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HIGH ORDER APPROXIMATION THEORY FOR BANACH SPACE VALUED FUNCTIONS

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Abstract. Here we study quantitatively the high degree of approximation of sequences of linear operators acting on Banach space valued differentiable functions to the unit operator. These operators are bounded by real positive linear companion operators. The Banach spaces considered here are general and no positivity assumption is made on the initial linear operators whose we study their approximation properties. We derive pointwise and uniform estimates which imply the approximation of these operators to the unit assuming differentiability of functions. At the end we study the special case where the high order derivative of the on hand function fulfills a convexity condition resulting into sharper estimates.

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1. MOTIVATION

Let $(X, \|\cdot\|)$ be a Banach space, $N \in \mathbb{N}$. Consider $g \in C([0,1])$ and the classic Bernstein polynomials

$$(\widetilde{B}_N g)(t) = \sum_{k=0}^{N} g\left(\tfrac{k}{N}\right)\tbinom{N}{k} t^k (1-t)^{N-k}, \quad \forall\, t \in [0,1]. \tag{1.1}$$

Let also $f \in C([0,1], X)$ and define the vector valued in X Bernsein linear operators

$$(B_N f)(t) = \sum_{k=0}^{N} f\left(\tfrac{k}{N}\right)\tbinom{N}{k} t^k (1-t)^{N-k}, \quad \forall\, t \in [0,1]. \tag{1.2}$$

That is $(B_N f)(t) \in X$.

Clearly here $\|f\| \in C([0,1])$.

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We notice that
(1.3)

$$\left\| \left(B_N f \right)(t) \right\| \leq \sum_{k=0}^{N} \left\| f\left(\tfrac{k}{N}\right) \right\| \tbinom{N}{k} t^k \left(1-t\right)^{N-k} = \left(\widetilde{B}_N \left(\|f\| \right) \right)(t), \forall t \in [0,1]$$

The property

$$\left\| \left(B_N f \right)(t) \right\| \leq \left(\widetilde{B}_N \left(\|f\| \right) \right)(t), \quad \forall t \in [0,1], \tag{1.4}$$

is shared by almost all summation/integration similar operators and motivates our work here.

If $f(x) = c \in X$ the constant function, then

$$(B_N c) = c. \tag{1.5}$$

If $g \in C([0,1])$ and $c \in X$, then $cg \in C([0,1], X)$ and

$$(B_N (cg)) = c\widetilde{B}_N (g). \tag{1.6}$$

Again (1.5), (1.6) are fulfilled by many summation/integration operators.

In fact here (1.6) implies (1.5), when $g \equiv 1$.

The above can be generalized from $[0,1]$ to any interval $[a,b] \subset \mathbb{R}$. All this discussion motivates us to consider the following situation.

Let $L_N : C([a,b], X) \hookrightarrow C([a,b], X)$, $(X, \|\cdot\|)$ a Banach space, L_N is a linear operator, $\forall\ N \in \mathbb{N}$, $x_0 \in [a,b]$. Let also $\widetilde{L}_N : C([a,b]) \hookrightarrow C([a,b])$, a sequence of positive linear operators, $\forall\ N \in \mathbb{N}$.

We assume that

$$\left\| \left(L_N (f) \right)(x_0) \right\| \leq \left(\widetilde{L}_N \left(\|f\| \right) \right)(x_0), \tag{1.7}$$

$\forall\ N \in \mathbb{N}$, $\forall\ x_0 \in X$, $\forall\ f \in C([a,b], X)$.

When $g \in C([a,b])$, $c \in X$, we assume that

$$(L_N (cg)) = c\widetilde{L}_N (g). \tag{1.8}$$

The special case of

$$\widetilde{L}_N (1) = 1, \tag{1.9}$$

implies

$$L_N (c) = c, \quad \forall\ c \in X. \tag{1.10}$$

We call $\widetilde{L}_N$ the companion operator of L_N.

Based on the above fundamental properties we study the high order approximation properties of the sequence of linear operators $\{L_N\}_{N \in \mathbb{N}}$, i.e. their high speed convergence to the unit operator. No kind of positivity property of $\{L_N\}_{N \in \mathbb{N}}$ is assumed. Other important motivation comes from [1], [2], [3]-[6].

Our vector valued differentiation here resembles completely the numerical one, see [7, pp. 83-84].

2. MAIN RESULTS

We need vector Taylor's formula

THEOREM 2.1. [7, pp. 93]. Let $n \in \mathbb{N}$ and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space. Then

$$f(b) = \sum_{i=0}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a) + \frac{1}{(n-1)!} \int_a^b (b-t)^{n-1} f^{(n)}(t)\, dt. \tag{2.1}$$

Above the integral is the usual vector valued Riemann integral, see [7, p. 86].

We also need

THEOREM 2.2. *Let $n \in \mathbb{N}$ and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space. Then*

$$f(a) = \sum_{i=0}^{n-1} \frac{(a-b)^i}{i!} f^{(i)}(b) + \frac{1}{(n-1)!} \int_b^a (a-t)^{n-1} f^{(n)}(t)\, dt. \tag{2.2}$$

Proof. Let

$$F(x) := \sum_{i=0}^{n-1} \frac{(a-x)^i}{i!} f^{(i)}(x), \quad x \in [a,b].$$

Here $F \in C([a,b], X)$. Notice that $F(a) = f(a)$, and

$$F(b) = \sum_{i=0}^{n-1} \frac{(a-b)^i}{i!} f^{(i)}(b). \tag{2.3}$$

We have

$$F'(x) = \frac{(a-x)^{n-1}}{(n-1)!} f^{(n)}(x), \quad \forall\, x \in [a,b].$$

Clearly $F' \in C([a,b], X)$.

By [7, pp. 92] we get

$$F(b) - F(a) = \int_a^b F'(t)\, dt. \tag{2.4}$$

That is we have

(2.5)

$$\sum_{i=0}^{n-1} \frac{(a-b)^i}{i!} f^{(i)}(b) - f(a) = \int_a^b \frac{(a-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt = -\int_b^a \frac{(a-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt,$$

proving (2.2). □

Based on the above Theorems 2.1, 2.2, we have

COROLLARY 2.3. *Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^n([a,b], X)$, then we have the vector valued Taylor's formula*

$$f(y) - \sum_{i=0}^{n-1} f^{(i)}(x) \frac{(y-x)^i}{i!} = \frac{1}{(n-1)!} \int_x^y (y-t)^{n-1} f^{(n)}(t)\, dt, \tag{2.6}$$

$\forall\ x, y \in [a, b]$ *or*

$$f(y) - \sum_{i=0}^{n} f^{(i)}(x) \tfrac{(y-x)^i}{i!} = \tfrac{1}{(n-1)!} \int_x^y (y-t)^{n-1} \left(f^{(n)}(t) - f^{(n)}(x)\right) dt, \tag{2.7}$$

$\forall\ x, y \in [a, b]$.

We need

Definition 2.4. *Let $f \in C([a, b], X)$, where $(X, \|\cdot\|)$ is a Banach space. We define*

$$\omega_1(f, \delta) := \sup_{\substack{x,y:\\ |x-y| \leq \delta}} \|f(x) - f(y)\|, \quad 0 < \delta \leq b - a, \tag{2.8}$$

the first modulus of continuity of f.

Remark 2.5. We study the remainder of (2.7):

$$R_n(x, y) := \tfrac{1}{(n-1)!} \int_x^y (y-t)^{n-1} \left(f^{(n)}(t) - f^{(n)}(x)\right) dt, \tag{2.9}$$

$\forall\ x, y \in [a, b]$.

We estimate $R_n(x, y)$.

Case of $y \geq x$. We have

$$\|R_n(x, y)\| = \tfrac{1}{(n-1)!} \left\| \int_x^y (y-t)^{n-1} \left(f^{(n)}(t) - f^{(n)}(x)\right) dt \right\| \overset{[7,\text{ p. } 88]}{\leq}$$

$$\leq \tfrac{1}{(n-1)!} \int_x^y (y-t)^{n-1} \left\| f^{(n)}(t) - f^{(n)}(x) \right\| dt \tag{2.10}$$

$$\leq \tfrac{1}{(n-1)!} \int_x^y (y-t)^{n-1} \omega_1(f^{(n)}, |t-x|) dt \overset{\text{let } h>0}{=}$$

$$= \tfrac{1}{(n-1)!} \int_x^y (y-t)^{n-1} \omega_1(f^{(n)}, \tfrac{|t-x|}{h} h) dt$$

$$\leq \tfrac{\omega_1\left(f^{(n)}, h\right)}{(n-1)!} \int_x^y (y-t)^{n-1} (1 + \tfrac{|t-x|}{h}) dt \tag{2.11}$$

$$= \tfrac{\omega_1\left(f^{(n)}, h\right)}{(n-1)!} \int_x^y (y-t)^{n-1} (1 + \tfrac{(t-x)}{h}) dt$$

$$= \tfrac{\omega_1\left(f^{(n)}, h\right)}{(n-1)!} \Big[\int_x^y (y-t)^{n-1} dt + \tfrac{1}{h} \int_x^y (y-t)^{n-1} (t-x)^{2-1} dt \Big]$$

$$= \tfrac{\omega_1\left(f^{(n)}, h\right)}{(n-1)!} \Big[\tfrac{(y-x)^n}{n} + \tfrac{1}{h} \tfrac{\Gamma(n)\Gamma(2)}{\Gamma(n+2)} (y-x)^{n+1} \Big] \tag{2.12}$$

$$= \tfrac{\omega_1\left(f^{(n)}, h\right)}{(n-1)!} \left[\tfrac{(y-x)^n}{n} + \tfrac{(y-x)^{n+1}}{n(n+1)h} \right] = \tfrac{\omega_1\left(f^{(n)}, h\right)}{n!} (y-x)^n \left[1 + \tfrac{(y-x)}{(n+1)h} \right].$$

We have found that

$$\|R_n(x, y)\| \leq \tfrac{\omega_1\left(f^{(n)}, h\right)}{n!} (y-x)^n \left[1 + \tfrac{(y-x)}{(n+1)h} \right], \tag{2.13}$$

for $y \geq x$, and $h > 0$.

Case of $y \leq x$.

Then

$$\|R_n(x,y)\| = \frac{1}{(n-1)!}\left\|\int_x^y (y-t)^{n-1}\left(f^{(n)}(t)-f^{(n)}(x)\right)dt\right\|$$

$$(2.14)\qquad = \frac{1}{(n-1)!}\left\|\int_y^x (t-y)^{n-1}\left(f^{(n)}(t)-f^{(n)}(x)\right)dt\right\|$$

$$\leq \frac{1}{(n-1)!}\int_y^x (t-y)^{n-1}\left\|f^{(n)}(t)-f^{(n)}(x)\right\|dt$$

$$\leq \frac{1}{(n-1)!}\int_y^x (t-y)^{n-1}\,\omega_1\left(f^{(n)},\tfrac{|t-x|}{h}h\right)dt$$

$$\leq \frac{\omega_1\left(f^{(n)},h\right)}{(n-1)!}\int_y^x (t-y)^{n-1}\left(1+\tfrac{(x-t)}{h}\right)dt$$

$$(2.15)\qquad = \frac{\omega_1\left(f^{(n)},h\right)}{(n-1)!}\left[\int_y^x (t-y)^{n-1}\,dt+\frac{1}{h}\int_y^x (x-t)^{2-1}(t-y)^{n-1}\,dt\right]$$

$$= \frac{\omega_1\left(f^{(n)},h\right)}{(n-1)!}\left[\frac{(x-y)^n}{n}+\frac{(x-y)^{n+1}}{n(n+1)h}\right]$$

$$= \frac{\omega_1\left(f^{(n)},h\right)}{n!}(x-y)^n\left[1+\frac{(x-y)}{(n+1)h}\right].$$

Hence

$$(2.16)\qquad \|R_n(x,y)\| \leq \frac{\omega_1\left(f^{(n)},h\right)}{n!}(x-y)^n\left[1+\frac{(x-y)}{(n+1)h}\right],$$

when $y \leq x$, $h>0$.

We have proved that

$$\|R_n(x,y)\| = \left\|\frac{1}{(n-1)!}\int_x^y (y-t)^{n-1}\left(f^{(n)}(t)-f^{(n)}(x)\right)dt\right\| \leq$$

$$(2.17)\qquad \leq \frac{\omega_1\left(f^{(n)},h\right)}{n!}|x-y|^n\left[1+\frac{|x-y|}{(n+1)h}\right],$$

$\forall\, x,y \in [a,b]$, $h>0$.

We have established

THEOREM 2.6. *Let $(X,\|\cdot\|)$ be a Banach space and $f \in C^n([a,b],X)$, $n \in \mathbb{N}$. Then*

$$(2.18)\qquad \left\|f(y)-\sum_{i=0}^{n} f^{(i)}(x)\frac{(y-x)^i}{i!}\right\| \leq \frac{\omega_1\left(f^{(n)},h\right)}{n!}|x-y|^n\left[1+\frac{|x-y|}{(n+1)h}\right],$$

$\forall\, x,y \in [a,b]$, $h>0$.

It follows our first main result

THEOREM 2.7. *Let $N \in \mathbb{N}$ and $L_N : C([a,b],X) \to C([a,b],X)$, where $(X,\|\cdot\|)$ is a Banach space and L_N is a linear operator. Let the positive linear operators $\widetilde{L}_N : C([a,b]) \hookrightarrow C([a,b])$, such that*

$$(2.19)\qquad \|(L_N(f))(x_0)\| \leq \left(\widetilde{L}_N(\|f\|)\right)(x_0),$$

$\forall\ N \in \mathbb{N}, \forall\ f \in C([a,b], X), \forall\ x_0 \in [a,b]$.

Furthermore assume that

$$L_N(cg) = c\widetilde{L}_N(g), \quad \forall\ g \in C([a,b]), \forall\ c \in X. \tag{2.20}$$

Let $n \in \mathbb{N}$, *here we deal with* $f \in C^n([a,b], X)$.

Then

1)

$$\left\| (L_N(f))(x_0) - \sum_{i=0}^{n} \frac{f^{n(i)}(x_0)}{i!} (\widetilde{L}_N((\cdot - x_0)^i))(x_0) \right\| \leq$$
$$\leq \frac{\omega_1\left(f^{(n)}, \left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}\right)}{n!} \left(\left(\widetilde{L}_N(|\cdot - x_0|^{n+1}))(x_0)\right)^{\left(\frac{n}{n+1}\right)} \times$$
$$\times \left[((\widetilde{L}_N(1))(x_0))^{\frac{1}{n+1}} + \frac{1}{n+1}\right], \tag{2.21}$$

2)

$$\|(L_N(f))(x_0) - f(x_0)\| \leq$$
$$\leq \|f(x_0)\| \left|(\widetilde{L}_N(1))(x_0) - 1\right| +$$
$$+ \sum_{k=0}^{n} \frac{\|f^{(k)}(x_0)\|}{k!} ((\widetilde{L}_N(|\cdot - x_0|^k))(x_0)) +$$
$$+ \frac{\omega_1(f^{(n)}, ((\widetilde{L}_N(|\cdot - x_0|^{n+1}))(x_0))^{\frac{1}{n+1}})}{n!} ((\widetilde{L}_N(|\cdot - x_0|^{n+1}))(x_0))^{\left(\frac{n}{n+1}\right)} \times$$
$$\times \left[((\widetilde{L}_N(1))(x_0))^{\frac{1}{n+1}} + \frac{1}{n+1}\right], \tag{2.22}$$

from (2.22), *and as* $(\widetilde{L}_N(1))(x_0) \to 1$, $(\widetilde{L}_N(|\cdot - x_0|^{n+1}))(x_0) \to 0$, *we obtain* $(L_N(f))(x_0) \to f(x_0)$, *as* $N \to \infty$,

3) if $f^{(k)}(x_0) = 0$, $k = 0, 1, ..., n$, *we get that*

$$\|(L_N(f))(x_0) - f(x_0)\| \leq$$
$$\leq \frac{\omega_1\left(f^{(n)}, \left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)^{\frac{1}{n+1}}\right)}{n!} \times$$
$$\times ((\widetilde{L}_N(|\cdot - x_0|^{n+1}))(x_0))^{\left(\frac{n}{n+1}\right)} \left[((\widetilde{L}_N(1))(x_0))^{\frac{1}{n+1}} + t\frac{1}{n+1}\right], \tag{2.23}$$

an extreme high speed of convergence,

4) one also derives

$$\left\|\left\|\left(L_N(f)\right)-f\right\|\right\|_{\infty,[a,b]} \leq$$
$$\leq \left\|\left\|f\right\|\right\|_{\infty,[a,b]} \left\|\left(\widetilde{L}_N(1)\right)-1\right\|_{\infty,[a,b]} +$$
$$+\sum_{k=0}^{n} \frac{\left\|\left\|f^{(k)}\right\|\right\|_{\infty,[a,b]}}{k!} \left\|\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^k\right)\right)(x_0)\right\|_{\infty,x_0\in[a,b]} +$$
$$+\frac{\omega_1\left(f^{(n)}, \left\|\left(\widetilde{L}_N\left(\left|\cdot-x_0\right|^{n+1}\right)\right)(x_0)\right\|_{\infty,x_0\in[a,b]}^{\frac{1}{n+1}}\right)}{n!} \times$$
$$\times \left\|\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^{n+1}\right)\right)(x_0)\right\|_{\infty,x_0\in[a,b]}^{\left(\frac{n}{n+1}\right)} \left[\left\|\widetilde{L}_N(1)\right\|^{\frac{1}{n+1}} + \frac{1}{n+1}\right], \tag{2.24}$$

if $\widetilde{L}_N(1) \xrightarrow{u} 1$*, uniformly, and* $\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^{n+1}\right)\right)(x_0) \xrightarrow{u} 0$*, uniformly in* $x_0 \in [a,b]$*, by (2.24), we obtain* $L_N(f) \xrightarrow{u} f$*, uniformly, as* $N \to \infty$*.*

Proof. 1) One can rewrite (2.18) as follows

$$\left\|f(\cdot) - \sum_{i=0}^{n} f^{(i)}(x_0) \frac{(\cdot - x_0)^i}{i!}\right\| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{n!} \left[\left|\cdot - x_0\right|^n + \frac{\left|\cdot - x_0\right|^{n+1}}{(n+1)h}\right], \tag{2.25}$$

for a fixed $x_0 \in [a,b]$, $h > 0$.

We observe that ($N \in \mathbb{N}$)

$$\left\|\left(L_N(f)\right)(x_0) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}\left(\widetilde{L}_N\left((\cdot - x_0)^i\right)\right)(x_0)\right\| =$$
$$= \left\|\left(L_N\left[f(\cdot) - \sum_{i=0}^{n} f^{(i)}(x_0) \frac{(\cdot - x_0)^i}{i!}\right]\right)(x_0)\right\| \tag{2.26}$$
$$\leq \left(\widetilde{L}_N\left(\left\|f(\cdot) - \sum_{i=0}^{n} f^{(i)}(x_0) \frac{(\cdot - x_0)^i}{i!}\right\|\right)\right)(x_0) \overset{\text{(by (2.25))}}{\leq}$$
$$\leq \frac{\omega_1\left(f^{(n)}, h\right)}{n!} \left[\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^n\right)\right)(x_0) + \frac{\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^{n+1}\right)\right)(x_0)}{(n+1)h}\right] =: (\xi_1). \tag{2.27}$$

Above notice that $\left(f(\cdot) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(\cdot - x_0)^i\right) \in C([a,b], X)$.

By Hölder's inequality and Riesz representation theorem we obtain

$$\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^n\right)\right)(x_0) \leq \left(\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^{n+1}\right)\right)(x_0)\right)^{\left(\frac{n}{n+1}\right)} \left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{(n+1)}}. \tag{2.28}$$

Therefore

$$(\xi_1) \leq \frac{\omega_1\left(f^{(n)}, h\right)}{n!} \left[\left(\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^{n+1}\right)\right)(x_0)\right)^{\left(\frac{n}{n+1}\right)} \left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{(n+1)}}\right.$$
$$\left. + \frac{1}{h} \frac{\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^{n+1}\right)\right)(x_0)}{(n+1)}\right] =: (\xi_2). \tag{2.29}$$

We choose

$$h := \left(\left(\widetilde{L}_N \left(|\cdot - x_0|^{n+1} \right) \right) (x_0) \right)^{\frac{1}{(n+1)}}, \tag{2.30}$$

in case of $\left(\widetilde{L}_N \left(|\cdot - x_0|^{n+1} \right) \right) (x_0) > 0$.

Then it holds

$$(\xi_2) = \frac{\omega_1 \left(f^{(n)}, \left(\left(\widetilde{L}_N \left(|\cdot - x_0|^{n+1} \right) \right) (x_0) \right)^{\frac{1}{(n+1)}} \right)}{n!} \tag{2.31}$$

$$\times \left[\left(\left(\widetilde{L}_N \left(|\cdot - x_0|^{n+1} \right) \right) (x_0) \right)^{\left(\frac{n}{n+1} \right)} \left(\left(\widetilde{L}_N (1) \right) (x_0) \right)^{\frac{1}{(n+1)}} + \frac{\left(\left(\widetilde{L}_N \left(|\cdot - x_0|^{n+1} \right) \right) (x_0) \right)^{\left(\frac{n}{n+1} \right)}}{(n+1)} \right] =$$

$$= \frac{\omega_1 \left(f^{(n)}, \left(\left(\widetilde{L}_N \left(|\cdot - x_0|^{n+1} \right) \right) (x_0) \right)^{\frac{1}{(n+1)}} \right)}{n!}$$

$$\times \left(\left(\widetilde{L}_N \left(|\cdot - x_0|^{n+1} \right) \right) (x_0) \right)^{\left(\frac{n}{n+1} \right)} \left[\left(\left(\widetilde{L}_N (1) \right) (x_0) \right)^{\frac{1}{(n+1)}} + \frac{1}{n+1} \right]. \tag{2.32}$$

We have proved that

$$\left\| (L_N (f)) (x_0) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} \left(\widetilde{L}_N \left((\cdot - x_0)^i \right) \right) (x_0) \right\| \leq$$

$$\leq \frac{\omega_1 \left(f^{(n)}, \left(\left(\widetilde{L}_N \left(|\cdot - x_0|^{n+1} \right) \right) (x_0) \right)^{\frac{1}{(n+1)}} \right)}{n!}$$

$$\times \left(\left(\widetilde{L}_N \left(|\cdot - x_0|^{n+1} \right) \right) (x_0) \right)^{\left(\frac{n}{n+1} \right)} \left[\left(\left(\widetilde{L}_N (1) \right) (x_0) \right)^{\frac{1}{(n+1)}} + \frac{1}{n+1} \right]. \tag{2.33}$$

By Riesz representation theorem we have

$$\left(\widetilde{L}_N (g) \right) (x_0) = \int_{[a,b]} g(t)\, d\mu_{x_0}(t), \quad \forall\ g \in C([a,b]), \tag{2.34}$$

where μ_{x_0} is a positive finite measure on $[a,b]$.

That is

$$\left(\widetilde{L}_N (1) \right) (x_0) = \mu_{x_0} ([a,b]) =: M. \tag{2.35}$$

We have that $\mu_{x_0} ([a,b]) > 0$, because otherwise, if $\mu_{x_0} ([a,b]) = 0$, then $\left(\widetilde{L}_n (g) \right) (x_0) = 0$, $\forall\ g \in C([a,b])$, and the whole theory here becomes trivial.

Therefore it holds $\left(\widetilde{L}_N (1) \right) (x_0) > 0$.

In case of

$$\left(\widetilde{L}_N \left(|\cdot - x_0|^{n+1} \right) \right) (x_0) = 0, \tag{2.36}$$

we have

$$\int_{[a,b]} |t - x_0|^{n+1} d\mu_{x_0}(t) = 0. \tag{2.37}$$

The last implies $|t - x_0|^{n+1} = 0$, a.e, hence $|t - x_0| = 0$, a.e, then $t - x_0 = 0$ a.e., and $t = x_0$, a.e. on $[a,b]$. Consequently $\mu_{x_0} (\{t \in [a,b] : t \neq x_0\}) = 0$.

That is $\mu_{x_0} = \delta_{x_0} M$, where δ_{x_0} is the Dirac measure at $\{x_0\}$. In that case holds

$$\left(\widetilde{L}_N(g)\right)(x_0) = g(x_0)\, M, \qquad \forall g \in C([a,b]). \tag{2.38}$$

Under (2.36), the right hand side of (2.33) equals zero. Furthermore it holds

$$\left(\widetilde{L}_N\left(\left\| f(\cdot) - \sum_{i=0}^{n} f^{(i)}(x_0) \frac{(\cdot - x_0)^i}{i!} \right\|\right)\right)(x_0) \overset{(2.38)}{=} \| f(x_0) - f(x_0) \| M = 0. \tag{2.39}$$

So that by (2.26) to have

$$\left\| (L_N(f))(x_0) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} \left(\widetilde{L}_N\left((\cdot - x_0)^i\right)\right)(x_0) \right\| =$$
$$= \| (L_N(f))(x_0) - f(x_0)\, M \| = 0, \tag{2.40}$$

also implying

$$(L_N(f))(x_0) = M f(x_0). \tag{2.41}$$

So we have proved that inequality (2.33) will be always true.

2) Next we see that

$$\| (L_N(f))(x_0) - f(x_0) \| =$$
$$= \Bigg\| (L_N(f))(x_0) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0) +$$
$$+ \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0) - f(x_0) \Bigg\| \le \tag{2.42}$$
$$\le \left\| (L_N(f))(x_0) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0) \right\| +$$
$$+ \left\| \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!} \left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0) + f(x_0)\left(\widetilde{L}_N(1)\right)(x_0) - f(x_0) \right\| \overset{(2.33)}{\le}$$

$$\le \| f(x_0) \| \left| \left(\widetilde{L}_N(1)\right)(x_0) - 1 \right| + \sum_{k=1}^{n} \frac{\left\| f^{(k)}(x_0) \right\|}{k!} \left(\widetilde{L}_N\left(|\cdot - x_0|^k\right)\right)(x_0) +$$
$$+ \frac{\omega_1\left(f^{(n)}, \left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)^{\frac{1}{(n+1)}}\right)}{n!}$$
$$\times \left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)^{\left(\frac{n}{n+1}\right)} \left[\left(\left(\widetilde{L}_N(1)\right)(x_0)\right)^{\frac{1}{(n+1)}} + \frac{1}{n+1}\right]. \tag{2.43}$$

We have proved that

$$\begin{aligned}
&\|(L_N(f))(x_0) - f(x_0)\| \\
&\leq \|f(x_0)\| \left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| + \\
&+ \sum_{k=1}^{n} \frac{\|f^{(k)}(x_0)\|}{k!} (\widetilde{L}_N(|\cdot - x_0|^k))(x_0) + \\
&+ \frac{\omega_1\left(f^{(n)}, \left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)^{\frac{1}{(n+1)}}\right)}{n!} \\
&\times \left((\widetilde{L}_N(|\cdot - x_0|^{n+1}))(x_0)\right)^{\left(\frac{n}{n+1}\right)} \left[\left((\widetilde{L}_N(1))(x_0)\right)^{\frac{1}{(n+1)}} + \frac{1}{n+1}\right]. \qquad (2.44)
\end{aligned}$$

By Hölder's inequality for $k = 1, ..., n$, we obtain

(2.45)

$$(\widetilde{L}_N(|\cdot - x_0|^k))(x_0) \leq ((\widetilde{L}_N(|\cdot - x_0|^{n+1}))(x_0))^{\left(\frac{k}{n+1}\right)} ((\widetilde{L}_N(1))(x_0))^{\left(\frac{n+1-k}{n+1}\right)}.$$

Clealry by (2.44) and (2.45), when $(\widetilde{L}_N(1))(x_0) \to 1$ and $(\widetilde{L}_N(|\cdot - x_0|^{n+1}))(x_0) \to 0$, we obtain $(L_N(f))(x_0) \to f(x_0)$, as $N \to \infty$. Notice that $(\widetilde{L}_N(1))(x_0)$ will be bounded.

3) If $f^{(k)}(x_0) = 0$, $k = 0, 1, ..., n$, we get that

$$\begin{aligned}
&\|(L_N(f))(x_0) - f(x_0)\| \overset{(54)}{\leq} \\
&\leq \frac{\omega_1\left(f^{(n)}, \left(\left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)^{\frac{1}{(n+1)}}\right)}{n!} \\
&\times \left((\widetilde{L}_N(|\cdot - x_0|^{n+1}))(x_0)\right)^{\left(\frac{n}{n+1}\right)} \left[\left((\widetilde{L}_N(1))(x_0)\right)^{\frac{1}{(n+1)}} + \frac{1}{n+1}\right], \qquad (2.46)
\end{aligned}$$

an extreme high speed of convergence.

4) One also derives from (2.44) that

$$\begin{aligned}
&\|\||(L_N(f)) - f\|\|_{\infty,[a,b]} \leq \\
&\leq \|\|f\|\|_{\infty,[a,b]} \left\|\widetilde{L}_N(1) - 1\right\|_{\infty,[a,b]} + \\
&+ \sum_{k=1}^{n} \frac{\|\|f^{(k)}\|\|_{\infty,[a,b]}}{k!} \left\|(\widetilde{L}_N(|\cdot - x_0|^k))(x_0)\right\|_{\infty, x_0 \in [a,b]} + \\
&+ \frac{\omega_1\left(f^{(n)}, \left\|\left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in [a,b]}^{\frac{1}{(n+1)}}\right)}{n!} \\
&\times \left\|(\widetilde{L}_N(|\cdot - x_0|^{n+1}))(x_0)\right\|_{\infty, x_0 \in [a,b]}^{\left(\frac{n}{n+1}\right)} \left[\|\widetilde{L}_N(1)\|^{\frac{1}{(n+1)}} + \frac{1}{n+1}\right]. \qquad (2.47)
\end{aligned}$$

Inequality (2.45), for $k = 1, ..., n$, implies

$$\left\| \left(\widetilde{L}_N \left(|\cdot - x_0|^k \right) \right) (x_0) \right\|_{\infty, x_0 \in [a,b]}$$
$$\leq \left\| \left(\widetilde{L}_N \left(|\cdot - x_0|^{n+1} \right) \right) (x_0) \right\|_{\infty, x_0 \in [a,b]}^{\left(\frac{k}{n+1} \right)} \left\| \widetilde{L}_N (1) \right\|_{\infty, x_0 \in [a,b]}^{\left(\frac{n+1-k}{n+1} \right)}. \tag{2.48}$$

Consequently, if $\widetilde{L}_N (1) \overset{u}{\to} 1$, uniformly, and $\left(\widetilde{L}_N \left(|\cdot - x_0|^{n+1} \right) \right) (x_0) \overset{u}{\to} 0$, uniformly in $x_0 \in [a, b]$, by (2.47) and (2.48), we obtain $L_N (f) \overset{u}{\to} f$, uniformly, as $N \to \infty$.

Here the assumption $\widetilde{L}_N (1) \overset{u}{\to} 1$, uniformly, as $N \to \infty$, implies that $\| \widetilde{L}_N (1) \|$ is bounded.

The proof of the theorem now is complete. □

We make

Remark 2.8. Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^n ([a, b], X)$, $n \in \mathbb{N}$, and $x_0 \in (a, b)$ be fixed. Then

$$f (y) - \sum_{i=0}^{n} f^{(i)} (x_0) \tfrac{(y-x_0)^i}{i!} = \tfrac{1}{(n-1)!} \int_{x_0}^{y} (y - t)^{n-1} \left(f^{(n)} (t) - f^{(n)} (x_0) \right) dt$$
$$=: R_n (x_0, y), \ \ \forall \ y \in [a, b] \tag{2.49}$$

We assume that $g (t) := \| f^{(n)} (t) - f^{(n)} (x_0) \|$ is convex in $t \in [a, b]$.

We consider $0 < h \leq \min (x_0 - a, b - x_0)$. Obviously $g (x_0) = 0$. Then by Lemma 8.1.1, p. 243 of [1], we obtain

$$g (t) \leq \tfrac{\omega_1(g,h)}{h} |t - x_0|, \ \ \forall \ t \in [a, b]. \tag{2.50}$$

For any $t_1, t_2 \in [a, b] : |t_1 - t_2| \leq h$ we get

$$\left| \| f^{(n)} (t_1) - f^{(n)} (x_0) \| - \| f^{(n)} (t_2) - f^{(n)} (x_0) \| \right|$$
$$\leq \| f^{(n)} (t_1) - f^{(n)} (t_2) \| \leq \omega_1 (f^n, h). \tag{2.51}$$

That is

$$\omega_1 (g, h) \leq \omega_1 (f^{(n)}, h). \tag{2.52}$$

The last implies

$$\| f^{(n)} (t) - f^{(n)} (x_0) \| \leq \frac{\omega_1 \left(f^{(n)}, h \right)}{h} |t - x_0|, \quad \forall \ t \in [a, b]. \tag{2.53}$$

We estimate $R_n (x_0, y)$.

Case of $y \geq x_0$. We have

$$\|R_n(x_0,y)\| = \frac{1}{(n-1)!}\left\|\int_{x_0}^{y}(y-t)^{n-1}\left(f^{(n)}(t)-f^{(n)}(x_0)\right)dt\right\| \overset{([7,\text{ pp. }88])}{\leq}$$

$$\leq \frac{1}{(n-1)!}\int_{x_0}^{y}(y-t)^{n-1}\left\|f^{(n)}(t)-f^{(n)}(x_0)\right\|dt \overset{(2.53)}{\leq}$$

$$\leq \frac{\omega_1\left(f^{(n)},h\right)}{(n-1)!h}\int_{x_0}^{y}(y-t)^{n-1}(t-x_0)^{2-1}dt =$$

$$= \frac{\omega_1\left(f^{(n)},h\right)}{(n-1)!h}\frac{\Gamma(n)\Gamma(2)}{\Gamma(n+2)}(y-x_0)^{n+1} = \frac{\omega_1\left(f^{(n)},h\right)}{(n+1)!h}(y-x_0)^{n+1}. \tag{2.54}$$

We proved that

$$\|R_n(x_0,y)\| \leq \frac{\omega_1\left(f^{(n)},h\right)}{(n+1)!h}(y-x_0)^{n+1}, \quad \forall\, y \geq x_0. \tag{2.55}$$

Case of $y \leq x_0$. Then

$$\|R_n(x_0,y)\| = \frac{1}{(n-1)!}\left\|\int_{x_0}^{y}(y-t)^{n-1}\left(f^{(n)}(t)-f^{(n)}(x_0)\right)dt\right\|$$

$$= \frac{1}{(n-1)!}\left\|\int_{y}^{x_0}(t-y)^{n-1}\left(f^{(n)}(t)-f^{(n)}(x_0)\right)dt\right\|$$

$$\leq \frac{1}{(n-1)!}\int_{y}^{x_0}(t-y)^{n-1}\left\|f^{(n)}(t)-f^{(n)}(x_0)\right\|dt \overset{(2.53)}{\leq}$$

$$\leq \frac{\omega_1\left(f^{(n)},h\right)}{h(n-1)!}\int_{y}^{x_0}(x_0-t)^{2-1}(t-y)^{n-1}dt \tag{2.56}$$

$$= \frac{\omega_1\left(f^{(n)},h\right)}{h(n+1)!}(x_0-y)^{n+1}.$$

That is proving

$$\|R_n(x_0,y)\| \leq \frac{\omega_1\left(f^{(n)},h\right)}{h(n+1)!}(x_0-y)^{n+1}, \quad \forall\, y \in [a,b] : y \leq x_0. \tag{2.57}$$

We have established that

$$\|R_n(x_0,y)\| \leq \frac{\omega_1\left(f^{(n)},h\right)}{h(n+1)!}|y-x_0|^{n+1}, \quad \forall\, y \in [a,b], \tag{2.58}$$

where $0 < h \leq \min(x_0-a, b-x_0)$, $x_0 \in (a,b)$, and $\|f(\cdot)-f(x_0)\|$ is convex over $[a,b]$.

We have proved

Theorem 2.9. *Let $(X,\|\cdot\|)$ be a Banach space and $f \in C^n([a,b],X)$, $n \in \mathbb{N}$, and $x_0 \in (a,b)$ be fixed. Let $0 < h \leq \min(x_0-a, b-x_0)$, and assume that $\left\|f^{(n)}(\cdot)-f^{(n)}(x_0)\right\|$ is convex over $[a,b]$. Then*

$$\left\|f(y)-\sum_{i=0}^{n}f^{(i)}(x_0)\frac{(y-x_0)^i}{i!}\right\| \leq \frac{\omega_1\left(f^{(n)},h\right)}{h(n+1)!}|y-x_0|^{n+1}, \quad \forall\, y \in [a,b]. \tag{2.59}$$

We give our second main result under convexity.

Theorem 2.10. *Let* $N \in \mathbb{N}$ *and* $L_N : C([a,b], X) \to C([a,b], X)$, *where* $(X, \|\cdot\|)$ *is a Banach space and* L_N *is a linear operator. Let the positive linear operators* $\widetilde{L}_N : C([a,b]) \hookrightarrow C([a,b])$, *such that*

$$\|(L_N(f))(x_0)\| \leq \left(\widetilde{L}_N(\|f\|)\right)(x_0), \tag{2.60}$$

$\forall\ N \in \mathbb{N}, \forall\ f \in C([a,b], X)$, *where* $x_0 \in (a,b)$ *is fixed.*

Furthermore assume that

$$L_N(cg) = c\widetilde{L}_N(g), \quad \forall\ g \in C([a,b]), \forall\ c \in X. \tag{2.61}$$

Let $n \in \mathbb{N}$, *here we deal with* $f \in C^n([a,b], X)$.

We further assume that $\left\|f^{(n)}(\cdot) - f^{(n)}(x_0)\right\|$ *is convex over* $[a,b]$, *and*

$$0 \leq \left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0) \leq \min(x_0 - a, b - x_0). \tag{2.62}$$

Then

1)

$$\left\|(L_N(f))(x_0) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}\left(\widetilde{L}_N\left((\cdot - x_0)^i\right)\right)\right\|(x_0) \leq \frac{\omega_1\left(f^{(n)}, \left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)}{(n+1)!}, \tag{2.63}$$

2)

$$\begin{aligned} \|(L_N(f))(x_0) - f(x_0)\| \leq \|f(x_0)\| \left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| + \\ + \sum_{k=1}^{n} \frac{\left\|f^{(k)}(x_0)\right\|}{k!}\left(\left(\widetilde{L}_N\left(|\cdot - x_0|^k\right)\right)(x_0)\right) + \\ + \frac{\omega_1\left(f^{(n)}, \left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)}{(n+1)!}, \end{aligned} \tag{2.64}$$

as $\left(\widetilde{L}_N(1)\right)(x_0) \to 1$, *and* $\left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0) \to 0$, *we obtain* $(L_N(f))(x_0) \to f(x_0)$, *as* $N \to \infty$,

3) if $f^{(k)}(x_0) = 0$, $k = 0, 1, ..., n$, *we get that*

$$\|(L_N(f))(x_0) - f(x_0)\| \leq \frac{\omega_1\left(f^{(n)}, \left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)}{(n+1)!}, \tag{2.65}$$

a high speed of convergence.

Proof. 1) One can rewrite (2.59) as follows

$$\left\|f(\cdot) - \sum_{i=0}^{n} f^{(i)}(x_0) \frac{(\cdot - x_0)^i}{i!}\right\| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} |\cdot - x_0|^{n+1}, \quad \forall\ y \in [a,b]. \tag{2.66}$$

We observe that

$$\left\| \left(L_N\left(f\right)\right)\left(x_0\right) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}\left(\widetilde{L}_N\left(\left(\cdot - x_0\right)^i\right)\right)\left(x_0\right) \right\| = \tag{2.67}$$

$$= \left\| \left(L_N\Big[f\left(\cdot\right) - \sum_{i=0}^{n} f^{(i)}\left(x_0\right) \tfrac{(\cdot - x_0)^i}{i!}\Big]\right)\left(x_0\right) \right\| \leq$$

$$\leq \left(\widetilde{L}_N\left(\left\| f\left(\cdot\right) - \sum_{i=0}^{n} f^{(i)}\left(x_0\right) \tfrac{(\cdot - x_0)^i}{i!}\right\|\right)\right)\left(x_0\right) \overset{(\text{by } (2.66))}{\leq} \tag{2.68}$$

$$\leq \frac{\omega_1\left(f^{(n)},h\right)}{h(n+1)!}\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^{n+1}\right)\right)\left(x_0\right) =$$

$$= \frac{\omega_1\left(f^{(n)},\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^{n+1}\right)\right)\left(x_0\right)\right)}{(n+1)!}, \tag{2.69}$$

by choosing

$$h := \left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^{n+1}\right)\right)\left(x_0\right), \tag{2.70}$$

if $\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^{n+1}\right)\right)\left(x_0\right) > 0$.

We have proved that

$$\left\| \left(L_N\left(f\right)\right)\left(x_0\right) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}\left(\widetilde{L}_N\left(\left(\cdot - x_0\right)^i\right)\right)\left(x_0\right) \right\|$$

$$\leq \frac{\omega_1\left(f^{(n)},\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^{n+1}\right)\right)\left(x_0\right)\right)}{(n+1)!}. \tag{2.71}$$

By Riesz representation theorem we have

$$\left(\widetilde{L}_N\left(g\right)\right)\left(x_0\right) = \int_{[a,b]} g\left(t\right) d\mu_{x_0}\left(t\right), \quad \forall\ g \in C\left([a,b]\right), \tag{2.72}$$

where μ_{x_0} is a positive finite measure on $[a,b]$.

That is

$$\left(\widetilde{L}_N\left(1\right)\right)\left(x_0\right) = \mu_{x_0}\left([a,b]\right) =: M. \tag{2.73}$$

Without loss of generality we assume that $M > 0$, if $M = 0$, then our theory is trivial.

In case of

$$\left(\widetilde{L}_N\left(\left|\cdot - x_0\right|^{n+1}\right)\right)\left(x_0\right) = 0, \tag{2.74}$$

we have

$$\int_{[a,b]} \left|t - x_0\right|^{n+1} d\mu_{x_0}\left(t\right) = 0. \tag{2.75}$$

The last implies $\left|t - x_0\right|^{n+1} = 0$, a.e, hence $\left|t - x_0\right| = 0$, a.e, then $t - x_0 = 0$ a.e., and $t = x_0$, a.e. on $[a,b]$. Consequently $\mu_{x_0}\left(\{t \in [a,b] : t \neq x_0\}\right) = 0$. That is $\mu_{x_0} = \delta_{x_0} M$, where δ_{x_0} is the Dirac measure at $\{x_0\}$. In that case holds

$$\left(\widetilde{L}_N\left(g\right)\right)\left(x_0\right) = g\left(x_0\right) M, \tag{2.76}$$

$\forall\, g \in C([a,b])$.

Under (2.74), the right hand side of (2.71) equals zero. Furthermore it holds

$$\left(\widetilde{L}_N\left(\left\| f(\cdot) - \sum_{i=0}^{n} f^{(i)}(x_0)\frac{(\cdot - x_0)^i}{i!}\right\|\right)\right)(x_0) \overset{(2.76)}{=} \|f(x_0) - f(x_0)\| M = 0. \tag{2.77}$$

So that by (2.67) to have

$$\left\|(L_N(f))(x_0) - \sum_{i=0}^{n}\frac{f^{(i)}(x_0)}{i!}\left(\widetilde{L}_N\left((\cdot - x_0)^i\right)\right)(x_0)\right\| = 0. \tag{2.78}$$

Therefore inequality (2.71) is true again and always.

2) Next again we see that

$$\begin{aligned}
&\|(L_N(f))(x_0) - f(x_0)\| = \\
&= \left\|(L_N(f))(x_0) - \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0) + \right.\\
&\quad \left. + \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0) - f(x_0)\right\| \le
\end{aligned} \tag{2.79}$$

$$\begin{aligned}
&\le \left\|(L_N(f))(x_0) - \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0)\right\| + \\
&\quad + \left\|\sum_{k=1}^{n}\frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0) + f(x_0)\left(\widetilde{L}_N(1)\right)(x_0) - f(x_0)\right\| \overset{(2.71)}{\le} \\
&\le \|f(x_0)\| \left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| + \sum_{k=1}^{n}\frac{\|f^{(k)}(x_0)\|}{k!}\left(\widetilde{L}_N\left(|\cdot - x_0|^k\right)\right)(x_0) + \\
&\quad + \frac{\omega_1\left(f^{(n)}, \left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)}{(n+1)!}.
\end{aligned} \tag{2.80}$$

We have proved that

$$\begin{aligned}
\|(L_N(f))(x_0) - f(x_0)\| \le \|f(x_0)\| &\left|\left(\widetilde{L}_N(1)\right)(x_0) - 1\right| + \\
&+ \sum_{k=1}^{n}\frac{\|f^{(k)}(x_0)\|}{k!}\left(\widetilde{L}_N\left(|\cdot - x_0|^k\right)\right)(x_0) + \\
&+ \frac{\omega_1\left(f^{(n)}, \left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)}{(n+1)!}.
\end{aligned} \tag{2.81}$$

Clearly by (2.45) and (2.81), when $\left(\widetilde{L}_N(1)\right)(x_0) \to 1$ and $\left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0) \to 0$, we obtain $(L_N(f))(x_0) \to f(x_0)$, as $N \to \infty$. Notice that $\left(\widetilde{L}_N(1)\right)(x_0)$ will be bounded.

3) If $f^{(k)}(x_0) = 0$, $k = 0, 1, ..., n$, we get that

$$\|(L_N(f))(x_0) - f(x_0)\| \overset{(2.81)}{\leq} \frac{\omega_1\left(f^{(n)}, \left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)}{(n+1)!}, \tag{2.82}$$

a high speed of convergence.

The theorem is proved. □

Theorem 2.11. *All as in Theorem* 2.10. *Consider $n \in \mathbb{N}$ an odd number. Then inequalities* (2.63)–(2.65) *are sharp, in fact they are attained by $f_*(t) = \vec{i}\,|t - x_0|^{n+1}$, where $\vec{i} \in X$, $\|\vec{i}\| = 1$, $\forall\, t \in [a, b]$.*

Proof. Let n an odd natural number, then $n + 1$ is even. We consider $f_*(t) = \vec{i}\,|t - x_0|^{n+1} = \vec{i}\,(t - x_0)^{n+1} \in X$, where $\vec{i} \in X$, $\|\vec{i}\| = 1$. We have that $f_* \in C^n([a, b], X)$ and

$$f_*^{(n)}(t) = \vec{i}\,(n + 1)!\,(t - x_0), \quad \forall t \in [a, b], \tag{2.83}$$

along with $f_*^{(k)}(x_0) = 0$, $k = 0, 1, ..., n$.

Furthermore it holds

$$\left\| f_*^{(n)}(t) - f_*^{(n)}(x_0) \right\| = (n + 1)!\,|t - x_0|, \tag{2.84}$$

which is a convex function in $t \in [a, b]$. So we apply f_* to inequalities (2.63)-(2.65).

1) The left hand side of (2.63) equals $\left(\widetilde{L}_N(|t - x_0|^{n+1})\right)(x_0)$. The right hand side of (2.63) is

$$\frac{\omega_1\left(f_*^{(n)}, \left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)}{(n+1)!} = \tag{2.85}$$

$$= \frac{\omega_1\left(\vec{i}(n+1)!(t - x_0), \left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)\right)}{(n+1)!}$$

$$= \frac{(n+1)! \sup\limits_{\substack{t_1, t_2 \in [a,b]: \\ |t_1 - t_2| \leq (\widetilde{L}_N(|\cdot - x_0|^{n+1}))(x_0)}} \left\| \vec{i}\,(t_1 - x_0) - \vec{i}\,(t_2 - x_0) \right\|}{(n+1)!} =$$

$$= \sup_{\substack{t_1, t_2 \in [a,b]: \\ |t_1 - t_2| \leq (\widetilde{L}_N(|\cdot - x_0|^{n+1}))(x_0)}} |t_1 - t_2| = \left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0). \tag{2.86}$$

Hence the right hand side of (2.63) equals also $\left(\widetilde{L}_N\left(|\cdot - x_0|^{n+1}\right)\right)(x_0)$. That is (2.63) is an attained inequality by f_*.

2) The left hand side of (2.64) equals $\left(\widetilde{L}_N\left(|t - x_0|^{n+1}\right)\right)(x_0)$, which equals its right hand side. That is (2.64) is an attained inequality by f_*.

3) Same as above, inequality (2.65) is attained by f_*.

We have proved that (2.63)-(2.65) are sharp inequalities. □

Corollary 2.12. *(to Theorem 2.7) Let $(X, \|\cdot\|)$ be a Banach space and any $f \in C^1([0,1], X)$. Then*

1)

(2.87)

$$\|(B_N(f))(x) - f(x)\| \leq \frac{3}{2}\omega_1\left(f', \sqrt{\frac{x(1-x)}{N}}\right)\sqrt{\frac{x(1-x)}{N}} \leq \frac{0.75}{\sqrt{N}}\omega_1(f', \frac{1}{2\sqrt{N}}),$$

$\forall x \in [0,1], \forall N \in \mathbb{N}$, and

2)

$$\||B_N(f) - f\||_{\infty,[0,1]} \leq \frac{0.75}{\sqrt{N}}\omega_1(f', \frac{1}{2\sqrt{N}}). \tag{2.88}$$

Hence $B_n(f) \xrightarrow{u} f$, uniformly, as $N \to \infty$.

Proof. The operators B_N, $\widetilde{B}_N$ fulfill (2.19), (2.20). We have that $\widetilde{B}_N(1) = 1$, $(\widetilde{B}_N(id))(x) = x$,

$$(\widetilde{B}_N((\cdot - x)))(x) = 0, \tag{2.89}$$

and

$$(\widetilde{B}_N((\cdot - x)^2))(x) = \frac{x(1-x)}{N} \leq \frac{1}{4N}, \quad \forall\, x \in [0,1].$$

We use (2.21) for $n = 1$. We have (by use of (2.89)) that

$$\|(B_N(f))(x) - f(x)\| \leq$$

$$\leq \frac{3}{2}\omega_1(f', ((\widetilde{B}_N((\cdot - x)^2))(x))^{\frac{1}{2}})((\widetilde{B}_N((\cdot - x)^2))(x))^{\frac{1}{2}} = \tag{2.90}$$

$$= \frac{3}{2}\omega_1\left(f', \sqrt{\frac{x(1-x)}{N}}\right)\sqrt{\frac{x(1-x)}{N}} \leq \frac{3}{2}\omega_1\left(f', \frac{1}{2\sqrt{N}}\right)\frac{1}{2\sqrt{N}} =$$

$$= \frac{3}{4\sqrt{N}}\omega_1\left(f', \frac{1}{2\sqrt{N}}\right) = \frac{0.75}{\sqrt{N}}\omega_1\left(f', \frac{1}{2\sqrt{N}}\right). \tag{2.91}$$

□

We finish with

Corollary 2.13. (to *Theorem* 2.10) *Let $(X, \|\cdot\|)$ be a Banach space and any $f \in C^1([0,1], X)$ such that $\|f'(t) - f'(x_0)\|$ is convex function in $t \in [0,1]$, where $x_0 \in (0,1)$ is a fixed number. Then*

(2.92)

$$\|(B_N(f))(x_0) - f(x_0)\| \leq \frac{1}{2}\omega_1(f', \frac{x_0(1-x_0)}{N}) \leq \frac{1}{2}\omega_1\left(f', \frac{1}{4N}\right), \quad \forall\, N \in \mathbb{N}.$$

Above notice the high speed of convergence $\frac{1}{N}$ under the convexity assumption.

Inequalities (2.92) are sharp. The first part of (2.92) is attained by $\vec{i}(t - x_0)^2$, $\vec{i} \in X$, $\|\vec{i}\| = 1$, $\forall\, t \in [0,1]$. The second part of (2.92) is equality at $x_0 = \frac{1}{2}$.

As $N \to \infty$, we have that $(B_N(f))(x_0) \to f(x_0)$.

Proof. Let $x_0 \in (0,1)$, then $x_0(1-x_0) \leq x_0$ and $x_0(1-x_0) \leq 1-x_0$, hence $x_0(1-x_0) \leq \min(x_0, 1-x_0)$ and

$$\frac{x_0(1-x_0)}{N} \leq \min(x_0, 1-x_0), \quad \forall\, N \in \mathbb{N}. \tag{2.93}$$

The last shows that (see (2.89))

$$0 < \left(\widetilde{B}_N\left((\cdot - x_0)^2\right)\right)(x_0) \leq \min(x_0, 1-x_0), \quad \forall\, N \in \mathbb{N}. \tag{2.94}$$

Let here $f \in C^1([0,1], X)$ such that $\|f'(\cdot) - f'(x_0)\|$ is convex over $[0,1]$. Then, by (2.63), we get

$$\begin{aligned} \|(B_N(f))(x_0) - f(x_0)\| &\leq \frac{\omega_1\left(f', \left(\widetilde{B}_N\left((\cdot - x_0)^2\right)\right)(x_0)\right)}{2} \\ &= \frac{\omega_1\left(f', \frac{x_0(1-x_0)}{N}\right)}{2} \leq \frac{1}{2}\omega_1\left(f', \frac{1}{4N}\right). \end{aligned} \tag{2.95}$$

□

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