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L^p-APPROXIMATION AND GENERALIZED GROWTH OF GENERALIZED BIAXIALLY SYMMETRIC POTENTIALS ON HYPER SPHERE

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Abstract. The generalized order of growth and generalized type of an entire function $F^{\alpha,\beta}$ (generalized biaxisymmetric potentials) have been obtained in terms of the sequence $E_n^p(F^{\alpha,\beta}, \Sigma_r^{\alpha,\beta})$ of best real biaxially symmetric harmonic polynomial approximation on open hyper sphere $\Sigma_r^{\alpha,\beta}$. Moreover, the results of McCoy [8] have been extended for the cases of fast growth as well as slow growth.

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1. INTRODUCTION

Let $F^{\alpha,\beta}$ be a real valued regular solution to the generalized biaxisymmetric potential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + \frac{2\beta+1}{y}\frac{\partial}{\partial y}\right) F^{\alpha,\beta} = 0, \quad \alpha > \beta > -\tfrac{1}{2},$$

where (α, β) are fixed in a neighbourhood of the origin and the analytic Cauchy data $F_x^{\alpha,\beta}(0, y) = F_y^{\alpha+\beta}(x, 0) = 0$ is satisfied along the singular lines in the open hyper sphere $\Sigma_r^{\alpha,\beta} : x^2 + y^2 < r^2$. Such functions with even harmonic functions are referred to as generalized biaxisymmetric potentials $(GBASP's)$ having local expansions of the form

$$F^{\alpha,\beta}(x, y) = \sum_{n=0}^{\infty} a_n R_n^{\alpha,\beta}(x, y)$$

such that

$$R_n^{\alpha,\beta}(x, y) = (x^2 + y^2)^n P_n^{\alpha,\beta}(x^2 - y^2/x^2 + y^2)/P_n^{\alpha,\beta}(1), \quad n = 0, 1, 2, \ldots$$

where the $P_n^{\alpha,\beta}$ are Jacobi polynomials [1], [18].

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Let K be a compact subset of the complex plane. Let the one to one operator mapping between the space $L^p(\Sigma_r^{\alpha,\beta})$ of real valued GBASP's with finite p-norm

$$\begin{aligned}\|\cdot\|_p &= \left(\frac{1}{A}\iint_{\Sigma_r^{\alpha,\beta}} |\cdot|^p dxdy\right)^{\frac{1}{p}}, \quad p\in[1,\infty),\\ \|\cdot\|_\infty &= \sup_{\Sigma_r^{\alpha,\beta}} |\cdot|, \quad \|1\|_p = 1\end{aligned}$$

and the space $l^p(K)$ of associated functions

$$f(z)=\sum_{n=0}^{\infty} a_n z^{2n}, \quad R_n^{\alpha,\beta}(z,0)=z^{2n}, \qquad n=0,1,2,\ldots$$

continuous on K with finite p-norm. Following McCoy [14] for Koornwinder's integral for Jacobi polynomials and the inverse operator have been defined as:

$$\begin{aligned}F^{\alpha,\beta}(x,y) &= K_{\alpha,\beta}(f)=\int_0^1\int_0^\pi f(\zeta)\mu_{\alpha,\beta}(t,s)dsdt\\ \mu_{\alpha,\beta}(t,s) &= \gamma_{\alpha,\beta}(1-t^2)^{\alpha-\beta-1}t^{2\beta+1}(\sin\ s)^{2\alpha}\\ \zeta^2 &= x^2-y^2t^2-i2xyt\cos\ s\\ \gamma_{\alpha,\beta} &= 2\Gamma(\alpha+1)/\Gamma(\tfrac{1}{2})\Gamma(\alpha-\beta)\Gamma(\beta+\tfrac{1}{2}),\end{aligned}$$

and

$$\begin{aligned}f(z) &= K^{-1}_{\alpha,\beta}(F^{\alpha,\beta})=\int_{-1}^1 F^{\alpha,\beta}(r\xi, r(1-\xi^2)^{\frac{1}{2}})\nu_{\alpha,\beta}((z/r^2)^2,\xi)d\xi\\ \nu_{\alpha,\beta}(\tau,\xi) &= S_{\alpha,\beta}(\tau,\xi)(1-\xi)^\alpha(1+\xi)^\beta\\ S_{\alpha,\beta}(\tau,\xi) &= \eta_{\alpha,\beta}\tfrac{1-\tau}{(1+\tau)^{\alpha+\beta+2}}F\left(\tfrac{\alpha+\beta+2}{2};\tfrac{\alpha+\beta+3}{2};\beta+1;\tfrac{2\tau(1+\xi)}{(1+\tau)^2}\right)\\ \eta_{\alpha,\beta} &= \Gamma(\alpha+\beta+2)/2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1).\end{aligned}$$

The normalizations $K_{\alpha,\beta}(1)=K^{-1}_{\alpha,\beta}(1)=1$ are taken. The kernel $S_{\alpha,\beta}(\tau,\xi)$ is analytic on $\|\tau\|<1$ for $-1\le\xi\le 1$. The local function elements $F^{\alpha,\beta}$ and f are continued harmonically/analytically by contour deformation using the Envelope Method [3].

Let K be a compact subset of the complex plane with Card. $K=0$ and let $u_1,u_2,\ldots,u_n\in K$. Following [4, p.285] we put

$$\begin{aligned}&V(u_1,u_2,\ldots,u_n)=\prod_{k,l(k<l)}^{n}(u_k-u_l),\\ &V_n=\max\{|V(u_1,u_2,\ldots,u_n)| : u_k\in K, 1\le k\le n\}.\end{aligned}$$

Set $d=\max\{|z|:z\in K\}$. Also, let $\mu_n(z)=z^n+a_1z^{n-1}+\cdots+a_n$ denote the Chebyshev polynomial for K such that all zeros of μ_n belong to K. We set

$$m_n^*=\max\{|\mu_n(z)|, z\in K\}.$$

Then we have [4, pp. 287–289],

$$m_n^* \leq \tfrac{V_{n+1}}{V_n} \leq (n+1)m_n^*,$$
$$\lim_{n\to\infty}\left(\tfrac{V_{n+1}}{V_n}\right)^{1/n} = \widetilde{d},$$

where $\widetilde{d}$ is the transfinite diameter of K.

Using the Koornwinder's integral and inverse operator the information concerning the approximation and growth of analytic functions can be transfer to GBASP $F^{\alpha,\beta}$.

The essential properties of $F^{\alpha,\beta} \in L^p(\Sigma_r^{\alpha,\beta})$ that are the restrictions of entire GBASP functions are drawn from approximation on sets of polynomials

$$P_{2n}^{\alpha,\beta} = \{K_{\alpha,\beta}(h) : h \in p_{2n}\},$$

and

$$p_{2n} = \left\{\sum_{k=0}^{n} a_k z^{2k} : a_k \text{ - real}, 0 \leq k \leq n\right\}, \quad n = 0, 1, 2, \ldots.$$

It is the Bernstein limits of the optimal approximates,

$$E_{2n}^p = E_{2n}^p(F^{\alpha,\beta}, \Sigma_r^{\alpha,\beta}) = \min\left\{\|F^{\alpha,\beta} = H\|_p : H \in P_{2n}^{\alpha,\beta}\right\},$$

and

$$e_{2n}^p = e_{2n}^p(f;k) = \min\left\{\|f-h\|_p : h \in p_{2n}\right\},$$

and provide the characterizations. The set p_{2n} contains all real polynomials of degree at most $2n$ and set $P_{2n}^{\alpha,\beta}$ contains all real biaxisymmetric harmonic polynomials of degree at most $2n$. The operators $K_{\alpha,\beta}$ and $K_{\alpha,\beta}^{-1}$ establish one-one equivalence of sets p_{2n} and $P_{2n}^{\alpha,\beta}$.

Several authors such as Harfaoui [5], Kumar [10], Harfaoui and Kumar [6] and others obtained generalized characteristics of growth of entire functions by using the best polynomial approximation and interpolation in L_p-norm. The growth characteristics of solutions of certain linear partial differential equations have been studied by Kumar and Basu [12], [13], Kumar [11], Khan and Ali [9].

McCoy [15, Th. 2] obtained the necessary and sufficient conditions for the entire GBASP $F^{\alpha,\beta} \in L^p(D), p \geq 2$ to be the restriction to D of order and type in terms of the errors $E_n^p(F^{\alpha,\beta})$, here D is parabolic convex set. To the best of our knowledge, these characterizations leave an important class of growth of entire function GBASP $F^{\alpha,\beta}$ such as fast and slow growth. In this paper we have tried to fill this gap. Moreover, we have extended the results of McCoy to generalized orders and generalized types which will cover the cases of fast growth as well as slow growth. Here we replace D by open hypersphere $\Sigma_r^{\alpha,\beta}$.

2. GENERALIZED ORDER AND GENERALIZED TYPE

Seremeta [17] defined the generalized order and generalized type with the help of general functions as follows.

Let L^0 denote the class of functions h satisfying the following conditions:

(i) $\phi(x)$ is defined on $[a,\infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$.

(ii) It holds

$$\lim_{x\to\infty} \frac{\phi\left\{\left(1+\frac{1}{\varphi(x)}\right)(x)\right\}}{\phi(x)} = 1$$

for every function $\varphi(x)$ such that $\varphi(x) \to \infty$, as $x \to \infty$.

Let Δ denote the class of functions ϕ satisfying condition (i) and the following:

(iii) It holds

$$\lim_{x\to\infty} \frac{\phi(cx)}{\phi(x)} = 1$$

for every $c > 0$, that is, $\phi(x)$ is slowly increasing.

For an entire function $f(z)$ and functions $\alpha(x) \in \Delta, \beta(x) \in L^0$, Seremeta [10, Th. 1] proved that

$$\rho(\alpha,\beta,f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r,f)]}{\beta(\log r)} = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\log|a_n|\right)}.$$

Further, for $\alpha(x) \in L^0, \beta^{-1}(x) \in L^0$ and $\gamma(x) \in L^0$, we have

$$T(\alpha,\beta,f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r,f)]}{\beta[(\gamma(r))^\rho]} = \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\left[\gamma(e^{\frac{1}{\rho}}|a_n|^{-\frac{1}{n}})\right]^\rho},$$

where $0 < \rho < \infty$ is a fixed number.

The above relations were obtained under certain conditions which do not hold if $\alpha = \beta$. In 1968, Seremeta [16] obtained the results connected with slow growth of entire functions. The characteristic for slow growth entire functions $f(z) = \sum_{k=0}^{\infty} c_k(f)z^k$ has the following form:

$$\rho_\alpha = \limsup_{n\to\infty} \frac{\alpha\left(\log\log M(r,f)\right)}{\alpha(\log\log r)},$$

where $M(r,f) = \max_{|z|=r}|f(z)|$ and a function $\alpha \in \Delta$. Let us define $F(x,c) = \alpha^{-1}(c\alpha(x))$, where $c \in (0,\infty)$ is any constant. It was shown in [16] that if for any $c \in (0,\infty)$ the inequality

$$0 \le \frac{dF(x,c)}{dx} \le A\big(\exp(F(x,c))^B\big)$$

is realized for any $x \ge x_1$, where A and B are some constants ($0 < A, B < \infty$), then we obtain

$$\rho_\alpha = \max(\rho'_\alpha, 1).$$

Here

$$\rho'_\alpha = \limsup_{k\to\infty} \frac{\alpha(\log k)}{\alpha\big(\log\big(\frac{1}{k}\log|c_k(f)|^{-1}\big)\big)}.$$

To refine this scale, Kapoor and Nautiyal [7] introduced a new class of functions as follows: A function $\phi(t) \in \Omega$ if $\phi(t)$ satisfies (ii) and:

(iv) There exists a function $\delta(t) \in \Delta$ and t_0, K_1 and K_2 such that for all $t > t_0$

$$0 < K_1 \leq \frac{d(\phi(t))}{d(\delta(\log t))} \leq K_2 < \infty.$$

Further a function $\phi(t) \in \overline{\Omega}$ if $\phi(t)$ satisfies (ii) and

(v)

$$\lim_{t\to\infty} \frac{d(\phi(t))}{d(\log(t))} = K, \quad 0 < K < \infty.$$

Kapoor and Nautiyal [7, p. 66] showed that $\Omega, \overline{\Omega} \subseteq \Delta$ and $\Omega \cap \overline{\Omega} = \Phi$. Let $\alpha(t) \in \Omega$ or $\overline{\Omega}$. Then following Kapoor and Nautiyal [7, p. 66], for entire GBASP $F^{\alpha,\beta}$ and associate we define the generalized order and generalized type as

$$\rho = \rho(\alpha, \alpha, f) = \limsup_{r\to\infty} \frac{\alpha(\log M(R,f))}{\alpha(\log r)} \tag{2.1}$$

$$T = T(\alpha, \alpha, f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r,f)]}{[\alpha(\log r)^{\rho}]} \tag{2.2}$$

$$\rho^* = \rho^*(\alpha, \alpha, F^{\alpha,\beta}) = \limsup_{r\to\infty} \frac{\alpha(\log M(r,F^{\alpha,\beta}))}{\alpha(\log r)} \tag{2.3}$$

$$T^* = T^*(\alpha, \alpha, F^{\alpha,\beta}) = \limsup_{r\to\infty} \frac{\alpha[\log M(r,F^{\alpha,\beta})]}{[\alpha(\log r)^{\rho}]}, \tag{2.4}$$

where

$$M(r,f) = \max_{|z|=r} |f(z)|, M(r, F^{\alpha,\beta}) = \max_{x^2+y^2=r^2} |F^{\alpha,\beta}(x,y)|.$$

Let K_r be the largest equipotential curve of K defined by $K_r = \{z \in \mathbb{C} : |\gamma(z)|d = r\}$, where $w = \gamma(z)$ is holomorphic and maps the unbounded component of the complement of K on $|w| > 1$ such that $\gamma(\infty) = \infty$ and $\gamma'(\infty) > 0$. When $r = d = 1, K_r = K$. So we take $r > d, r > 1$. We set $\overline{M}(r, F^{\alpha,\beta}) = \sup_{z\in K_r} |F^{\alpha,\beta}(z, o)|$ for $r > 1$.

McCoy [14] proved the following result:

THEOREM 2.1. *For each GBASP $F^{\alpha,\beta}$ regular in the hyper sphere $\Sigma_r^{\alpha,\beta}$ there is a unique $K_{\alpha,\beta}$ associated even function f analytic in the disk D_r and conversely.*

Now we prove the following Lemmas:

LEMMA 2.1. *Let $F^{\alpha,\beta}$ be real valued entire function GBASP with $K_{\alpha,\beta}$ associate f. Then the generalized order and generalized type of $F^{\alpha,\beta}$ respectively are identical.*

Proof. Let us consider the relation $F^{\alpha,\beta}(x,y) = K_{\alpha,\beta}(f)$. The nonnegativity and the normalization of the measure leads directly to the bound

$$M(r, F^{\alpha,\beta}) \leq M(r, f). \tag{2.5}$$

The inverse relation

$$f(z) = K^{-1}_{\alpha,\beta}(F^{\alpha,\beta})$$

leads to the inequality

$$|f(z)| \leq M(r, F^{\alpha,\beta}) N_{\alpha,\beta}(\tau), \quad \tau = \left(\tfrac{z}{r}\right)^2,$$

where

$$N_{\alpha,\beta}(\tau) = \max\{\eta^{-1}_{\alpha,\beta}|S_{\alpha,\beta}(\tau,\xi)|; -1 \leq \xi \leq 1\}.$$

However, for $z = \varepsilon r e^{i\theta}$($\varepsilon$ real),

$$M(\varepsilon r, f) \leq M(r, F^{\alpha,\beta}) N_{\alpha,\beta}(\tau)$$

it gives

$$M(r, f) \leq M(\varepsilon^{-1} r, F^{\alpha,\beta}) N_{\alpha,\beta}(\varepsilon^2). \tag{2.6}$$

Using inequalities (2.5), (2.6) and definitions (2.1)–(2.4), the proof is completed. □

LEMMA 2.2. *Let $F^{\alpha,\beta}$ be a real entire function GBASP of generalized order and generalized type. Then*

$$\rho = \limsup_{r\to\infty} \tfrac{\alpha(\log \overline{M}(r, F^{\alpha,\beta}))}{\alpha(\log r)} \tag{2.7}$$

$$T = \limsup_{r\to\infty} \tfrac{\alpha[\log \overline{M}(r, F^{\alpha,\beta})]}{[\alpha(\log r)^{\rho}].} \tag{2.8}$$

Proof. Using the definitions of generalized order and generalized type of entire GBASP and proof proceeds on the lines of Lemma 1 [6]. □

LEMMA 2.3. *Let $\alpha(x) \in \overline{\Omega}$ and $K \subseteq \mathbb{C}$ be an arbitrary compact set with* $\operatorname{card} K = \infty$. *Let $f \in L^p(K), 1 \leq p \leq \infty$, be an entire function. Then f has generalized order $\rho(f), 1 \leq \rho(f) \leq \infty$, if and only if,*

$$\rho(f) = \Theta(L(f)), \tag{2.9}$$

where

$$L(f) = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha[\log\{e^p_n(f,K)/m^*_{n+1}\}^{-\frac{1}{n}}]}, \tag{2.10}$$

and

$$\Theta(L(f)) = \begin{cases} \max\{1, L(f)\}, & \textit{if } \ \alpha(x) \in \Omega, \\ 1 + L(f), & \textit{if } \ \alpha(x) \in \overline{\Omega}. \end{cases}$$

Proof. Let $d = \max\{|z|; z \in K\}$. From the definition of $e_n^p(f,K)$, since $h \in p_n$, we have

$$e_n^p(f,K) \le \|f-h\|_p \le A^{\frac{1}{p}} \max_{z\in K} |f(z)-h(z)| = A^{\frac{1}{p}} e_n(f,K), \tag{2.11}$$

where A is the area of K.

Now using Lemma 2 [2, p. 923], it has been shown that for $R > d$,

$$e_n(f,K) \le \tfrac{Rm^*_{n+1}}{(R-d)^{n+2}} \tfrac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\theta})| d\theta. \tag{2.12}$$

Using (2.12) in (2.11) we get

$$e_n^p(f,K)/m^*_{n+1} \le A^{\frac{1}{p}} R(R-d)^{-(n+2)} M(R,f). \tag{2.13}$$

Suppose $\alpha(x) \in \overline{\Omega}$ and $\rho < \infty$. Then by the definition of ρ, we have for any given $\varepsilon > 0$ and $R > R_0(\varepsilon), R(\varepsilon) \in (0,\infty)$,

$$\alpha(\log M(R,f)) \le \alpha(\log R)\overline{\rho}, \quad \overline{\rho} = \rho + \varepsilon. \tag{2.14}$$

In view of (2.13) and (2.14) we have

$$e_n^p(f,K)/m^*_{n+1} < A^{\frac{1}{p}} R(R-d)^{-(n+2)} \exp[\alpha^{-1}\{\overline{\rho}\alpha(\log R)\}]. \tag{2.15}$$

Since d is finite and fixed and the above inequality holds for all $R > R_0(\varepsilon)$, we can choose

$$R = R(n) = \exp\left[\alpha^{-1}\left\{\tfrac{\alpha(n)}{\overline{\rho}-1}\right\}\right] = \exp\left[F\big(n, \tfrac{1}{\rho-1}\big)\right].$$

Substituting this value of R in (2.15), we obtain

$$\begin{aligned} e_n^p(f,K)/m^*_{n+1} &< A^{\frac{1}{p}} \exp\left[-(n+1)F\big(n,\tfrac{1}{\rho-1}\big)\right] \exp\left[\alpha^{-1}\left\{\overline{\rho} + \tfrac{\alpha(n)}{\overline{\rho}-1}\right\}\right] \\ &< A^{\frac{1}{p}} \exp\left[-n\left\{F\big(n,\tfrac{1}{\overline{\rho}-1}\big) - 1\right\}\right], \end{aligned}$$

since $F\big(n,\tfrac{1}{\overline{\rho}-1}\big) \to \infty$ as $n \to \infty$. Hence

$$\begin{aligned} &\log\left[\{e_n^p(f,K)/m^*_{n+1}\}^{-\frac{1}{n}}\right] > \\ &> F\big(n,\tfrac{1}{\overline{\rho}-1}\big) - 1 - \tfrac{1}{np}\log A \\ &= \alpha^{-1}\left\{\tfrac{\alpha(n)}{\overline{\rho}-1}\right\}\left\{1 - \Big(F\big(n,\tfrac{1}{\overline{\rho}-1}\big)\Big)^{-1}\left[1 + \tfrac{1}{np}\log A\right]\right\} > \tfrac{\alpha(n)}{\overline{\rho}-1}. \end{aligned}$$

Since $\alpha(x) \in \overline{\Omega}$, as $n \to \infty$, we have

$$\rho(f) \ge 1 + \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\left[\log[e_n^p(K,f)/m^*_{n+1}]^{-\frac{1}{n}}\right]}. \tag{2.16}$$

In order to prove reverse inequality, let us put

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\left[\log\{e_n^p(f,K)/m^*_{n+1}\}^{-\frac{1}{n}}\right]} = L(f). \tag{2.17}$$

Suppose $L(f) < \infty$. Then for given $\varepsilon > 0$ and all $n > n_0(\varepsilon)$, we have

$$e_n^p(f,K) < m^*_{n+1} \exp\left[-nF\big(n,\tfrac{1}{\overline{L}(f)}\big)\right], \quad \overline{L}(f) = L(f) + \varepsilon.$$

Now we consider the function

$$h^*(z) = \sum_{n=n_0}^{\infty} a_{n+1,R_0}^{n+1} z^{n+1} \exp\Big[-nF\Big(n, \tfrac{1}{L(f)}\Big)\Big]$$

where $R_0 > d$ and

$$a_{n+1,R_0} = \left\{ \Big(1 + \tfrac{d}{R_0}\Big)^{n+2} \left[\frac{2(n+2)}{1-\frac{d}{R_0}} \right] \right\}^{\frac{1}{(n+1)}}.$$

Suppose $\{p_n(z)\}_0^\infty$ be the best polynomial approximating for the function f on K. Let D_R denote the disk of radius R centered at the origin and Γ_R be the boundary of D_R. Let

$$S(z) = \sum_{n=0}^{\infty} \{p_{n+1}(z) - p_n(z)\} + p_0(z). \tag{2.18}$$

In view of proof of [2, Th. 1, p. 924], it can be easily seen that the series (2.18) is uniformly convergent on Γ_R for any arbitrary $R > 0$. Thus the sum represent an entire function. Now

$$\begin{aligned} S(z) &= \lim_{n\to\infty} \left\{ \sum_{m=0}^{n} \{p_{m+1}(z) - p_m(z)\} \right\} + p_0(z), \\ &= \lim_{n\to\infty} p_{n+1}(z) = f(z). \end{aligned}$$

On $K \subseteq \mathbb{C}$ we have the inequality

$$\begin{aligned} \|p_{n+1}(z) - p_n(z)\|_p &\le \|p_{n+1}(z) - f(z)\|_p + \|p_n(z) - f(z)\|_p \\ &\le 2e_n^p(f,K). \end{aligned}$$

Dovgoshei [2, p. 924] shown that

$$\begin{aligned} \max_{z\in\Gamma_R} |p_{n+1}(z) - p_n(z)|^{\frac{1}{n}} &\le \Big[[R^{n+1} e_n^\infty(f,K)/m_{n+1}^*] \Big]^{\frac{1}{n}} \\ &\quad \cdot \left[2(n+2) \frac{(1+(\frac{d}{R}))^{n+2}}{1-\frac{d}{R}} \right]^{\frac{1}{n}}. \end{aligned}$$

For $1 \le p \le \infty$, we have

$$\|p_{n+1}(z) - p_n(z)\|_p \le \left[2AR^{n+1}(n+2) \frac{\big(1+(\frac{d}{R})\big)^{n+2}}{m_{n+1}^*(1-\frac{d}{R})} \right]^{\frac{1}{np}} [e_n^p(f,K)]^{\frac{1}{n}}$$

leading to the relation

$$\begin{aligned} \max_{z\in\Gamma_R} |h^*(z)| &= \sum_{n=n'}^{\infty} a_{n+1,R_0}^{n+1} R^{n+1} \exp\Big[-nF\big(n, \tfrac{1}{L(f)}\big)\Big] \\ &\ge \max_{z\in\Gamma_R} |S(z) - p_{n'}(z)| \ge A^{-\frac{1}{p}} \|S(z) - p_{n'}(z)\|_p. \end{aligned}$$

From the last inequality, using [7, Th. 4], the relationship between generalized order ρ for $\alpha(x) \in \overline{\Omega}$ and the Taylor coefficients of the function $h^*(z)$, we obtain generalized order of $h^*(z) \geq$ generalized order of f.

If ρ_1 denotes the generalized order of $h^*(z)$ then

$$\rho_1 = 1 + \limsup_{n\to\infty} \frac{\alpha(n+1)}{\alpha\Big[\log\Big\{\big|a_{n+1,R_0}^{n+1}\big|\exp\big\{-nF\big(n,\frac{1}{\overline{L}(f)}\big)\big\}\Big\}^{-\frac{1}{(n+1)}}\Big]}.$$

Since

$$\log(a_{n+1,R_0}) = \tfrac{n+2}{n+1}\log(1+\tfrac{d}{R_0}) + \tfrac{1}{n+1}\log\left(\frac{2(n+1)}{1-\frac{d}{R_0}}\right) = \mathcal{O}(1), \quad \text{as } n\to\infty.$$

Hence

$$\begin{aligned}&\log\big\{|a_{n+1,R_0}^{n+1}|\exp\big\{-nF\big(n,\tfrac{1}{\overline{L}(F)}\big)\big\}\big\}^{-\frac{1}{(n+1)}} =\\ &= -\tfrac{1}{n+1}\log\big[\exp\big(-nF\big(n,\tfrac{1}{\overline{L}(F)}\big)\big)\big] - \mathcal{O}(1)\\ &= \tfrac{n}{n+1}F\big(n,\tfrac{1}{\overline{L}(f)}\big) \simeq \alpha^{-1}\Big[\tfrac{\alpha(n)}{\overline{L}(f)}\Big] - \mathcal{O}(1).\end{aligned}$$

Since $\alpha(x) \in \overline{\Omega}$, we finally get

$$\rho_1 = 1 + \limsup_{n\to\infty}\frac{\alpha(n+1)\overline{L}(f)}{\alpha(n)} = 1 + \overline{L}(f) = 1 + L(f) + \varepsilon.$$

Since ε was arbitrary, we get $\rho_1 \geq \rho(f)$. Combining this with (2.16) we get (2.9). ☐

LEMMA 2.4. *Let $\alpha(x) \in \overline{\Omega}$ and $K \subseteq \mathbb{C}$ be an arbitrary compact set with Card. $K = \infty$. Let $f \in L^p(K), 1 \leq p \leq \infty$, be an entire function. Then f has generalized order $\rho(f)$ and finite generalized type $T(f)$ if, and only if,*

$$T = T(f) = \limsup_{n\to\infty}\frac{\alpha(\log M(R,f))}{[\alpha(\log R)]^{\rho}} = \limsup_{n\to\infty}\frac{\alpha(\frac{n}{\rho})}{\Big\{\alpha\Big[\frac{\rho}{\rho-1}\log\Big[\frac{e_n^p(f,K)}{m_{m+1}^*}\Big]^{-\frac{1}{n}}\Big]\Big\}^{\rho+1}} \tag{2.19}$$

provided $\frac{d[\alpha^{-1}\{(T+\varepsilon)[\alpha(x)]^{T/\rho}\}]}{d(\log x)} = \mathcal{O}(1)$ *as* $x\to\infty$, *for* $T, 0 < T < \infty$.

Proof. First we assume that f is of generalized type T with respect to the finite number ρ *i.e.*, $\rho < \infty$ and $\alpha(x) \in \overline{\Omega}$. Let $T < \infty$. Then for arbitrary $\varepsilon > 0$ and $R > R'(\varepsilon)$,

$$M(R,f) < \exp[\alpha^{-1}\{\overline{T}[\alpha(\log R)]^{\rho}\}].$$

Using (2.13), we get

$$e_n^p(f,K)/m_{n+1}^* \leq A^{1/p}R(R-d)^{-(n+2)}\exp[\alpha^{-1}\{\overline{T}[\alpha(\log R)]^{\rho}\}, \quad \overline{T} = T+\varepsilon. \tag{2.20}$$

The above inequality holds for all n and $R > R'(\varepsilon)$. To minimize the right hand side of (2.20) taking $R = R(n)$ to be the unique root of the equation

$$n = \tfrac{\rho}{\log R}(\alpha^{-1}\{\overline{T}[\alpha(\log R)]^{\rho}\}), \quad n = 2,3,\ldots,$$

or

$$\log R = \alpha^{-1}[(\tfrac{1}{\overline{T}}\alpha(\tfrac{n}{\rho}))^{\frac{1}{(\rho-1)}}] \quad \text{and} \quad (R-d)^{-n} \cong R^{-n},$$

substituting these values in (2.20), we get

$$\log(e_n^p(f,K)/m_{n=1}^*) \leq -n\alpha^{-1}[(\tfrac{1}{\overline{T}}\alpha(\tfrac{n}{\rho}))^{\frac{1}{(\rho-1)}}] + \tfrac{n}{\rho}\alpha^{-1}[(\tfrac{1}{\overline{T}}\alpha(n\rho))^{\frac{1}{(\rho-1)}}]$$

or

$$\tfrac{\rho}{\rho-1}\log(e_n^p(f,K)/m_{n+1}^*)^{\frac{1}{n}} \geq \alpha^{-1}[(\tfrac{\alpha(\frac{n}{\rho})}{\overline{T}})^{\frac{1}{(\rho-1)}}]$$

or

$$\overline{T} \geq \frac{\alpha(\frac{n}{\rho})}{[\alpha(\frac{\rho}{\rho-1}\log(e_n^p(f,K)/m_{n+1}^*)^{-\frac{1}{n}})]^{\rho-1}}.$$

Since $\alpha(x) \in \overline{\Omega}$, as $n \to \infty$, we have

$$T \geq \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{[\alpha(\frac{\rho}{\rho-1}\log(e_n^p(f,K)/m_{n+1}^*)^{-\frac{1}{n}})]^{\rho-1}}. \tag{2.21}$$

To prove the reverse inequality, we follow the method of proof of Lemma 2.3. Hence let

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{[\alpha(\frac{\rho}{\rho-1}\log(e_n^p(f,K)/m_{n+1}^*)^{\frac{1}{n}})]^{\rho-1}} = \sigma^*.$$

Then for a given $\varepsilon > 0$ and all $n > n_0(\varepsilon)$, we have

$$e_n^p(f,K) < m_{n+1}^* \exp\{-\tfrac{n}{\rho}(\rho-1)\alpha^{-1}([\tfrac{\alpha(\frac{n}{\rho})}{\overline{\sigma}^*}]^{\frac{1}{(\rho-1)}})\}, \quad \overline{\sigma}^* = \sigma^* + \varepsilon.$$

Now consider the function $g(z)$ defined by the infinite series

$$\begin{aligned} g(z) &= \sum_{n=n_0}^{\infty} a_{n+1,R_0}^{n+1} z^{n+1} \exp\{-\tfrac{n}{\rho}(\rho-1)\alpha^{-1}([\tfrac{\alpha(\frac{n}{\rho})}{\overline{\sigma}^*}]^{\frac{1}{(\rho-1)}})\}, \\ &= \sum_{n+n_0}^{\infty} b_{n+1} z^{n+1}, \text{ say,} \end{aligned}$$

where the sequence $\{a_{n+1,R_0}^{n+1}\}$ is as defined before. Since $\alpha^{-1}\{[\alpha^{-1}(\sigma\alpha(x))]^{\frac{1}{\rho}}\} \to \infty$ as $x \to \infty$, we get

$$\limsup_{n\to\infty}[a_{n+1,R_0}^{n+1} \exp\{\tfrac{-n}{\rho}(\rho-1)\alpha^{-1}([\tfrac{\alpha(\frac{n}{\rho})}{\overline{\sigma}^*}]^{\frac{1}{(\rho-1)}})\}]^{\frac{1}{\rho}} = 0$$

and therefore $g(z)$ represents an entire function. Now

$$\begin{aligned} \max_{z\in\Gamma_R}|g(z)| &= \sum_{n=n_0}^{\infty} a_{n+1,R_0}^{n+1} R^{n+1} \exp\{-\tfrac{n}{\rho}(\rho-1)\alpha^{-1}([\tfrac{\alpha(\frac{n}{\rho})}{\overline{\sigma}^*}]^{\frac{1}{(\rho-1)}})\} \\ &\geq \sum_{n=n_0}^{\infty} a_{n+1,R_0}^{n+1} R^{n+1} e_n^p(f,K)/m_{n+1}^* \\ &\geq A^{\frac{1}{p}} \|S(z) - p_{n'}(z)\|_p. \end{aligned}$$

Hence if $g(z)$ is an entire function of generalized type T' with respect to the finite number ρ then from above the entire function $g(z)$, we have

$$T' = \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\left[\alpha\left(\frac{\rho}{\rho-1}\log|b_n|^{-\frac{1}{n}}\right)\right]^{\rho-1}}. \tag{2.22}$$

Now

$$\begin{aligned}
&|b_n|^{\frac{1}{n}} \simeq (a_{n+1,R_0})\exp\{\tfrac{\rho-1}{\rho}\alpha^{-1}([\tfrac{\alpha(\frac{n}{\rho})}{\sigma^*}]^{\frac{1}{(\rho-1)}})\},\\
&\tfrac{\rho}{\rho-1}\log|b_n|^{-\frac{1}{n}} \simeq \alpha^{-1}([\tfrac{\alpha(\frac{n}{\rho})}{\sigma^*}]^{\frac{1}{(\rho-1)}})(1+\mathcal{O}(1))\\
&[\alpha(\tfrac{\rho}{\rho-1}\log|b_n|^{-\frac{1}{n}})]^{\rho-1} \simeq \tfrac{\alpha(\frac{n}{\rho})}{\sigma^*}.
\end{aligned}$$

Putting these values in (2.22), we get $T' = \sigma^* + \varepsilon$. As stated above we have $T^* \geq T$. Hence we get for arbitrary $\varepsilon > 0, \sigma^* + \varepsilon \geq T$, *i.e.*, $T \leq \sigma^*$. Combining this with (2.21), the proof is immediate. □

3. MAIN RESULTS

Now we shall prove our main results.

Following on the lines of proof of [15, Th. 1] we obtain the following inequalities for $1 \leq p \leq \infty$,

$$\begin{aligned}
\|F^{\alpha,\beta} - H\|_p &\leq w^{\frac{1}{p}}\|f-h\|_p, \qquad w = w(\alpha,\beta,p;K),\\
\|f-h\|_p &\leq \delta^{\frac{1}{p}}\|F^{\alpha,\beta} - H\|_p, \qquad \delta = \delta(\alpha,\beta,p;K)
\end{aligned}$$

for $H \in P_{2n}^{\alpha,\beta}$ and each $h \in K_{\alpha,\beta}^{-1}(H) \in p_{2n}$, $n = 0,1,2,\ldots$.

Hence we get optimal approximates

$$E_{2n}^{p/2n}(F^{\alpha,\beta}, \Sigma_r^{\alpha,\beta}) \leq w^{\frac{1}{(2np)}} e_{2n}^{\frac{p}{(2n)}}(f,K), \tag{3.23}$$

and

$$e_{2n}^{\frac{p}{(2n)}}(f,K) \leq \delta^{\frac{1}{(2np)}} E_{2n}^{\frac{p}{(2n)}}(F, \Sigma_r^{\alpha,\beta}). \tag{3.24}$$

THEOREM 3.1. *Let $\alpha(x) \in \overline{\Omega}$. For fixed $p \geq 1$, let the $F^{\alpha,\beta} \in L^p(\Sigma_r^{\alpha,\beta})$ be the restriction to $\Sigma_r^{\alpha,\beta}$ of an entire GBASP function. Then $F^{\alpha,\beta}$ has generalized order ρ if, and only if,*

$$\rho = \Theta(L^*)$$

where

$$L^* = \limsup_{n\to\infty} \frac{\alpha(2n)}{\alpha(\log[E_{2n}^p(F^{\alpha,\beta}, E_r^{\alpha,\beta})/m_{2n+2}^*]^{-\frac{1}{(2n)}})} \tag{3.25}$$

and $\Theta(L^*)$ is defined as in Lemma 3.

Proof. Let $F \in L^p(\Sigma_r^{\alpha,\beta})$ be the restriction to $\Sigma_r^{\alpha,\beta}$ of an entire GBASP function of generalized order ρ and let $\varepsilon > 0$ be given. From (2.10), the appraisal

$$L(f) - \varepsilon < \frac{\alpha(2n)}{\alpha(\log[e^p_{2n}(f,K)/m^*_{2n+2}]^{-\frac{1}{(2n)}})} < L(f) + \varepsilon, \tag{3.26}$$

applies to the $K_{\alpha,\beta}$ associate with the lower bound holding for $n \in n_1(\varepsilon)$, an infinite sequence of indices, and the upper bound for $n \in n_2(\varepsilon)$, as sequence of all but a finite number of indices. From (3.24), we have

$$\frac{\alpha(2n)}{\alpha(\log[\delta^{1/p}E^p_{2n}(\Sigma_r^{\alpha,\beta})/m^*_{2n+2}]^{-\frac{1}{(2n)}})} > \frac{\alpha(2n)}{\alpha(\log[e^p_{2n}(f,K)/m^*_{2n+2}]^{-\frac{1}{(2n)}})} > L(f) - \varepsilon, \tag{3.27}$$

$n \in n_1(\varepsilon)$. For an upper estimates, (3.23) gives

$$\begin{aligned} L(f) + \varepsilon \quad &> \quad \frac{\alpha(2n)}{\alpha(\log(w^{\frac{1}{p}}\, e^p_{2n}(f,K)/m^*_{2n+2})^{-\frac{1}{(2n)}})} \\ &> \quad \frac{\alpha(2n)}{\alpha(\log(E^p_{2n}(F^{\alpha,\beta},\Sigma_r^{\alpha,\beta})/m^*_{2n+2})^{-\frac{1}{(2n)}})}, \quad n \in n_2(\varepsilon). \end{aligned} \tag{3.28}$$

Thus,

$$L(f) - \varepsilon \le \limsup_{n\to\infty} \frac{\alpha(2n)}{\alpha(\log(E^p_{2n}(F^{\alpha,\beta},\Sigma_r^{\alpha,\beta})/m^*_{2n+2})^{-\frac{1}{(2n)}})} \le L(f) + \varepsilon.$$

Hence the proof is immediate. $\square$

THEOREM 3.2. *Let $\alpha(x) \in \overline{\Omega}$. For fixed $p \ge 1$, let the $F^{\alpha,\beta} \in L^p(\Sigma_r^{\alpha,\beta})$ be the restriction to $\Sigma_r^{\alpha,\beta}$ of an entire GBASP function. Then $F^{\alpha,\beta}$ has generalized order ρ and finite generalized type $T(F^{\alpha,\beta})$ if, and only if,*

$$T = \limsup_{n\to\infty} \frac{\alpha(\frac{2n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\log\{E^p_{2n}(F^{\alpha,\beta},\Sigma_r^{\alpha,\beta})/m^*_{2n+2}\}^{-\frac{1}{(2n)}}]\}^{\rho-1}}$$

provided

$$\frac{d(\alpha^{-1}\{(T+\varepsilon)[\alpha(x)]^{T/\rho}\})}{d(\log x)} = \mathcal{O}(1), \quad \text{as } x \to \infty, \text{for } T, 0 < T < \infty.$$

Proof. From Lemma 4, for $\varepsilon > 0$ given,

$$T - \varepsilon < \frac{\alpha(\frac{2n}{\rho})}{\left\{\alpha\left[\frac{\rho}{\rho-1}\log\left(\frac{e^p_{2n}(f,K)}{m^*_{2n+2}}\right)^{-\frac{1}{(2n)}}\right]\right\}^{\rho-1}} < T + \varepsilon$$

with the lower bound for $n \in n_1(\varepsilon)$ and the upper bound for $n \in n_2(\varepsilon)$. Now, using (3.24) we get

$$\begin{aligned} T - \varepsilon \quad &< \quad \frac{\alpha(\frac{2n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\log\{\frac{e^p_{2n}(f,K)}{m^*_{2n+2}}\}^{-\frac{1}{(2n)}}]\}^{\rho-1}} \\ &< \quad \frac{\alpha(\frac{2n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\log\{\delta^{1/p}E^p_{2n}(F^{\alpha,\beta},\Sigma_r^{\alpha,\beta})/m^*_{2n+2}\}^{-\frac{1}{(2n)}}]\}^{\rho-1}} \end{aligned} \tag{3.29}$$

for $n \in n_1(\varepsilon)$. The upper bound can be found by using (3.23)

$$\begin{aligned} & \frac{\alpha(\frac{2n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\log\{E^p_{2n}(F^{\alpha,\beta},\Sigma^{\alpha,\beta}_r)/m^*_{2n+2}\}^{-\frac{1}{(2n)}}]\}^{\rho-1}} < \\ (3.30) \qquad & < \frac{\alpha(\frac{2n}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\log\{w^{1/p}e^p_{2n}(f,K)/m^*_{2n+2}\}^{-\frac{1}{(2n)}}]\}^{\rho-1}} \\ & < T + \varepsilon. \end{aligned}$$

Taking limit supremum and combining (3.29) and (3.31) we get the required result. □

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