n+1]+\beta)^2}\right) \\
&+(3+2\alpha)\tfrac{[n]}{([n+1]+\beta)^2}+\left(\alpha^2+\tfrac{2\alpha}{[2]}+\tfrac{1}{[3]}\right)\tfrac{1}{([n+1]+\beta)^2} \qquad (30)
\end{aligned}
$$

Now, for a given $\epsilon>0$, let us define the following sets:

$$
\begin{aligned}
K &:= \{n : \|R_{n,q_n}^{(\alpha,\beta)}(e_2,.)-e_2(.)\| \geq \epsilon\}, \\
K_1 &:= \{n : \left(1-q\tfrac{[n][n-1]}{([n+1]+\beta)^2}\right) \geq \tfrac{\epsilon}{3}\} \\
K_2 &:= \{n : (3+2\alpha)\tfrac{[n]}{([n+1]+\beta)^2} \geq \tfrac{\epsilon}{3}\} \\
K_3 &:= \{n : \left(\alpha^2+\tfrac{2\alpha}{[2]}+\tfrac{1}{[3]}\right)\tfrac{1}{([n+1]+\beta)^2} \geq \tfrac{\epsilon}{3}\}
\end{aligned}
$$

From (30) it is clear that $K\subseteq K_1\cup K_2\cup K_3$. Therefore we have,

$$\delta(K)\leq\delta(K_1)+\delta(K_2)+\delta(K_3). \qquad (31)$$

Taking the conditions given in (25) into account, one has

$$
\begin{aligned}
st-\lim_n\left(1-q\tfrac{[n][n-1]}{([n+1]+\beta)^2}\right) &= 0 \\
st-\lim_n(3+2\alpha)\tfrac{[n]}{([n+1]+\beta)^2} &= 0 \qquad (32)\\
st-\lim_n\left(\alpha^2+\tfrac{2\alpha}{[2]}+\tfrac{1}{[3]}\right)\tfrac{1}{([n+1]+\beta)^2} &= 0
\end{aligned}
$$

From (32) the right hand side of (31) becomes zero and hence we get

$$\delta\{n : \|R_{n,q_n}^{(\alpha,\beta)}(e_2,.)-e_2(.)\|\geq\epsilon\}=0$$

i.e.,

$$st-\lim_n\|R_{n,q_n}^{(\alpha,\beta)}(e_2,.)-e_2(.)\|=0. \qquad (33)$$

Now by (26), (29) and (33) we conclude from Theorem A that for all $f\in C[0,1]$

$$st-\lim_n\|R_{n,q_n}^{(\alpha,\beta)}(f,.)-f(.)\|=0.$$

□

Example 10. Taking $f(x)=x^2$, (curve 4), we compute the error estimation of $q-$Lupaş Kantorovich operators given by (10) for $q=0.5$ (curve 3), $q=0.7$ (curve 2) and $q=0.85$ (curve 1).

x	Error bound for $q = 0.5$	Error bound for $q = 0.7$	Error bound for $q = 0.85$
0	0.6574686544	0.6508325077	0.4855162530
0.3	0.0569703701	0.3717804595	0.2747085315
0.5	0.1057573294	0.0748125294	0.0569703701
0.8	0.0476530351	0.0418758717	0.0211787652
1	0.1345946106	0.1266259064	0.05804583687

Table 1. Error estimates of $R_{n,q_n}^{(\alpha,\beta)}(f,x)$ for different values of q. ($n = 30$, $\alpha = 1$ and $\beta = 4$.)

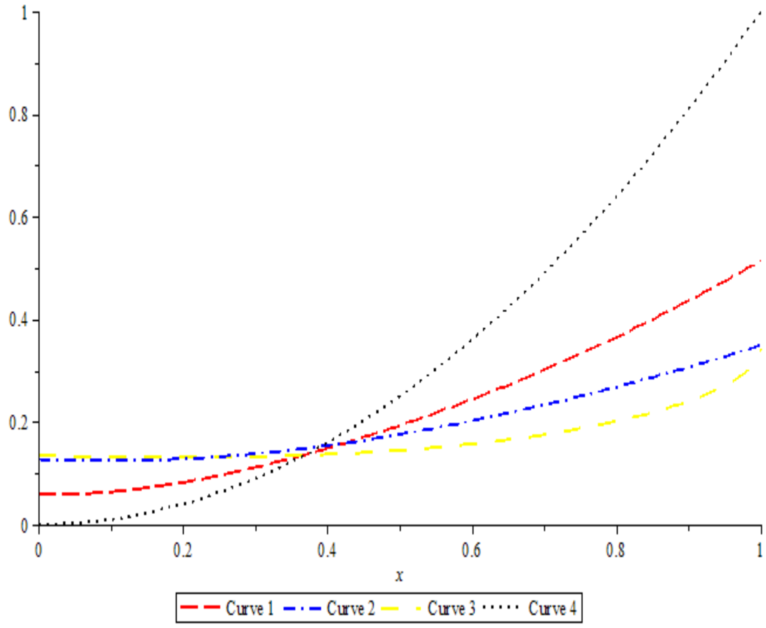

Fig. 1. Estimation of the q-Lupaş-Kantorovich operators to the function $f(x) = x^2$ for $q = 0.5; 0.7$ and 1.

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