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ON BASKAKOV OPERATORS PRESERVING THE EXPONENTIAL FUNCTION

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Abstract. In this paper, we are concerned about the King-type Baskakov operators defined by means of the preserving functions e_0 and e^{2ax}, $a>0$ fixed. Using the modulus of continuity, we show the uniform convergence of the new operators to f. Also, by analyzing the asymptotic behavior of King-type operators with a Voronovskaya-type theorem, we establish shape preserving properties using the generalized convexity.

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1. INTRODUCTION

Approximation theory is one of the important subjects that is frequently used by scientific community. It is divided into many fields one of which is linear positive operators that have an essential role in approximation theory. Some linear positive operators such as Bernstein operators are defined within finite intervals, there are also many linear positive operators that are defined in infinite intervals such as Baskakov operators [7] defined as

$$K_n\left(f,x\right)=\tfrac{1}{(1+x)^n}\sum_{k=0}^{\infty}f\left(\tfrac{k}{n}\right)\binom{n+k-1}{k}\tfrac{x^k}{(1+x)^k}$$

where $n\in\mathbb{N}$, $x\in[0,\infty)$ and $f\in C\,[0,\infty)$ whenever the above sum converges.

In recent years, several operators were appropriately modified, which preserve the test functions (or the linear combination of them) emerged in this field and many improvements have been made regarding this matter to obtain better approximation. One may see some of the results in the related direction in [2], [3], [4], [6], [8], [12], [13], [16], [18], [20] and [22] etc.

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In 2003, King [21] described the modified Bernstein operators which preserve e_0 and e_2 test functions on $[0,1]$ and examined their approximation properties. Later in 2007, Duman and Özarslan [15], introduced Szász-Mirakyan type operators preserving the test function e_0 and e_2 on the interval $[0,\infty)$ and achieved better approximation for the generalization of the classical Szász-Mirakyan operators.

In 2006, Morales et al. [11] have considered the King-type Bernstein polynomials which reproduce the linear combination of test functions for $\alpha > 0$, $e_2 + \alpha e_1$ and analyzed their shape preserving and approximation properties. They gave information about the order of approximation by comparing Bernstein and its modified operators; hence they obtained more general operators than those of King.

Besides preserving the test functions, in 2010 Aldaz and Render [5] defined linear positive operators which preserve exponential functions and in 2014 Birou [9] gave some examples about general King-type operators which reproduce polynomials and exponential functions.

Very recently, Acar et al. [1] proposed the modification of the classical Szász-Mirakyan operators which preserve the functions e_0 and $e^{2ax},\ a > 0$. They investigated the uniform convergence and shape preserving properties of these operators. Using weighted modulus of continuity and first and second order moduli of smoothness, they obtained important approximation properties of the modified operators. A number of results concerning Szász-Mirakyan type operators and generalization of them can be found in [16]. Also, it was pointed out recently in [17] that by finding the moment generation function, which is common in statistical analysis, one can easily obtain the moments. They obtained m.g.f. and moments of many operators in their recent paper [17].

The aim of this paper is to consider the modification of the Baskakov operators preserving e^{2ax}. We introduce our operators for $x \in [0,\infty)$, as

$$
\begin{aligned}
K_{n,\beta}(f,x) &= \sum_{k=0}^{\infty} f\left(\tfrac{k}{n}\right)\tbinom{n+k-1}{k}\tfrac{(\beta_n(x))^k}{(1+\beta_n(x))^{n+k}} \\
&= \sum_{k=0}^{\infty} f\left(\tfrac{k}{n}\right)\tbinom{-n}{k}\left(-\beta_n(x)\right)^k\left(1+\beta_n(x)\right)^{-n-k} \qquad (1)\\
&= K_n\left(f,\beta_n(x)\right),
\end{aligned}
$$

where

$$\beta_n(x) = \left(K_n(e^{2at},x)\right)^{-1} \circ e^{2ax}. \qquad (2)$$

In particular, if we take $\beta_n(x) = x$ in (1), the King-type Baskakov operators return to classical Baskakov operators.

Let's find the function β_n, satisfying the condition,

$$K_{n,\beta}(e^{2at},x)=e^{2ax},$$

which is satisfied for all $x\in[0,\infty)$ and $n\in\mathbb{N}$.

From the definition of King-type Baskakov Operators, we can write

$$\begin{aligned} e^{2ax} &= K_{n,\beta}(e^{2at},x) \\ e^{2ax} &= \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{\left(e^{\frac{2a}{n}}\beta_n(x)\right)^k}{(1+\beta_n(x))^{n+k}} \\ e^{\frac{-2ax}{n}} &= 1+\beta_n\left(x\right)\left(1-e^{\frac{2a}{n}}\right) \end{aligned}$$

and finally we get,

$$\beta_n\left(x\right)=\frac{e^{\frac{-2ax}{n}}-1}{\left(1-e^{\frac{2a}{n}}\right)}. \tag{3}$$

The structure of the paper is as follows. In Section 2, we give some lemmas about new operators which will be used in subsequent sections. In section 3, the uniform convergence of the operators is gained by means of modulus of continuity. The order of approximation is obtained with the help of Voronovskaya-type theorem in Section 4. In the last section, we investigate some results about shape preserving property of King-type Baskakov operators by comparing them to classical operators.

2. PRELIMINARIES

Here, we mention some lemmas about our operator $K_{n,\beta}$ which will be used to prove theorems in other sections.

LEMMA 1. *Let $a>0$ and $n\in\mathbb{N}$. For $f(t)=e^{at}$, the operators get,*

$$\begin{aligned} K_{n,\beta}(e^{at},x) &= \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{\left(e^{\frac{a}{n}}\beta_n(x)\right)^k}{(1+\beta_n(x))^{n+k}} \\ &= \left(\beta_n\left(x\right)\left(1-e^{\frac{a}{n}}\right)+1\right)^{-n} \\ &= \left(\frac{e^{-\frac{2ax}{n}}-1}{\left(1-e^{\frac{2a}{n}}\right)}\left(1-e^{\frac{a}{n}}\right)+1\right)^{-n} \\ &= \left(\frac{e^{\frac{a}{n}}+e^{-\frac{2ax}{n}}}{1+e^{\frac{a}{n}}}\right)^{-n}. \end{aligned}$$

Moreover,

$$\lim_{n\to\infty}\left(\frac{e^{\frac{a}{n}}+e^{-\frac{2ax}{n}}}{1+e^{\frac{a}{n}}}\right)^{-n}=e^{ax}.$$

LEMMA 2. *Let $e_i\left(t\right)=t^i$, $i=0,1,2$. For $a>0$, we have,*

$$\begin{aligned} K_{n,\beta}(e_0,x) &= 1, \\ K_{n,\beta}(e_1,x) &= \beta_n\left(x\right), \\ K_{n,\beta}(e_2,x) &= \left(\beta_n\left(x\right)\right)^2+\frac{\beta_n(x)(1+\beta_n(x))}{n}. \end{aligned}$$

Proof. Let $f(t) = e^{\theta t}$, $\theta \in \mathbb{R}$, then

$$K_{n,\beta}(e^{\theta t}, x) = \left(1 + \beta_n(x) - \beta_n(x) e^{\frac{\theta}{n}}\right)^{-n}. \tag{4}$$

Using the software Mathematica, we get the expansion of right side of (4) in powers of θ as follows:

$$\begin{aligned} & K_{n,\beta}(e^{\theta t}, x) \\ = \; & 1 + \beta_n(x)\, \theta + \left(\tfrac{n(\beta_n(x))^2 + (\beta_n(x))^2 + \beta_n(x)}{n}\right) \tfrac{\theta^2}{2!} \\ & + \left(\tfrac{(\beta_n(x))^3 + 3n(\beta_n(x))^3 + 2(\beta_n(x))^3 + 3n(\beta_n(x))^2 + 3(\beta_n(x))^2 + \beta_n(x)}{n^2}\right) \tfrac{\theta^3}{3!} + \mathcal{O}(\theta^4). \end{aligned}$$

Following [17], we can say that $K_{n,\beta}(e^{\theta t}, x)$ may be treated as moment generating function of the operators $K_{n,\beta}$, which may be utilized to obtain the moments of (1). Thus using the following equivalence between moments and moment generating function:

$$\begin{aligned} K_{n,\beta}(e_r, x) &= \left[\tfrac{\partial^r}{\partial \theta^r} K_{n,\beta}(e^{\theta t}, x)\right]_{\theta=0} \\ &= \left[\tfrac{\partial^r}{\partial \theta^r}\left\{\left(1 + \beta_n(x) - \beta_n(x) e^{\frac{\theta}{n}}\right)^{-n}\right\}\right]_{\theta=0}, \end{aligned} \tag{5}$$

we obtain the moments immediately. $\square$

LEMMA 3. *Let $e_i^x(t) = (t-x)^i$, $i = 0, 1, 2$. From Lemma 2, we immediately have*

$$\begin{aligned} K_{n,\beta}(e_0^x, x) &= 1, \\ K_{n,\beta}(e_1^x, x) &= \beta_n(x) - x, \\ K_{n,\beta}(e_2^x, x) &= (\beta_n(x) - x)^2 + \tfrac{\beta_n(x)(1+\beta_n(x))}{n}, \end{aligned}$$

and also using the software Mathematica, we can compute the following limits,

$$\lim_{n \longrightarrow \infty} n K_{n,\beta}(e_1^x, x) = -ax(x+1), \tag{6}$$

$$\lim_{n \longrightarrow \infty} n K_{n,\beta}(e_2^x, x) = x(x+1). \tag{7}$$

3. MAIN RESULT

In this section we deal with the uniform convergence of $K_{n,\beta}(f, x)$ to f on $[0, \infty)$ by means of the modulus of continuity. First we will give the theorem about uniform convergence for linear positive operators in terms of exponential functions and second theorem is about the rate of the uniform convergence.

Let us denote by $C^*[0, \infty)$, the Banach space of all real-valued continuous functions on $[0, \infty)$ with the property $\lim_{x \longrightarrow \infty} f(x)$ exists and finite, given with the uniform norm.

THEOREM 4. [10] *If the sequence $A_n : C^*[0,\infty) \rightharpoonup C^*[0,\infty)$ of positive linear operators satisfies the conditions*

$$\lim_{n \longrightarrow \infty} A_n(e^{-kt}, x) = e^{-kx}, \quad k = 0, 1, 2.$$

uniformly in $[0,\infty)$ *then for* $f\in[0,\infty)$,

$$\lim_{n\longrightarrow\infty} A_n(f,x)=f(x)$$

uniformly in $[0,\infty)$.

Now we mention the theorem which is about the estimate for the rate of the uniform convergence. In this theorem, we will use the definition of the modulus of continuity of f in [19], for any $\delta>0$ and each $x\in[0,\infty)$ is defined by,

$$\omega^*(f,\delta)=\sup_{\substack{x,t\geq 0\\ |e^{-x}-e^{-t}|\leq\delta}}|f(x)-f(t)|.$$

This modulus of continuity has the property

$$|f(t)-f(x)|\leq\Big(1+\tfrac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\Big)\omega^*(f,\delta),\quad \delta>0.$$

Theorem 5. [19] *Let* $A_n: C^*[0,\infty)\rightharpoonup C^*[0,\infty)$ *be a sequence of linear positive operators with*

$$\begin{aligned}\|A_n(1,x)-1\|_{[0,\infty)} &= a_n,\\ \|A_n(e^{-t},x)-e^{-x}\|_{[0,\infty)} &= b_n,\\ \|A_n(e^{-2t},x)-e^{-2x}\|_{[0,\infty)} &= c_n.\end{aligned}$$

Then

$$\|A_n(f,x)-f(x)\|_{[0,\infty)}\leq\|f\|_{[0,\infty)}a_n+(2+a_n)\,\omega^*(f,\sqrt{a_n+2b_n+c_n})$$

for every function of $f\in C^*[0,\infty)$.

Now we will give the main result about uniform convergence of $K_{n,\beta}$ by means of the modulus of continuity which is defined above.

Theorem 6. *For* $f\in C^*[0,\infty)$, *we have*

$$\|K_{n,\beta}f-f\|_{[0,\infty)}\leq 2\omega^*(f,\sqrt{2b_n+c_n}),$$

where

$$\begin{aligned}\|K_{n,\beta}(e^{-t},x)-e^{-x}\|_{[0,\infty)} &= b_n,\\ \|K_{n,\beta}(e^{-2t},x)-e^{-2x}\|_{[0,\infty)} &= c_n.\end{aligned}$$

Here when $n\longrightarrow\infty$, b_n *and* c_n *tend to zero and* $\{K_{n,\beta}f\}$ *converges uniformly to the function* f.

Proof. It is enough to show that the conditions of the Theorem 4. From Lemma 2, we can write that

$$\|K_{n,\beta}(1,x)-1\|_{[0,\infty)}=a_n=0.$$

Now we can examine that for the images of the e^{-kt}, $k=1,2.$ under the operators $K_{n,\beta}f$. To show it, we will use the (1) and (3).

For $f(t) = e^{-t}$,

$$\begin{aligned} K_{n,\beta}(e^{-t}, x) &= \sum_{k=0}^{\infty} \tbinom{n+k-1}{k} \tfrac{\left(e^{\frac{-1}{n}}\beta_n(x)\right)^k}{(1+\beta_n(x))^{n+k}} \\ &= \sum_{k=0}^{\infty} \tbinom{-n}{k} \Big(-e^{\frac{-1}{n}}_n \beta_n(x)\Big)^k (1+\beta_n(x))^{-n-k} \\ &= \left(\beta_n(x)\left(1-e^{-\frac{1}{n}}\right)+1\right)^{-n} \\ &= \Big(\tfrac{e^{-\frac{2ax}{n}}-1}{\left(1-e^{\frac{2a}{n}}\right)}\left(1-e^{-\frac{1}{n}}\right)+1\Big)^{-n}. \end{aligned}$$

Expanding the right hand side of above expression using Mathematica, we find that

$$\begin{aligned} K_{n,\beta}(e^{-t}, x) =& e^{-x} + \tfrac{e^{-x}}{2n} x(1+x)(1+2a) \\ &+ \tfrac{e^{-x}}{24n^2}(1+2a)e^{-x}x(1+x)[-4-4a-5x-2ax+3x^2+6ax^2] \\ &+ \mathcal{O}(n^{-3}). \end{aligned}$$

Next, for $f(t) = e^{-2t}$, after simple computation, we have

$$K_{n,\beta}(e^{-2t}, x) = \Big(\tfrac{e^{-\frac{2ax}{n}}-1}{\left(1-e^{\frac{2a}{n}}\right)}\left(1-e^{-\frac{2}{n}}\right)+1\Big)^{-n}.$$

Finally, expanding the right hand side of above expression using Mathematica, we get

$$\begin{aligned} K_{n,\beta}(e^{-2t}, x) =& e^{-2x} + \tfrac{e^{-2x}}{n} 2(1+a)x(1+x) \\ &+ \tfrac{2e^{-2x}}{3n^2} x(1+x)(1+a)(-2-a-x+ax+3x^2+3ax^2) \\ &+ \mathcal{O}(n^{-3}). \end{aligned}$$

The result follows immediately from the above identities. $\square$

4. AYMPTOTIC EXPRESSION

Voronovskaya-type theorem is the main tool to talk about the order of the approximation in approximation theory. Now, we will analyse the asymptotic behavior of our operators $K_{n,\beta}$ in view of the Voronovskaya-type theorem.

THEOREM 7. *Let $f,\ f'' \in C^*[0,\infty)$. Then we have,*

$$\begin{aligned} &\left|n\left[K_{n,\beta}(f,x) - f(x)\right] + ax(x+1)f'(x) - \tfrac{x(x+1)}{2} f''(x)\right| \le \\ &\le |p_n(x)|\,|f'(x)| + |q_n(x)|\,|f''(x)| + 2\left(2q_n + x(x+1) + r_n(x)\right)\omega^*(f'', \tfrac{1}{\sqrt{n}}), \end{aligned}$$

where

$$\begin{aligned} p_n(x) &= nK_{n,\beta}\left(e_1^x(t), x\right) + ax(x+1), \\ q_n(x) &= \tfrac{1}{2}\left(nK_{n,\beta}\left(e_2^x(t), x\right) - x(x+1)\right), \\ r_n(x) &= n^2\sqrt{K_{n,\beta}\left(\left(e^{-x}-e^{-t}\right)^4, x\right)}\sqrt{K_{n,\beta}((t-x)^4, x)}\,. \end{aligned}$$

Proof. From the Taylor's expansion of f at the point x, we have

$$f(t)=f(x)+f'(x)(t-x)+\tfrac{f''(x)}{2}(t-x)^2+h(t,x)(t-x)^2,$$

where,

$$h(t,x)=\tfrac{f''(\eta)-f''(x)}{2},$$

and η is a number lying between x and t.

If we apply the operators $K_{n,\beta}$ on both sides of the above Taylor's expansion and use the linearity property of $K_{n,\beta}$, we can write

$$\begin{aligned}&K_{n,\beta}\left(f,x\right)-f(x)-f'(x)K_{n,\beta}\left(e_1^x(t),x\right)-\tfrac{f''(x)}{2}K_{n,\beta}\left(e_2^x(t),x\right)=\\ &=K_{n,\beta}\left(h(t,x)e_2^x(t),x\right).\end{aligned}\tag{8}$$

We multiply both sides of the (8) with n and applying Lemma 3, we have

$$\begin{aligned}&\left|n\left[K_{n,\beta}\left(f,x\right)-f(x)\right]+ax(x+1)f'(x)-\tfrac{x(x+1)}{2}f''(x)\right|\le\\ &\le\left|\left(nK_{n,\beta}\left(e_1^x(t),x\right)+ax(x+1)\right)\right|\left|f'(x)\right|\\ &\quad+\tfrac{1}{2}\left|nK_{n,\beta}\left(e_2^x(t),x\right)-x(x+1)\right|\left|f''(x)\right|+\left|nK_{n,\beta}\left(h(t,x)e_2^x(t),x\right)\right|.\end{aligned}$$

If we define p_n and q_n with following equations,

$$\begin{aligned}p_n(x)&=nK_{n,\beta}\left(e_1^x(t),x\right)+ax(x+1),\\ q_n(x)&=\tfrac{1}{2}\left(nK_{n,\beta}\left(e_2^x(t),x\right)-x(x+1)\right).\end{aligned}$$

we can write

$$\begin{aligned}&\left|n\left[K_{n,\beta}\left(f,x\right)-f(x)\right]+ax(x+1)f'(x)-\tfrac{x(x+1)}{2}f''(x)\right|\le\\ &\le\left|p_n(x)\right|\left|f'(x)\right|+\tfrac{1}{2}\left|q_n(x)\right|\left|f''(x)\right|+\left|nK_{n,\beta}\left(h(t,x)e_2^x(t),x\right)\right|.\end{aligned}\tag{9}$$

Using (6) and (7) , we can see that if we let the n goes the infinity, $p_n(x)$ and $q_n(x)$ tend to zero at any point x.

To finish the proof, we have to calculate the term $\left|nK_{n,\beta}\left(h(t,x)e_2^x(t),x\right)\right|$. We know that,

$$\left|f(t)-f(x)\right|\le\Big(1+\tfrac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\Big)\omega^*\left(f,\delta\right).$$

From this equality, we get

$$\left|h(t,x)\right|\le\Big(1+\tfrac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\Big)\omega^*\left(f'',\delta\right).$$

Here for $\delta>0$, if $\left|e^{-x}-e^{-t}\right|\le\delta$ then $|h(t,x)|\le 2\omega^*(f'',\delta)$ and if

$$\left|e^{-x}-e^{-t}\right|>\delta$$

then $|h(t,x)|\le 2(\frac{(e^{-x}-e^{-t})^2}{\delta^2})\omega^*\left(f'',\delta\right)$ so we can write,

$$|h(t,x)|\le 2\Big(1+\tfrac{(e^{-x}-e^{-t})^2}{\delta^2}\Big)\omega^*\left(f'',\delta\right).$$

If we use it (9) , we obtain

$$\begin{aligned} nK_{n,\beta}\left(h(t,x)e_2^x(t),x\right) &\leq 2n\omega^*\left(f'',\delta\right)K_{n,\beta}\left(e_2^x(t),x\right) \\ &\quad +\tfrac{2n}{\delta^2}\omega^*\left(f'',\delta\right)K_{n,\beta}\left(\left(e^{-x}-e^{-t}\right)^2 e_2^x(t),x\right). \end{aligned}$$

By applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &nK_{n,\beta}\left(h(t,x)e_2^x(t),x\right) \leq \\ &\leq 2n\omega^*\left(f'',\delta\right)K_{n,\beta}\left(e_2^x(t),x\right) \\ &\quad + \tfrac{2n}{\delta^2}\omega^*\left(f'',\delta\right)\sqrt{K_{n,\beta}\left(\left(e^{-x}-e^{-t}\right)^4,x\right)}\sqrt{K_{n,\beta}\left(e_4^x(t),x\right)}. \end{aligned}$$

If we choose $\delta = \frac{1}{\sqrt{n}}$ and define

$$r_n(x) = \sqrt{n^2 K_{n,\beta}\left(\left(e^{-x}-e^{-t}\right)^4,x\right)}\sqrt{n^2(K_{n,\beta}\left(t-x\right)^4,x)},$$

we achieve

$$\begin{aligned} &\left|n\left[K_{n,\beta}(f,x)-f(x)\right]+ax(x+1)f'(x)-\tfrac{x(x+1)}{2}f''(x)\right| \leq \\ &\leq \left|p_n(x)\right|\left|f'(x)\right|+\left|q_n(x)\right|\left|f''(x)\right|+2\left(2q_n+x(x+1)+r_n(x)\right)\omega^*\left(f'',\tfrac{1}{\sqrt{n}}\right). \end{aligned}$$

Hence we have the desired result. □

THEOREM 8. *Let $x \in [0,\infty)$ and $f,\ f'' \in C^*[0,\infty)$, we have*

$$\lim_{n\longrightarrow\infty} n\left[K_{n,\beta}(f,x)-f(x)\right] = -ax(x+1)f'(x)+\tfrac{x(x+1)}{2}f''(x). \tag{10}$$

Proof. By the Taylor's expansion of f, we have

$$f(t) = f(x)+f'(x)(t-x)+\tfrac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2, \tag{11}$$

where $\lim\limits_{t\to x} r(t,x) = 0$. Operating $K_{n,\beta}$ to the identity (11), we obtain

$$\begin{aligned} K_{n,\beta}(f,x)-f(x) =& K_{n,\beta}(t-x,x)f'(x)+K_{n,\beta}\left((t-x)^2,x\right)\tfrac{f''(x)}{2} \\ &+K_{n,\beta}\left(r\left(t,x\right)(t-x)^2,x\right). \end{aligned}$$

Using the Cauchy-Schwarz inequality, we have

$$K_{n,\beta}\left(r\left(t,x\right)(t-x)^2,x\right) \leq \sqrt{K_{n,\beta}\left(r^2\left(t,x\right),x\right)}\sqrt{K_{n,\beta}\left((t-x)^4,x\right)}. \tag{12}$$

Also, we have

$$\lim_{n\to\infty} K_{n,\beta}\left(r^2\left(t,x\right),x\right) = r^2\left(x,x\right) = 0. \tag{13}$$

Now from (12), (13) and (5), we get

$$\lim_{n\to\infty} nK_{n,\beta}\left(r\left(t,x\right)(t-x)^2,x\right) = 0.$$

Thus we have

$$\begin{aligned} &\lim_{n\to\infty} n\left(K_{n,\beta}(f,x)-f(x)\right) = \\ &= \lim_{n\to\infty} n\left[K_{n,\beta}(t-x,x)f'(x)+\tfrac{1}{2}f''(x)K_{n,\beta}((t-x)^2,x)+K_{n,\beta}\left(r\left(t,x\right)(t-x)^2,x\right)\right]. \end{aligned}$$

The result follows immediately by (6) and (7). □

5. BETTER ESTIMATION BY MEANS OF GENERALIZED CONVEXITY

Now we deal with the comparison with the operators $K_{n,\beta}$ with the classical Baskakov operators by way of generalized convexity. To do this, first of all we give the definition of generalized convexity. We can see these definitions [9], [23].

DEFINITION 9. *A function $f \in C[0,\infty)$ is said to be convex with respect to the function τ if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ \tau(x_0) & \tau(x_1) & \tau(x_2) \\ f(x_0) & f(x_1) & f(x_2) \end{vmatrix} \geq 0 \quad , \qquad 0 < x_0 < x_1 < x_2 < \infty.$$

We take account of the convexity in the classical sense is obtained for $\tau = e_1$.

DEFINITION 10. *A function $f \in C^k[0,\infty)$ (the space of k times continuously differentiable functions) is said to be τ convex of order $k \in \mathbb{N}$ whenever*

$$D^k(f \circ \tau^{-1}) \circ \tau \geq 0.$$

The classical convexity is obtained for $\tau = e_1$ and $k = 2$.

For the operators $K_{n,\beta}$, we are talking about generalized convexity with respect to the function $\tau = e^{2ax}$.

Next we will give the following remark about τ convexity which is given in [9].

REMARK 11. The function $f \in C^2[0,\infty)$ is convex with respect to the function τ if and only if

$$f''(x) \geq \tfrac{\tau''(x)}{\tau'(x)} f'(x), \quad x \in [0,\infty).$$

According to the remark which is above, we can write that the function $f \in C^2[0,\infty)$ is convex with respect to the function $\tau = e^{2ax}$ if and only if

$$f''(x) \geq 2af'(x), \quad x \in [0,\infty). \tag{14}$$

THEOREM 12. *If the function $f \in C^2[0,\infty)$ is strictly φ-convex with respect to $\tau(x) = e^{2ax}$, $a > 0$, then for all $x \geq 0$ there exists $n_0 = n_0(x) \in \mathbb{N}$ such that for $n \geq n_0$ there holds*

$$f(x) \leq K_{n,\beta}(f,x) \tag{15}$$

Proof. Using (10) and (14), the proof is obtained immediately. □

THEOREM 13. *Let $f \in C[0,\infty)$ and K_n be a classical Baskakov operator which is defined on $C[0,\infty)$, we can write*

i) If f is convex, then $f(x) \leq K_n(f,x)$.

ii) If f is convex, then $K_{n+1}(f,x) \leq K_n(f,x)$.

iii) If f is increasing (decreasing), then $K_n(f,x)$ is increasing (decreasing).

THEOREM 14. *Let $f \in C[0,\infty)$ be a decreasing and convex function. Then we have*

$$f(x) \leq K_{n+1,\beta}(f,x) \leq K_{n,\beta}(f,x).$$

Proof. From Theorem 13, we can write $\beta_n(x) \leq \beta_{n+1}(x)$. Using this inequality,

$$\begin{aligned} K_{n,\beta}(f,x) - K_{n+1,\beta}(f,x) &= K_n(f,\beta_n(x)) - K_{n+1}(f,\beta_{n+1}(x)) \\ &= K_n(f,\beta_n(x)) - K_{n+1}(f,\beta_n(x)) \\ &\quad + K_{n+1}(f,\beta_n(x)) - K_{n+1}(f,\beta_{n+1}(x)). \end{aligned}$$

Since f is decreasing and convex, we get

$$\begin{aligned} K_{n,\beta}(f,x) - K_{n+1,\beta}(f,x) &\geq 0 \\ K_{n,\beta}(f,x) &\geq K_{n+1,\beta}(f,x). \end{aligned}$$

Also from Theorem 8, there exists $n_1 = n_1(x) \in \mathbb{N}$, such that for $n \geq n_1$, we have

$$f(x) \leq K_{n+1,\beta}(f,x).$$

□

THEOREM 15. *Let $f \in C[0,\infty)$ be a increasing and convex with respect to the function $\tau = e^{2ax}$. Then*

$$f(x) \leq K_{n,\beta}(f,x) \leq K_n(f,x).$$

Proof. From [23], we can write

$$f(x) \leq K_{n,\beta}(f,x).$$

Since f is convex with respect to the function $\tau = e^{2ax}$, we have from Theorem 13, that

$$f(x) \leq K_n(f,x).$$

Also $\tau = e^{2ax}$ is a convex function, so that we can write τ instead of f

$$e^{2ax} \leq K_n(e^{2at},x).$$

Using the $(K_n(\tau))^{-1}$ as an increasing function,

$$K_n(e^{2at},x)^{-1} \circ e^{2ax} \leq K_n(e^{2at},x)^{-1} \circ K_n(e^{2at},x).$$

Hence from the (2), we have

$$\beta_n(x) \leq x. \tag{16}$$

Applying the operators K_n both sides of (16),

$$K_n(f,\beta_n(x)) = K_{n,\beta}(f,x) \leq K_n(f,x) \tag{17}$$

Combining (15) and (17), we get desired result. □

REMARK 16. We can also use the definition of Baskakov operators, considered by Chen [14]:

$$\widetilde{V}_n f(x) = \sum_{k=0}^{\infty} \frac{n(n+\alpha)(n+2\alpha)...[n+(k-1)\alpha]}{k!} \frac{x^k}{(1+\alpha x)^{\frac{n}{\alpha}+k}} f\left(\frac{k}{n}\right).$$

Using this definition, we can define another King type of Baskakov operators,

$$V_n f(x) = \sum_{k=0}^{\infty} \frac{n(n+\alpha)(n+2\alpha)...[n+(k-1)\alpha]}{k!} \frac{(b_n(x))^k}{(1+\alpha b_n(x))^{\frac{n}{\alpha}+k}} f\left(\frac{k}{n}\right),$$

$x \geq 0$, $n \in \mathbb{N}$.
In the last expression by setting $\alpha = 1$, we get the definition given in (1).

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