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EXTENDING THE RADIUS OF CONVERGENCE FOR A CLASS OF EULER-HALLEY TYPE METHODS

IOANNIS K. ARGYROS* and SANTHOSH GEORGE†

Abstract. The aim of this paper is to extend the radius of convergence and improve the ratio of convergence for a certain class of Euler-Halley type methods with one parameter in a Banach space. These improvements over earlier works are obtained using the same functions as before but more precise information on the location of the iterates. Special cases and examples are also presented in this study.

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1. INTRODUCTION

Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces and Ω be an open and convex subset of $\mathcal{B}_1$. The problem of finding a solution of equation

$$F(x) = 0. \tag{1}$$

where $F : \Omega \longrightarrow \mathcal{B}_2$ is differentiable in the sense of Fréchet is important problem in applied mathematics due its wide applications.

In this paper we study the local convergence of the Euler-Halley-type method (EHTM) defined for each $n = 0, 1, 2...$ by [9]–[14]

$$x_{n+1,\alpha} = T_{F,\alpha}(x_{n,\alpha}) \tag{2}$$

where

$$\begin{aligned} T_{F,\alpha}(x) &= x - [I + \frac{1}{2}(I - \alpha L_F(x))^{-1} L_F(x)] F'(x)^{-1} F(x) \\ K_F(x) &= F'(x)^{-1} F''(x) F'(x)^{-1} F(x) \end{aligned}$$

with x_0 being an initial guess and $\alpha \in (-\infty, +\infty)$.

Notice that, method (2) becomes Halley method when $\alpha = \frac{1}{2}$, becomes Chebyshev-Euler method when $\alpha = 0$ and super-Halley method when $\alpha = 1$.

*1Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA, e-mail: iargyros@cameron.edu.

†Department of Mathematical and Computational Sciences, NIT Karnataka, India-575 025, e-mail: sgeorge@nitk.ac.in.

The local convergence of the EHTM (2) was studied in [14], using the second-order generalized Lipschitz assumption with $L-$average (see Section2 in [14]). The radius of the optimal convergence ball and the error estimation of method (2) corresponding to the parameter α are also estimated for each $\alpha \in (-\infty, +\infty)$ in [14]. Huang and Guocham in [14] also shown that the method (2) with α is better than the one corresponding to $-\alpha$ for each $\alpha > 0$ and the Chebyshev-Euler method is best among all methods in the family with $\alpha \in (-\infty, 0]$ as far as the choice of initial point and error estimates are concerned.

In this study we use second-order generalized Lipschitz condition with K_0-average (to be precised in Definition 1) to study the local convergence of method (2). Using second-order generalized Lipschitz condition with K_0-average we improved the results in [14]. Moreover, our radius of convergence is better than the one in [14] and the information on the location of the iterates in our study is more precise than that of [14].

The paper is structured as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result not given in the earlier studies [1]–[17]. Special cases and numerical examples are presented in the concluding Section 3.

2. LOCAL CONVERGENCE

Denote by $S(\lambda, \xi), \bar{S}(\lambda, \xi)$, respectively the open and closed balls in $\mathcal{B}_1$ with center $\lambda \in \mathcal{B}_1$ and of radius $\xi > 0$.

Let $R > 0$. Set $R_0 = \sup\{t \in [0, R) : S(t, R) \subset \Omega\}$. Let also $K_0, \bar{K}, K$ be real valued C^1 functions defined on the interval $[0, R_0]$, increasing on $[0, R_0]$ with $K_0'(t) \geq 0, \bar{K}'(t) \geq 0, K'(t) \geq 0, K_0(0) > 0, \bar{K}(0) > 0$ and $K(0) > 0$. Denote by ρ the smallest positive solution of equation

$$\int_0^1 K_0(t)dt = 1. \tag{1}$$

Define function h_0 by

$$h_0(t) = -t + \int_0^t (t-u)K_0(u)du.$$

Notice that $h_0'(\rho) = 0$ and $h_0'(t) < 0$ for all $t \in [0, \rho)$. We need the notion of the second-order generalized center-Lipschitz condition with K_0-average in $S(p, \rho)$.

Definition 1. *We say that F satisfies the second-order generalized Lipschitz condition with K_0- average in $S(p, \rho)$, if there exists $p \in \Omega$ such that $F(p) = 0$ and $F'(p)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$;*

$$\|F'(p)^{-1}F''(p) \leq K_0(0)$$

and

$$\|F'(p)^{-1}(F''(x) - F''(p))\| \leq \int_0^{\|x-p\|} K_0'(u)du$$

for all $x \in S(p, \rho)$.

Definition 2. *We say that* F *satisfies the second-order generalized Lipschitz condition with* $K-$*average in* $S(p, R_0)$, *if there exists* $p \in \Omega$ *such that* $F(p) = 0$ *and* $F'(p)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$;

$$\|F'(p)^{-1}F''(p)\| \leq K(0)$$

and

$$\|F'(p)^{-1}(F''(x) - F''(p + \theta(x-p)))\| \leq \int_{\theta\|x-p\|}^{\|x-p\|} K'(u)du$$

for all $x \in S(p, R_0)$ *and* $\theta \in [0, 1]$.

Next, we introduce the notion of second-order generalized K_0-restricted Lipschitz condition with $\bar{K}-$average in $S(p, \rho)$.

Definition 3. *We say that* F *satisfies the second-order generalized* K_0-*restricted Lipschitz condition with* $\bar{K}-$*average in* $S(p, \rho)$, *if there exists* $p \in \Omega$ *such that* $F(p) = 0$ *and* $F'(p)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$;

$$\|F'(p)^{-1}F''(p)\| \leq \bar{K}(0)$$

and

$$\|F'(p)^{-1}(F''(x) - F''(p + \theta(x-p)))\| \leq \int_{\theta\|x-p\|}^{\|x-p\|} \bar{K}'(u)du$$

for all $x \in S(p, \rho)$ *and* $\theta \in [0, 1]$.

Remark 4. The introduction of function was not possible before, since $K = \bar{K}(K_0)$. Clearly, we have

$$K_0(t) \leq K(t) \tag{2}$$

$$\bar{K}(t) \leq K(t) \tag{3}$$

for all $t \in I \subseteq [0, R_0]$. We have noticed that iterates $\{x_n\}$ lie in $S(p, \rho)$ which is a more accurate location than $S(p, R_0)$, since $\rho \leq R_0$ and the estimate

$$\|F'(x)^{-1}F'(p)\| \leq -\tfrac{1}{h_0'(\|x-p\|)} \tag{4}$$

(obtained using Definition 1) is more precise than

$$\|F'(x)^{-1}F'(p)\| \leq -\tfrac{1}{h'(\|x-p\|)} \tag{5}$$

(using Definition 3 (see [14])), where

$$h(t) = -t + \int_0^t (t-u)K(u)du.$$

Define also function $\bar{h}$ by

$$\bar{h}(t) = -t + \int_0^1 (t-u)\bar{K}(u)du.$$

Then, we have that

$$h_0(t) \leq h(t) \tag{6}$$

and

$$\bar{h}(t) \leq h(t) \text{ for all } t \in I. \tag{7}$$

Suppose from now on that

$$h_0(t) \leq \bar{h}(t) \text{ for all } t \in I. \tag{8}$$

Then, the results in [14] can be written with $\bar{h}$ replacing h and estimate (4) replacing (5). If

$$\bar{h}(t) \leq h_0(t) \text{ for all } t \in I. \tag{9}$$

Then, the results in [14] can be written with h_0 replacing h. Hence, we arrived at:

THEOREM 5. *Suppose: F satisfies the second-order generalized K_0-restricted Lipschitz condition with $\bar{K}-$average in $S(p,\rho)$.*

(i) *Let $\alpha \leq 0$. Then, $\bar{\rho}_\alpha$ is the unique solution of equation*

$$1 + (\tfrac{1}{2} - \alpha)K_{\bar{h}}(t) = 0 \tag{10}$$

in $(0,\rho)$. Moreover, $\bar{\rho}_\alpha$ is the closest repelling extraneous fixed point of $T_{\bar{h},\alpha}(t)$ to zero for t being a real number. Furthermore, if $\bar{K}(t)$ exists and $\bar{h}(t)$ satisfies hypotheses of Definition 3 in $S(p,\rho) \subseteq \mathbb{C}$, then $\bar{\rho}_\alpha$ is the closest repelling extraneous fixed point of $\{T_{\bar{h},\alpha}(t)\}$ to 0 for $t \in S(0,\rho) \subseteq \mathbb{C}$.

(ii) *$\bar{\rho}_\alpha$ increases, if α increases in $(-\infty, 0]$.*

(iii) *$\bar{\rho}_{-\alpha} \leq \bar{\rho}_\alpha$ for all $\alpha > 0$.*

(iv) *Sequence $\{T^n_{F,\alpha}(x_{0,\alpha})\}$ defined by $x_{0,\alpha} = x_0 \in S(p, \bar{\rho}_{-\alpha}) - \{p\}$ converges to p such that for all $n = 0,1,2,\ldots, \alpha \in (-\infty,+\infty)$*

$$\|x_{n+1,\alpha} - p\| \leq y_{n+1,-|\alpha|} \leq \bar{q}_\alpha^{3^{n+1}-1} y_{0,\alpha}$$

where $y_{n+1,-|\alpha|} = T_{\bar{h},-|\alpha|}(y_{n,-|\alpha|}), y_{0,\alpha} = y_0 = \|x_0 - p\| \in S(0, \bar{\rho}_{-|\alpha|})$ and

$$\bar{q}_\alpha = \sqrt{\frac{T_{\bar{h},-|\alpha|}(y_0)}{y_0}} \in (0,1). \tag{11}$$

(v) *Sequence $\{y_{n,\alpha}\}$ converges optimaly to zero for all $\alpha < 0$. Moreover, if $\alpha_2 < \alpha_1 < 0$, then*

$$0 < y_{n,\alpha_1} < y_{n,\alpha_2}$$

holds for all $y_{0,\alpha_1} = y_{0,\alpha_2} = y_0 \in (0, \bar{\rho}_{\alpha_2})$.

REMARK 6. (a) Let ρ_α, q_α be the radius of convergence and ratio of convergence, respectively corresponding to $\bar{\rho}_\alpha, \bar{q}_\alpha$ *i.e.* ρ_α satisfies

$$1 + (\tfrac{1}{2} - \alpha)K_h(t) = 0 \tag{12}$$

and

$$q_\alpha = \sqrt{\frac{T_{h,-|\alpha|}(y_0)}{y_0}} \in (0,1). \tag{13}$$

Then, in view of (3) and (7), we have that

$$\rho_\alpha \leq \bar{\rho}_\alpha \tag{14}$$

and

$$\bar{q}_\alpha \leq q_\alpha. \tag{15}$$

Hence, (14) and (15) justify the advantages claimed in the introduction (see also the numerical examples).

(b) Radius ρ and function $\bar{K}$ can be introduced in a different way as follows: Suppose: There exists function w_0 defined on $[0, R_0)$ with $w_0(0) = 0$ such that

$$\|F'(p)^{-1}(F'(x) - F'(p))\| \leq w_0(\|x - p\|) \tag{16}$$

for all $x \in S(p, R_0)$. Let r be the smallest positive solution of equation

$$w_0(t) = 1. \tag{17}$$

If $x \in S(p, r)$, then we have $F'(x)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|F'(x)^{-1}F'(p)\| \leq \tfrac{1}{1 - w_0(\|x-p\|)}. \tag{18}$$

Suppose that

$$w_0(t) \leq 1 + h_0'(t) \text{ for all } t \in I \subseteq [0, \rho]. \tag{19}$$

Then, (18) gives a better upper bound on $\|F'(x)^{-1}F'(p)\|$ than (4). Then, since the iterates $\{x_n\}$ stay in $S(p, r)$ this ball can be used in Definition 3) to introduce function $K^1 = K^1(r)$ replacing $\bar{K}$. Then, clearly r, K^1 can replace $\rho, \bar{K}$ in Theorem 5. Let

$$1 + (\tfrac{1}{2} - \alpha)K^1_{h^1}(t) = 0 \tag{20}$$

in $(0, r)$ and

$$q^1_\alpha = \sqrt{\frac{T_{h^1,-|\alpha|}(y_0)}{y_0}}. \tag{21}$$

where

$$h^1(t) = -t + \int_0^t (t-u)K^1(u)du. \tag{22}$$

Suppose that $r \leq \rho$ and

$$K^1(t) \leq \bar{K}(t) \text{ for all } t \in I \subseteq [0, r], \tag{23}$$

then, we have

$$\bar{\rho}_\alpha \le r_\alpha \tag{24}$$

and

$$q^1_\alpha \le \bar{q}_\alpha. \tag{25}$$

Estimates (24) and (25) show that the radius of convergence can be enlarged even further and the error bounds can be improved even further too (see also the numerical examples).

3. SPECIAL CASES AND EXAMPLES

The numerical examples are presented in this section.

3.1. Special case: Kantorovich-type hypothesis. Let $K(t) = \beta t + \gamma$ for some $\beta \ge 0$ and $\gamma > 0$. The other "K" functions can be defined similarly (see also the numerical examples).

3.2. Special case: (Smale-Wang-type hypothesis). Let $K(t) = \frac{2\delta}{(1-\delta t)^3}$ for some $\delta > 0$. The other "K" functions can be defined similarly.

Example 7. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0,1), p = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \tfrac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case $K(t) = 2et + 1, K_0(t)(e-1)t + 1, \bar{K}(t) = 2e^\rho t + 1, w_0(t) = (e-1)t$, $K_1(t) = 2e^{\frac{1}{e-1}}t + 1, \rho = \frac{-1+\sqrt{1+2(e-1)}}{e-1}$.

Notice that $w_0(t) < K_0(t) < K_1(t) < \bar{K}(t) < K(t)$. Then the parameters are given in Table 1.

$-\alpha$	ρ_α	q_α	$\bar{\rho}_\alpha$	$\bar{q}_\alpha$	r_α	q^1_α
0.4	0.2236	0.89587717	0.2515	0.70428172	0.2566	0.67821226
0.5	0.2157	0.70376365	0.2422	0.58159283	0.2469	0.56343970
0.6	0.2085	0.51959637	0.2337	0.46428071	0.2382	0.45348383
1.0	0.1850	0.48909481	0.2063	0.29397157	0.2101	0.27074561
2.0	0.1473	0.82647906	0.1628	0.61788060	0.1655	0.59280075
3.0	0.1241	0.91927180	0.1368	0.70577233	0.1382	0.68001607
4.0	0.1079	0.96383321	0.1175	0.74855766	0.1194	0.72259296
5.0	0.0958	0.99009945	0.1040	0.77402979	0.1054	0.74796064
6.0	0.0863	1.60074361	0.0933	0.79090015	0.0945	0.76482714

Table 1. Comparison table for the parameters.

Clearly, the new results appearing in columns 4–7 are such that the radii are larger leading to a wider choice of initial points and the ratio is smaller implying fewer iterates to arrive at a desired error tolerance than in columns 2 and 3. It is worth noticing that these advantages are obtained under the same computational cost, since in practice the computation of K requires the computation of $w_0, K_0, K_1, \bar{K}$ as special cases. Hence, the claims made previously are justified.

REFERENCES

[1] S. Amat, S. Busquier, J.M. Gutiérrez, *Geometric constructions of iterative functions to solve nonlinear equations*, J. Comput. Appl. Math., **157** (2003), pp. 197–205.

[2] I.K. Argyros, *On an improved unified convergence analysis for a certain class of Euler-Halley-type methods*, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. **13**, (2006), pp. 207–215.

[3] I.K. Argyros, D. Chen, *Results on the Chebyshev method in Banach spaces*, Proyecciones **12**, (1993), pp. 119–128.

[4] I.K. Argyros, D. Chen, Q. Quian, *A convergence analysis for rational methods with a parameter in Banach space*, Pure Math. Appl. **5**, (1994), pp. 59–73.

[5] I.K. Argyros, A.A. Magreñán, *Improved local convergence analysis of the Gauss-Newton method under a majorant condition*, Computational Optimization and Applications,60,2(2015), pp. 423–439.

[6] D. Chen, I.K. Argyros, Q. Quian, *A local convergence theorem for the Supper-Halley method in a Banach space*, Appl. Math. Lett.**7** (1994), pp. 49–52.

[7] J.A. Ezquerro, M.A. Hernandez, *New Kantorovich-type conditions for Halley's method*, Appl. Numer. Anal. Comput. Math., bf 2(2005), pp. 70–77.

[8] J.A. Ezquerro, M.A. Hernandez, *On the $R-$order of the Halley method*, J. Math. Anal. Appl., **303** (2005), pp. 591–601.

[9] J.A. Ezquerro, M.A. Hernandez, *Halley's method for operators with unbounded second derivative*, Appl.Numer. Math., **57**(2007), pp. 113–130.

[10] J.M. Gutiérrez, M.A. Hernández, *A family of Chebyshev-Halley-type methods in Banach space*, Bull. Aust. Math.Soc. **55**,(1997), pp. 113–130.

[11] J.M. Gutiérrez, M.A. Hernández, *An acceleration of Newton's method: Super-Halley method*, Appl. Math. Comput., bf 117 (2001), pp. 223–239.

[12] M.A. Hernández, M.A. Salanova, *A family of Chebyshev-Halley type methods*, Int. J. Comput. MAth., **47** (1993), pp. 59–63.

[13] Z. Huang, *On a family of Chebyshev-Halley type methods in Banach space under weaker Smale condition*, Numer. Math. JCU **9** (2000), pp. 37–44.

[14] Z. Huang, M. Guochun, https://doi.org/10.1007/s11075-009-9284-1 *On the local convergence of a family of Euler-Halley type iteration with a parameter*, Numer. Algor., **52** (2009), pp. 419–433.

[15] A.A. Magrenãn, *Different anomalies in a Jarratt family of iterative root-finding methods*, Appl.Math.Comput.**233** (2014), pp. 29-38.

[16] A.A. Magrenãn, *A new tool to study real dynamics: The convergence plane*, Appl. Math. Comput. **248** (2014), pp. 215-224.

[17] X. Wang, *Convergence of the iteration of Halley's family and Smale operator class in Banach space*, Sci. China Ser. A 41, (1998), pp. 700–709.

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