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A CLASS OF TRANSFORMATIONS OF A QUADRATIC INTEGRAL GENERATING DYNAMICAL SYSTEMS

PAUL BRACKEN*

Abstract. A class of transformation is investigated which maps a quadratic integral back to its original form but under a redefinition of free parameters. When this process is iterated, a dynamical system is generated in the form of recursive sequences which involve the parameters of the integrand. The creation of this dynamical system and some of its convergence properties are investigated.

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1. INTRODUCTION.

Invariance of certain types of integrals under special transformations is a subject of great interest for both theoretical and practical reasons [1, 2]. These are often referred to as Landen transformations [3, 4]. Suppose $I(\mathbf{p})$ represents a definite integral which depends on a set of parameters $\mathbf{p} = (p_1, \cdots, p_m)$. A Landen transformation for integral $I(\mathbf{p})$ is a map φ defined on the parameters of I such that

$$\int_{x_0}^{x_1} f(x;\mathbf{p})\,dx = \int_{x_0}^{x_1} f(x;\varphi(\mathbf{p}))\,dx.$$

The classical example of a Landen transformation is given by [5]

$$\varphi(u,v) = (\tfrac{u+v}{2}, \sqrt{uv}).$$

It can be shown [6] that this operation preserves the elliptic integral,

$$G(u,v) = \int_0^{\pi/2} \frac{dx}{\sqrt{u^2\cos^2 x + v^2 \sin^2 x}}.$$

This transformation of the parameters u and v can also be represented in the form,

$$G(u,v) = G(\tfrac{u+v}{2}, \sqrt{uv}).$$

It can be shown that this process defines a sequence or dynamical system (u_n, v_n) inductively by

$$(u_n, v_n) = \varphi(u_{n-1}, v_{n-1}),$$

*Department of Mathematics, University of Texas, Edinburg, TX, 78540, USA, e-mail: paul.bracken@utrgv.edu.

with $(u_0, v_0) = (u, v)$ is known to converge to a limit [6],

$$\sigma = \lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n.$$

This limit is called the arithmetic-geometric mean of u and v, and is denoted $\sigma = \mathrm{AGM}(u, v)$. It also of interest in that it can be applied to calculate the elliptic integral G by iteration quickly and efficiently.

It is the objective here to study the invariance of the specific integral

$$I(a,b,c) = \int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c} \tag{1}$$

under two different but related types of transformation of parameters. Each transformation maps the integral back to its original form (1) but under a redefinition of the parameters a, b, c. Each of these procedures defines a different dynamical system, each with its own rate of convergence. After defining the dynamical system, some of the associated properties, such as its limit and convergence behavior, will be studied and several new proofs of these aspects will be presented. It will turn out to be more advantageous to work with (1) in trigonometric form. To obtain this form, make the substitution $x = \tan\vartheta$ in (1) to obtain,

$$I(a,b,c) = \int_{-\pi/2}^{\pi/2} \frac{d\vartheta}{a\sin^2\vartheta + b\sin\vartheta\cos\vartheta + c\cos^2\vartheta}. \tag{2}$$

For convergence of integral (1), it must be that $4ac - b^2 > 0$. Since $b^2 \geq 0$, a and c must have the same sign. When they are both negative, a factor of negative one can be extracted from the denominator. It suffices then to restrict attention to the case in which a and c are strictly positive.

2. THE TRANSFORMATION OF THE INTEGRAL.

The first step is to develop some algebraic techniques which are required for the transformation. These will be common to both transformations investigated here. The integrand will be scaled by multiplying both numerator and denominator by an appropriate polynomial. To this end, a basis set of polynomials $U(x)$ and $V(x)$ is introduced as follows,

$$U(x) = 2x, \qquad V(x) = 1 - x^2. \tag{3}$$

These two polynomials have the following property

$$U(\tan\vartheta) = \tfrac{\sin(2\vartheta)}{\cos^2\vartheta}, \qquad V(\tan\vartheta) = \tfrac{\cos(2\vartheta)}{\cos^2\vartheta}. \tag{4}$$

It is required to determine coefficients z_i and g_i which depend on a, b and c such that the following identity holds

$$(ax^2 + bx + c)(z_0x^2 + z_1x + z_2) = g_0\,U(x)^2 + g_1\,U(x)\cdot V(x) + g_2\,V(x)^2. \tag{5}$$

The coefficients of x obtained from (5) generate a system of linear equations in the set of unknowns $\{z_i, g_i\}$. The solution set for the $\{z_i\}$ is given as follows

$$z_0 = \tfrac{1}{a}\, g_2, \quad z_1 = -\tfrac{2}{a}\, g_1 - \tfrac{b}{a^2}\, g_2, \quad z_2 = \tfrac{4}{a}\, g_0 + 2\tfrac{b}{a^3}\, g_1 + (\tfrac{b^2}{a^3} - \tfrac{2}{a} - \tfrac{c}{a^2})\, g_2.$$

Two equations at order zero and one in x remain, hence substituting the z_i into them, g_1 and g_2 can be obtained in terms of g_0

$$g_1 = \tfrac{2b(a-c)}{(a+c)^2-b^2}\, g_0, \qquad g_2 = \tfrac{4ac}{(a+c)^2-b^2}\, g_0. \tag{6}$$

Putting (6) back into the set of z_i gives

$$z_0 = \tfrac{4c}{(a+c)^2-b^2}\, g_0, \qquad z_1 = \tfrac{4b}{(a+c)^2-b^2}\, g_0, \qquad z_2 = \tfrac{4a}{(a+c)^2-b^2}\, g_0. \tag{7}$$

This procedure transforms the integrand of (1) into the form

$$\frac{1}{ax^2+bx+c} = \frac{z_0x^2+z_1x+z_2}{g_0U^2+g_1U\cdot V+g_2V^2}. \tag{8}$$

Integral (2) can now be expressed as

$$I = \sum_{i=0}^{2} \int_{-\pi/2}^{\pi/2} \frac{z_ih_i(\vartheta)\, d\vartheta}{g_0\sin^2(2\vartheta)+g_1\sin(2\vartheta)\cos(2\vartheta)+g_2\cos^2(2\vartheta)}, \tag{9}$$

where $h_0(\vartheta) = \sin^2\vartheta$, $h_1(\vartheta) = \sin\vartheta\cos\vartheta$ and $h_2(\vartheta) = \cos^2\vartheta$. Now $h_0(\vartheta)$ and $h_2(\vartheta)$ can be replaced using the trigonometric identities $(1 \mp \cos(2\vartheta))/2$ and $2h_1(\vartheta) = \sin(2\vartheta)$. It is shown the following two integrals appearing in (9) vanish,

$$\begin{aligned} I_S &= \int_{-\pi/2}^{\pi/2} \frac{\sin(2\vartheta)\, d\vartheta}{g_0\sin^2(2\vartheta)+g_1\sin(2\vartheta)\cos(2\vartheta)+g_2\cos^2(2\vartheta)}, \\ I_C &= \int_{-\pi/2}^{\pi/2} \frac{\cos(2\vartheta)\, d\vartheta}{g_0\sin^2(2\vartheta)+g_1\sin(2\vartheta)\cos(2\vartheta)+g_2\cos^2(2\vartheta)}. \end{aligned} \tag{10}$$

Consider first the integral I_S under the transformation of the variable $\vartheta \to \vartheta+\pi/2$ and use the identities $\sin(2(\vartheta+\pi/2)) = -\sin(2\vartheta)$ and $\cos(2(\vartheta+\pi/2)) = -\cos\vartheta$. Next repeat this by shifting the variable $\vartheta \to \vartheta + 3\pi/2$ and use the identities $\sin 2(\vartheta + 3\pi/2) = -\sin(2\vartheta)$ and $\cos 2(\vartheta + 3\pi/2) \to -\cos(2\vartheta)$. Substituting these results into I_S, the following two equivalent expressions for I_S are obtained

$$\begin{aligned} I_S &= -\int_0^{\pi} \frac{\sin(2\vartheta)\, d\vartheta}{g_0\sin^2(2\vartheta)+g_1\sin(2\vartheta)\cos(2\vartheta)+g_2\cos^2(2\vartheta)} \\ &= -\int_{\pi}^{2\pi} \frac{\sin(2\vartheta)\, d\vartheta}{g_0\sin^2(2\vartheta)+g_1\sin(2\vartheta)\cos(2\vartheta)+g_2\cos^2(2\vartheta)}. \end{aligned}$$

Adding these two results for I_S, we find that

$$I_S = -\frac{1}{2}\int_0^{2\pi} \frac{\sin(2\vartheta)\, d\vartheta}{g_0\sin^2(2\vartheta)+g_1\sin(2\vartheta)\cos(2\vartheta)+g_2\cos^2(2\vartheta)}. \tag{11}$$

Now transform the original form of I_S by means of $\vartheta \to \vartheta - \pi$ so that

$$I_S = \int_{\pi/2}^{3\pi/2} \frac{\sin(2\vartheta)\, d\vartheta}{g_0 \sin^2(2\vartheta) + g_1 \sin(2\vartheta)\cos(2\vartheta) + g_2 \cos^2(2\vartheta)}.$$

Adding this result to the original form for I_S and using periodicity, it is also the case that

$$I_S = \frac{1}{2} \int_0^{2\pi} \frac{\sin(2\vartheta) d\vartheta}{g_0 \sin^2(2\vartheta) + g_1 \sin(2\vartheta)\cos(2\vartheta) + g_2 \cos^2(2\vartheta)}. \tag{12}$$

Comparing (11) and (12), it follows that $I_S = -I_S$ which implies that $I_S = 0$. A similar analysis shows that $I_C = 0$. Therefore, it follows that only the constant terms contribute and (9) takes the form,

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \frac{d\vartheta}{a \sin^2 \vartheta + b \sin\vartheta \cos\vartheta + c \cos^2 \vartheta} \\ &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \frac{(z_0 + z_2)\, d\vartheta}{g_0 \sin^2(2\vartheta) + g_1 \sin(2\vartheta)\cos(2\vartheta) + g_2 \cos^2(2\vartheta)}. \end{aligned} \tag{13}$$

Making a change of variable $\phi = 2\vartheta$ in the integral on the right-hand side, (13) returns to its original form, but with a redefinition of parameters

$$\begin{aligned} I &= \frac{1}{4}(z_0 + z_2) \int_{-\pi/4}^{\pi/4} \frac{d\phi}{g_0 \sin^2 \phi + g_1 \sin\phi \cos\phi + g_2 \cos^2 \phi} \\ &= \int_{-\pi/2}^{\pi/2} \frac{d\vartheta}{\frac{2}{z_0+z_2} \sin^2 \vartheta + \frac{2g_1}{z_0+z_2} \sin\vartheta \cos\vartheta + \frac{2g_2}{z_0+z_2} \cos^2 \vartheta} \end{aligned} \tag{14}$$

Therefore, this procedure has produced an integral which has the same structure as the original I, however, the parameters a, b and c of the quadratic have been redefined in the process as follows

$$a_1 = \frac{(a+c)^2 - b^2}{2(a+c)}, \qquad b_1 = \frac{b(a-c)}{a+c}, \qquad c_1 = 2\frac{ac}{a+c}. \tag{15}$$

This procedure can be iterated again with (15) the first $n = 1$ terms of an infinite sequence or dynamical system. Substituting (15) into $4a_1c_1 - b_1^2$ and simplifying, it is found that it reduces to the original discriminant, $4ac - b^2$. Consequently, by induction, the resulting dynamical system will preserve the discriminant of the quadratic at each phase of the iteration.

Theorem 1. *There exists a Landen transformation which preserves the structure of the integral* (1) *and defines a nontrivial dynamical system by means of the identification*

$$\int_{-\infty}^{\infty} \frac{dx}{a_{n+1}x^2 + b_{n+1}x + c_{n+1}} = \int_{-\infty}^{\infty} \frac{dx}{a_n x^2 + b_n x + c_n}. \tag{16}$$

The dynamical system is defined by the three recursions,

$$a_{n+1} = \frac{(a_n+c_n)^2 - b_n^2}{2(a_n+c_n)}, \qquad b_{n+1} = \frac{b_n(a_n-c_n)}{a_n+c_n}, \qquad c_{n+1} = 2\frac{a_n c_n}{a_n+c_n}, \tag{17}$$

where $(a_0, b_0, c_0) = (a, b, c)$*. Moreover, transformation* (17) *preserves the discriminant of the quadratic, so that*

$$4a_n c_n - b_n^2 = \cdots = 4a_1 c_1 - b_1^2 = 4ac - b^2.$$

3. CONVERGENCE PROPERTIES OF FIRST DYNAMICAL SYSTEM.

The convergence properties of the sequence given by (17) can be established. Let us define

$$\gamma = \sqrt{4ac - b^2}. \tag{18}$$

Then (18) appears in the definition of the error ϵ_n,

$$\epsilon_n = (a_n - \tfrac{1}{2}\gamma, b_n, c_n - \tfrac{1}{2}\gamma). \tag{19}$$

It will be shown that $\epsilon_n \to 0$ as $n \to \infty$. To do this it will be convenient to introduce the transformation

$$2a_n = x_n - z_n, \qquad b_n = y_n, \qquad 2c_n = x_n + z_n. \tag{20}$$

Substituting (20) into (17) and solving for x_{n+1} and z_{n+1} gives

$$x_{n+1} = \tfrac{1}{2x_n}(2x_n^2 - y_n^2 - z_n^2), \qquad y_{n+1} = -\tfrac{y_n z_n}{x_n}, \qquad z_{n+1} = \tfrac{y_n^2 - z_n^2}{2x_n}. \tag{21}$$

The sequence (x_n, y_n, z_n) satisifes $4a_n c_n - b_n^2 = x_n^2 - y_n^2 - z_n^2$ and is initialized by the values $(x_0, y_0, z_0) = (x, y, z) = (a + c, b, c - a)$.

THEOREM 2. *The sequence* (x_n, y_n, z_n) *in* (21) *has the following limits*

$$\lim_{n\to\infty} x_n = \sqrt{x^2 - y^2 - z^2}, \qquad \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0. \tag{22}$$

The convergence of the sequence in this case is quadratic.

Proof. To prove (22), it can be shown that it suffices to study a sequence in which the number of variables has been reduced from three to one. Note first that (22) can be put in the equivalent form

$$\lim_{n\to\infty} (x_n - \sqrt{x^2 - y^2 - z^2})^2 + y_n^2 + z_n^2 = 0. \tag{23}$$

By invariance of the discriminant $\gamma^2 = 4a_n c_n - b_n^2 = x_n^2 - z_n^2 - y_n^2 = x^2 - y^2 - z^2$. Consequently, using $2x_n^2 - 2\gamma x_n + \gamma^2 - x_n^2 + y_n^2 + z_n^2 = 2(x_n^2 - \gamma x_n)$, the limit (23) can be expressed in the equivalent form,

$$\lim_{n\to\infty} x_n(x_n - \gamma) = 0. \tag{24}$$

Using the fact that $\gamma^2 = x_n^2 - y_n^2 - z_n^2$, by invariance of the discriminant, the first equation in (21) for $\{x_n\}$ can be written in the equivalent form

$$x_{n+1} = \tfrac{x_n^2 + \gamma^2}{2x_n}. \tag{25}$$

By introducing the new sequence α_n obtained by scaling the x_n as $x_n = \gamma\alpha_n$, the new dynamical relation is obtained

$$\alpha_{n+1} = \tfrac{\alpha_n^2 + 1}{2\alpha_n} = f(\alpha_n), \tag{26}$$

where $f(x)$ is defined to be

$$f(x) = \frac{1}{2}(x + \frac{1}{x}).$$

To establish convergence, note that the sequence results by iteration of $f(x)$, $\alpha_{n+1} = f(\alpha_n)$. Since the derivative of $f(x)$ is negative when $x < 1$ and positive when $x > 1$, this function decreases monotonically on $(0, 1)$, increases monotonically on $(1, \infty)$ and has a global minimum at $x = 1$. In fact, $x = 1$ is a fixed point of $f(x)$ and $x_0 > 0$ means $\alpha_0 > 0$. If $\alpha_0 \in (0, 1)$ then $\alpha_1 \in (1, \infty)$, so it suffices to consider the case $\alpha_0 \in (1, \infty)$. Since $f(x) > 1$ for all $x \in (0, 1) \cup (1, \infty)$, it follows that $\alpha_{n+1} = f(\alpha_n) > 1$, so $\alpha_n > 1$ for all $n \in \mathbb{N}$ by induction. Moreover,

$$\alpha_{n+1} - \alpha_n = f(\alpha_n) - \alpha_n = \frac{1-\alpha_n^2}{2\alpha_n} < 0,$$

when $\alpha_n \in (1, \infty)$, so the sequence is bounded above and below and decreases monotonically. Therefore, the Monotone Convergence Theorem implies the sequence converges to a real number $\alpha \geq 1$. Since the limit exists, the limit can be calculated by letting $n \to \infty$ on both sides of the recursion. This gives $\lim_{n\to\infty} (2\alpha_n\alpha_{n+1} - \alpha_n^2) = 1$ which implies $\alpha = 1$. Since this is the case, we calculate that

$$\alpha_{n+1} - 1 = f(\alpha_n) - 1 = \frac{(\alpha_n - 1)^2}{2\alpha_n}. \tag{27}$$

This means the convergence is quadratic. This also means that sequence $\{x_n\}$ converges and $\lim_{n\to\infty} x_n = \gamma$, and is consistent with (24). □

4. THE SECOND INTEGRAL TRANSFORMATION.

Another transformation can be formulated for the same integral by simply changing the basis set of polynomials, $U(x), V(x)$. This change will result in a different dynamical system. This transformation has been introduced [7], and will be used here. To do this, introduce the two polynomials

$$U(x) = x^3 - 3x, \qquad V(x) = 3x^2 - 1. \tag{28}$$

These polynomials have the following property,

$$U(\tan\vartheta) = -\frac{\sin(3\vartheta)}{\cos^3\vartheta}, \qquad V(\tan\vartheta) = -\frac{\cos(3\vartheta)}{\cos^3\vartheta}. \tag{29}$$

It is required to determine a new set of coefficients z_0, z_1, z_2, z_3 and g_0, g_1, g_2 such that

$$(ax^2+bx+c)(z_0x^4+z_1x^3+z_2x^2+z_1x+z_0) = g_0U(x)^2+g_1U(x)\cdot V(x)+g_2V(x)^2. \tag{30}$$

When this is expanded out and like powers are collected, a set of six equations are obtained. Starting from order zero and solving, it is found that

$$\begin{aligned} z_4 &= \frac{1}{c}\, g_2, \\ z_3 &= \frac{1}{c^2}(3cg_1 - bg_2), \\ z_2 &= \frac{1}{c^3}(9c^2g_0 - 3bcg_1 + b^2g_2 - acg_2 - 6c^2g_2), \end{aligned}$$

$$
\begin{aligned}
(31)\qquad z_1 =& \tfrac{1}{c^4}(-9bc^2g_0 + (3b^2 - 3ac - 10c^2)cg_1 + (2ac - b^2 + 6c^2)bg_2),\\
z_0 =& \tfrac{1}{c^5}[(9b^2 - 9ac - 6c^2)c^2g_0 + (6ac^2 - 3b^2c + 10c^3 + b^3)bg_2\\
&+ (a^2c - 3ab^2 - 6b^2c)cg_2 + (6ac^2 + 9c^3)cg_2].
\end{aligned}
$$

This leaves two equations that arise from the two highest powers of x. These can be used to find the coefficients g_i. The fifth order equation for g_2 is solved in terms of a, b, c, g_1 and g_0, so that substituting this result into the equation for the leading term produces

$$b(b^2 - 3(a-c)^2)g_0 = a(3b^2 - (a+3c)^2)g_1.$$

Let us choose $g_0 = a((a+3c)^2 - 3b^2)$, then it follows that $g_1 = -b(b^2 - 3(a-c)^2)$, and this yields $g_2 = -c(3b^2 - (3+c)^2)$.

Using (29), the integrand of (2) is transformed into

$$
\begin{aligned}
&\frac{1}{a\sin^2\vartheta + b\cos\vartheta\sin\vartheta + c\,\cos^2\vartheta} =\\
&= \cos^2\vartheta \sum_{k=0}^{4} \frac{z_k h_k(\vartheta)}{g_0\sin^2(3\vartheta) + g_1\sin(3\vartheta)\cos(3\vartheta) + g_2\cos^2(3\vartheta)},
\end{aligned}
$$

where $h_k(\vartheta) = \sin^{4-k}\vartheta\cos^k\vartheta$. Substituting this into integral (2), it follows that

$$(32)\qquad I = \sum_{k=0}^{4} z_k \int_{-\pi/2}^{\pi/2} \frac{h_k(\vartheta)}{g_0\sin^2(3\vartheta) + g_1\sin(3\vartheta)\cos(3\vartheta) + g_2\cos^2(3\vartheta)}.$$

The following identities

$$\cos^4\vartheta = \tfrac{1}{8}\cos(4\theta) + \tfrac{1}{2}\cos(2\theta) + \tfrac{3}{8}, \quad \cos^3\vartheta\sin\vartheta = \tfrac{1}{8}\sin(4\theta) + \tfrac{1}{4}\sin(2\theta),$$

$$\cos^2\theta\sin^2\theta = \tfrac{1}{8} - \tfrac{1}{8}\cos(4\theta),$$

$$\cos\theta\sin^3\theta = \tfrac{1}{4}\sin(2\theta) - \tfrac{1}{8}\sin(4\theta), \quad \sin^4\theta = \tfrac{1}{8}\cos(4\theta) - \tfrac{1}{2}\cos(2\theta) + \tfrac{3}{8}$$

transform (32) into a linear combination of the following integrals,

$$(33)\quad S_k = \int_{-\pi/2}^{\pi/2} \frac{\sin(k\vartheta)\,d\vartheta}{g_0\sin^2(3\vartheta) + g_1\sin(3\vartheta)\cos(\vartheta) + g_2\cos^2(3\vartheta)}, \qquad k = 2, 4,$$

$$(34)\quad C_k = \int_{-\pi/2}^{\pi/2} \frac{\cos(k\vartheta)\,d\vartheta}{g_0\sin^2(3\vartheta) + g_1\sin(3\vartheta)\cos(3\vartheta) + g_2\cos^2(3\vartheta)}, \quad k = 0, 2, 4.$$

Both $\sin(3\vartheta)$ and $\cos(3\vartheta)$ are invariant under shifts of $2\pi/3$ and $4\pi/3$, hence

$$6C_k = \int_0^{2\pi} \frac{\cos(ku) + \cos(ku - \frac{2\pi}{3}k) + \cos(ku - \frac{4\pi}{3}k)}{g_0\sin^2(3u) + g_1\sin(3u)\cos(3u) + g_2\cos^2(3u)}\,du.$$

The numerator reduces to either 3 or 0 depending on whether 3 divides k or not. This means that only the terms that contribute to I are the constants in the identities at $k = 0, 2, 4$. Therefore, by periodicity, I becomes

$$I = \frac{1}{16} \int_0^{2\pi} \frac{3z_4 + z_2 + 3z_0}{g_0 \sin^2(3u) + g_1 \sin(3u)\cos(3u) + g_2 \cos^2(3u)}\, du.$$

Changing the variable according to $\vartheta = 3u$ leads to the same structure for I as in (2),

$$I = \frac{1}{8} \int_{-\pi/2}^{\pi/2} \frac{d\vartheta}{a_1 \sin^2 \vartheta + b_1 \sin \vartheta \cos \vartheta + g_2 \cos^2 \vartheta}. \tag{35}$$

This is the original form of the integral, but with different set of parameters. Substituting the constants z_i and g_i in terms of a, b and c, the new coefficents in (35) are given as

$$a_1 = a\frac{(3a+c)^2-3b^2}{(3a+c)(a+3c)-b^2}, \quad b_1 = b\frac{3(a-c)^2-b^2}{(3a+c)(a+3c)-b^2}, \quad c_1 = c\frac{(a+3c)^2-3b^2}{(3a+c)(a+3c)-b^2}. \tag{36}$$

Moreover, substituting (36) into $4a_1c_1 - b_1^2$ and simplifying, the original discriminant is recovered, $4ac - b^2$. This can be summarized in the following theorem.

THEOREM 3. *There exists a Landen transformation which preserves the form of integral* (2) *and defines a dynamical system by means of the correspondence*

$$\int_{-\infty}^{\infty} \frac{dx}{a_{n+1}x^2 + b_{n+1}x + c_{n+1}} = \int_{-\infty}^{\infty} \frac{dx}{a_n x^2 + b_n x + c_n},$$

where the sequence is defined to be

$$\begin{aligned} a_{n+1} &= a_n \frac{(a_n+3c_n)^2-3b_n^2}{(3a_n+c_n)(a_n+3c_n)-b_n^2}, \\ b_{n+1} &= b_n \frac{3(a_n-c_n)^2-b_n^2}{(3a_n+c_n)(a_n+3c_n)-b_n^2}, \\ c_{n+1} &= c_n \frac{(3a_n+c_n)^2-3b_n^2}{(3a_n+c_n)(a_n+3c_n)-b_n^2}, \end{aligned} \tag{37}$$

where $a_0 = a, b_0 = b, c_0 = c$. Moreover, the form of the discriminant of the quadratic is preserved under (37),

$$4ac - b^2 = 4a_1c_1 - b_1^2 = \cdots = 4a_nc_n - b_n^2.$$

5. CONVERGENCE PROPERTIES OF THE SECOND DYNAMICAL SYSTEM.

The convergence properties of the resulting dynamical system (37) will be studied. Define the error as was done in (19) and show that it approaches zero. Introduce the change of variables

$$x_n = a_n + c_n, \qquad y_n = b_n, \quad z_n = a_n - c_n, \tag{38}$$

so that the dynamical system (37) goes into the form

$$x_{n+1} = x_n\Big(\frac{4x_n^2-3y_n^2-3z_n^2}{4x_n^2-y_n^2-z_n^2}\Big), \; y_{n+1} = y_n\Big(\frac{3z_n^2-y_n^2}{4x_n^2-y_n^2-z_n^2}\Big), \; z_{n+1} = z_n\Big(\frac{z_n^2-3y_n^2}{4x_n^2-y_n^2-z_n^2}\Big), \tag{39}$$

with initial conditions $x_0 = x$, $y_0 = y$ and $z_0 = z$. Again, the discriminant of the quadratic is preserved by transformation (39).

THEOREM 4. *The sequence given in* (39) *has the following limits*

$$\lim_{n\to\infty} x_n = \sqrt{x^2 - y^2 - z^2}, \qquad \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0, \tag{40}$$

Moreover, the convergence is cubic in this case.

Proof. System (40) can be expressed equivalently as,

$$\lim_{n\to\infty} (x_n - \sqrt{x^2 - y^2 - z^2})^2 + y_n^2 + z_n^2 = 0. \tag{41}$$

The invariance of the discriminant can be exploited to put the first equation of (39) in the equivalent form

$$x_{n+1} = x_n\left(\tfrac{x_n^2+3\gamma^2}{3x_n^2+\gamma^2}\right), \tag{42}$$

where $x_0 = a + c$. Therefore, this invariance or symmetry of the discriminant has allowed us again to reduce the number of variables from three to one giving a simpler recursion. Thus, (41) is equivalent to

$$\lim_{n\to\infty} x_n(x_n - \gamma) = 0. \tag{43}$$

As before, let us scale the sequence x_n and define a new sequence $\{\alpha_n\}$ by setting $x_n = \gamma\alpha_n$, so sequence (42) becomes

$$\alpha_{n+1} = \alpha_n\left(\tfrac{\alpha_n^2+3}{3\alpha_n^2+1}\right). \tag{44}$$

The sequence is then generated by means of the iteration $\alpha_{n+1} = h(\alpha_n)$ where

$$h(x) = x\left(\tfrac{x^2+3}{3x^2+1}\right). \tag{45}$$

In this instance, both $x = 0, 1$ are fixed points of function (45) which has derivative

$$h'(x) = 3\tfrac{(x-1)^2(x+1)^2}{(3x^2+1)^2} > 0, \tag{46}$$

when $x \neq 1$. It follows that $h(x)$ is strictly increasing on $(0, 1) \cup (1, \infty)$.

Suppose first that $\alpha_0 > 1$. Then since h is strictly increasing $h(x) > 1$ when $x > 1$ and so $\alpha_n > 1$ then implies that $\alpha_{n+1} = h(\alpha_n) > 1$ for all $n \in \mathbb{N}$ and

$$\alpha_{n+1} - \alpha_n = h(\alpha_n) - \alpha_n = \tfrac{2\alpha_n(1-\alpha_n)(\alpha_n+1)}{3\alpha_n+1} < 0. \tag{47}$$

This implies that the sequence $\{\alpha_n\}$ is monotonically decreasing and is bounded below by one. Therefore, it must converge to a limit in $[1, \infty)$ by the Monotone Convergence Theorem.

Next, suppose now that $0 < \alpha_n < 1$, then since $h : (0, 1) \to (0, 1)$, we have $\alpha_{n+1} = h(\alpha_n) < 1$ and by (47), it follows that $\alpha_{n+1} - \alpha_n > 0$. Therefore, the sequence is monotonically increasing and bounded above by one, so the sequence again converges by the Monotone Convergence Theorem. Since the limit $\alpha = \lim_{n\to\infty} \alpha_n$ exists and satisfies the equation $\alpha = h(\alpha)$, the only positive solution of this gives $\alpha = 1$. Although the sequence in the first case was shown to have quadratic convergence, under this second transformation, it is found that

$$|\alpha_{n+1} - 1| = |h(\alpha_n) - 1| = \tfrac{|\alpha_n - 1|^3}{3\alpha_n^2 + 1}. \tag{48}$$

Therefore, this procedure generates a sequence in which the convergence is cubic. Moreover, from the definition of the x_n in terms of the α_n, it follows that $\lim_{n\to\infty} x_n = \gamma$ in agreement with (43). □

It is worth stating in summary that these kinds of transformations are going to be useful in fast numerical evaluation of integrals. For either transformation considered here, moving the limit $n \to \infty$ inside the integral, for example (16), allows the explicit calculation of the integral itself

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c} = \tfrac{2\pi}{\gamma}. \tag{49}$$

It may be conjectured that this iterative process can be continued over other basis sets of polynomials, for example Chebyshev polynomials, which could result in a hierarchy of dynamical systems.

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