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A NUMERICAL COMPARISON BETWEEN TWO EXACT SIMPLICIAL METHODS FOR SOLVING A CAPACITATED 4-INDEX TRANSPORTATION PROBLEM

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Abstract. In this paper, we deal with a numerical comparison between two exact simplicial methods for solving a capacitated four-index transportation problem. The first method was developed by R. Zitouni and A. Keraghel for solving this problem [Resolution of a capacitated transportation problem with four subscripts, Kybernetes, Emerald journals, 32, 9/10: 1450-1463 (2003)]. The second approach is the well-known simplex method. We show across some obtained numerical results that the first algorithm competes well with the simplex method.

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Keywords. Capacitated transportation problem, four-index transportation problem, linear programming, simplicial algorithms.

1. INTRODUCTION

The transportation problem is an important subject in real-world life that covers many important problems in economy, telecommunication and localization, among others. As a special case, the transportation problem with two-dimensional index has been extensively studied and solved by L. V. Kantorovich and M. K. Gavurin (1949, [4]) and G. B. Dantzig (1951, [2]). Next, the study and the solution have been extended to transportation problems where the dimension of the index is higher than two. Since the sixties, several papers have been published for uncapacitated problems with a three-dimensional index and a general multi-dimensional index, see for instance [3], [5], [6] and [7].

Recently, Zitouni [10], first introduced an algorithm for solving a capacitated transportation problem with 4-index (four subscripts). The choice for the index transportation problem allows us to getting an idea for the more general multi-dimensional transportation problem case.

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Our aim in this paper is to examine two exact simplicial methods for the solution of the capacited 4-index transportation problem, the method developed in [10] and the simplex method. Because the algorithm developed in [10], tackles directly the problem, and shows its elegant computation and its rapidity convergence to provide a solution of this problem. For the comparison purpose, the obtained numerical results are compared with those obtained by the classical simplex approach applied to the reformulation of this problem as a linear program.

The outline of the paper is as follows. In Section 2, the 4-index capacited transportation problem formulation is stated. In section 3, the transportation problem solution is given. In section 4, the detailed description of two exact simplicial algorithms and the convergence of the second algorithm are presented. In section 5, some computational results are reported, followed by an important numerical comparison between their performances. Finally, a conclusion and future researchers are drawn in the last section.

Some notation used throughout the paper is as follows. $\mathbb{R}^r$ denotes the space of r-dimensional real vectors whereas the set of all matrices with type (r_1, r_2) is denoted by $\mathbb{R}^{r_1 \times r_2}$. If $x, z \in \mathbb{R}^r$, then $x^T z$ denote their usual inner product. If $S \in \mathbb{R}^{r_1 \times r_2}$, then its rank is defined as $\operatorname{rank}(S) = r \leq \min(r_1, r_2)$.

2. THE CAPACITATED 4-INDEX TRANSPORTATION PROBLEM FORMULATION

The capacitated transportation problem with a four-dimensional index, denoted by (T_{4C}), is formulated as the following constrained optimization problem:

$$\text{(1)} \qquad \text{Minimize}\, Z = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} c_{ijkl}x_{ijkl}$$

subject to the constraints:

$$\text{(2)} \qquad \sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \alpha_i \quad \text{for all} \quad i = 1, ..., m$$

$$\text{(3)} \qquad \sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \beta_j \quad \text{for all} \quad j = 1, ..., n$$

$$\text{(4)} \qquad \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} = \gamma_k \quad \text{for all} \quad k = 1, ..., p$$

$$\text{(5)} \qquad \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} = \delta_l \quad \text{for all} \quad l = 1, ..., q$$

$$\text{(6)} \qquad 0 \leq x_{ijkl} \leq d_{ijkl} \quad \text{for all} \quad (i, j, k, l),$$

where α_i, β_j, γ_k, δ_l, d_{ijkl} and c_{ijkl} are given and such that for all i, j, k, and l, $\alpha_i > 0$, $\beta_j > 0$, $\gamma_k > 0$, $\delta_l > 0$, $d_{ijkl} > 0$ and $c_{ijkl} \geq 0$.

The T_{4C} problem can be also reformulated as the following linear program:

$$\min_x Z = c^T x \text{ s.t. } Ax = b,\ 0 \leq x \leq d, \tag{7}$$

with

- $x = (x_{1111}, ..., x_{mnpq})^T \in \mathbb{R}^N$,
- $c = (c_{1111}, ..., c_{mnpq})^T \in \mathbb{R}^N$,
- $d = (d_{1111}, ..., d_{mnpq})^T \in \mathbb{R}^N$,
- $b = (\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n, \gamma_1, ..., \gamma_p, \delta_1, ..., \delta_q)^T \in \mathbb{R}^M$,
- $A \in \mathbb{R}^{M\times N}$, where $M = m + n + p + q$ and $N = mnpq$.

In this representation $x = (x_{1111}, x_{1211}, ..., x_{mnpq})$ has been associated to a vector $x \in \mathbb{R}^N$. To do that, we associate to each $(i, j, k, l) \in \{1, .., m\} \times \{1, .., n\} \times \{1, .., p\} \times \{1, .., q\}$ a vector $P_{ijkl} \in \mathbb{R}^M$. Only four entries of the vector P_{ijkl} are nonzero, they are located on lines i, $m + j$, $m + n + k$ and $m + n + p + l$, and their common value is 1. Note that P_{ijkl} are the columns of A, they are called coefficients vectors.

In the sequel, we quote the following useful definitions (see [13]).

Definition 1. *A feasible solution x of T_{4C} is called basic solution if the columns of the sub-matrix A_x obtained from A by keeping only the columns corresponding to the variables x_{ijkl} such that*

$$0 < x_{ijkl} < d_{ijkl}$$

are linearly independent.

Definition 2. *A basic feasible solution is said to be non degenerate if*

$$\text{rank}(A_x) = \text{rank}(A).$$

Definition 3. *Given a basic feasible solution $x = (x_{ijkl})$, the 4-tuple (i, j, k, l) is called interesting if*

$$0 < x_{ijkl} < d_{ijkl}.$$

Throughout the paper, we assume that the following feasibility assumption for T_{4C},

$$\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j = \sum_{k=1}^{p} \gamma_k = \sum_{l=1}^{q} \delta_l = H \tag{8}$$

holds.

As a consequence, it results that

$$\text{rank}(A) = M - 3.$$

Unlike the transportation problem with two-dimensional index, the matrix A is not totally unimodular since some of its minors do not belong to $\{-1, 0, 1\}$.

It is useful to present the data of the problem thanks to the following transportation table. It consists of an array of M rows and N columns, three additional rows and an additional column. The entries of these N columns of the first, second, and third additional rows are reserved for the data of the quantities d_{ijkl}, c_{ijkl}, and x_{ijkl}, respectively. The additional column is for the data of quantities α_i, β_j, γ_k, and δ_l, respectively. Finally, the entry of the array on the line corresponding to $\alpha_{i'}$ and the column P_{ijkl} is 1 if $i = i'$ and 0 otherwise. Same things for $\beta_{j'}$, $\gamma_{k'}$, and $\delta_{l'}$. We illustrate that by the following example.

d_{1111} .	d_{1211} .	...	d_{mnpq} .	
c_{1111} .	c_{1211} .	...	c_{mnpq} .	
x_{1111} .	x_{1211} .	...	x_{mnpq} .	
1	1	...	0	α_1 .
:	:	...	:	:
0	0	...	1	α_m .
1	0	...	0	β_1 .
0	1	...	0	β_2 .
:	:	...	:	:
0	0	...	1	β_n .
1	1	...	0	γ_1 .
:	:	...	:	:
0	0	...	1	γ_p .
1	1	...	0	δ_1 .
:	:	...	:	:
0	0	...	1	δ_q .

Table 1. T_{4C} - Transportation table.

3. TRANSPORTATION PROBLEM SOLUTION

3.1. Feasibility conditions. We begin first to give a useful theorem that ensures the feasibility of T_{4C} problem (see [13]).

THEOREM 4. (Feasibility)
1. *A necessary condition for the feasibility of the problem T_{4C} i.e., it has a feasible solution if the condition* (8) *holds and the following conditions*

$$(9)\quad \begin{cases} \alpha_i \leq \sum\limits_{j=1}^{n}\sum\limits_{k=1}^{p}\sum\limits_{l=1}^{q} d_{ijkl} & \textit{for} \quad i = 1, ..., m, \\ \beta_j \leq \sum\limits_{i=1}^{m}\sum\limits_{k=1}^{p}\sum\limits_{l=1}^{q} d_{ijkl} & \textit{for} \quad j = 1, ..., n, \\ \gamma_k \leq \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\sum\limits_{l=1}^{q} d_{ijkl} & \textit{for} \quad k = 1, ..., p, \\ \delta_l \leq \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{p} d_{ijkl} & \textit{for} \quad l = 1, ..., q, \end{cases}$$

are satisfied.

2. *A sufficient condition for the feasibility of the problem T_{4C}, i.e., it has a feasible solution if the condition* (8) *holds and the following conditions*

$$\frac{\alpha_i \beta_j \gamma_k \delta_l}{H^3} \leq d_{ijkl}, \quad \textit{for all } (i,j,k,l) \tag{10}$$

are satisfied.

Proof. **1.** It is clear that if $x = (x_{ijkl})$ is a feasible solution for the problem T_{4C}, then conditions (8) and (9) hold.
2. Assume that $x = (x_{ijkl})$ is a vector of $\mathbb{R}^N$ such that

$$x_{ijkl} = \frac{\alpha_i \beta_j \gamma_k \delta_l}{d_{ijkl}}, \text{ for all } (i,j,k,l).$$

We can easily verify that x is a feasible solution for the problem T_{4C}. Note that if the set of feasible solutions of the problem T_{4C} is non empty, then it is a polytopes. Since the objective function is continuous, then the set of the solution of the problem T_{4C} is non empty, i.e., there exists at least an optimal solution of it. □

3.2. Optimality conditions. In this subsection, we give a theorem that ensures when a feasible solution, is an optimal solution of T_{4C}.

THEOREM 5. *Assume that the problem T_{4C} is feasible. Then a feasible solution x of T_{4C} is optimal if and only if there exists a vector*

$$y = (u_1, \ldots, u_m, v_1, \ldots, v_n, w_1, \ldots, w_p, t_1, \ldots, t_q)^T \in \mathbb{R}^M$$

such that:

$$\begin{cases} u_i + v_j + w_k + t_l \leq c_{ijkl} & \textit{if} \quad x_{ijkl} = 0, \\ u_i + v_j + w_k + t_l = c_{ijkl} & \textit{if} \quad 0 < x_{ijkl} < d_{ijkl}, \\ u_i + v_j + w_k + t_l \geq c_{ijkl} & \textit{if} \quad x_{ijkl} = d_{ijkl}. \end{cases}$$

Proof. For that, we consider the following formulation of the problem T_{4C}:

$$\min_x [c^T x : \; Ax = b, -x \geq -d, \, x \geq 0], \tag{11}$$

and its dual problem is given by

$$\max_{(y,\,z)} [b^T y - d^T z : A^T y - z \leq c, \, z \geq 0]. \tag{12}$$

Let (y, z) an optimal solution of the problem (12). Then a feasible solution x of the problem (11) is optimal if and only if the two following complementarity conditions are satisfied

$$(A^T y - z - c)_{ijkl} x_{ijkl} = 0, \tag{13}$$

and

$$(d - x)_{ijkl} z_{ijkl} = 0. \tag{14}$$

- If $x_{ijkl} = 0$, then $x_{ijkl} < d_{ijkl}$ and (14) imply $z_{ijkl} = 0$. So $(A^T y)_{ijkl} \le c_{ijkl}$.
- If $0 < x_{ijkl} < d_{ijkl}$ then (14) implies $z_{ijkl} = 0$ and (13) leads to $(A^T y - z - c)_{ijkl} = 0$. So $(A^T y)_{ijkl} = c_{ijkl}$.
- If $x_{ijkl} = d_{ijkl}$ then $x_{ijkl} > 0$ and (13) imply $(A^T y - z - c)_{ijkl} = 0$, consequently $(A^T y - c)_{ijkl} = z_{ijkl}$. Finally, we get $(A^T y)_{ijkl} \ge c_{ijkl}$.

□

4. DESCRIPTION OF TWO SIMPLICIAL ALGORITHMS FOR T_{4C}

4.1. The simplex algorithm. In this subsection, in order to apply the technical of the simplex algorithm for solving T_{4C} problem, we need first to put T_{4C} in the framework of a special linear program. The simplex algorithm, is referred to us as ALGORITHM 1. It is well-known that the problem T_{4C}, according to (7), can be easily reformulated as the following linear program:

$$\min_{\hat{x}} Z = \hat{c}^T \hat{x} \text{ s.t. } \hat{A}\hat{x} = \hat{b},\ \hat{x} \ge 0, \tag{15}$$

in adding to the constraints those of the form: $x_{ijkl} + y_r = d_{ijkl}$, (see (6) above), for $i = 1, \dots, m$, $j = 1, \dots, n$, $k = 1, \dots, p$, $l = 1, \dots, q$, $r = 1, ..., N$, where

- y_r is the gap variable,
- $\hat{x} = (x^T, y^T)^T \in \mathbb{R}^{2N}$ with $y = (y_1, \dots, y_N)^T$,
- $\hat{c} = (c^T, 0^T)^T \in \mathbb{R}^{2N}$ where 0 is a N-null vector,
- $\hat{b} = (b^T, d^T)^T \in \mathbb{R}^{M+N}$, with d is the vector of N components d_{ijkl}.
- $\hat{A} = \left[\begin{array}{c|c} A & 0 \\ \hline I_N & I_N \end{array}\right] \in \mathbb{R}^{M+N\times 2N}$ with $\operatorname{rank}(\hat{A}) = M + N - 3$.
- $I_N \in \mathbb{R}^{N\times N}$ is the identity matrix.

4.2. Algorithm 2. According to the particularities of T_{4C} problem, Zitouni [10], proposed a modification of the simplex algorithm. This algorithm is referred here as ALGORITHM 2, which shares with the simplex algorithm, a structure consists also of two phases such as the finite convergence and the use of the pivot principle. The advantage of this new algorithm is that it tackles the T_{4C} directly without passing by other reformulations. For more comprehension of this new algorithm, we detail its fundamental ingredients (steps) as follows.

Phase 1. (*It finds a basic feasible solution or declare that T_{4C} is not solvable*)

Step 1:

Initialization: for all (i, j, k, l), $\hat{\alpha}_i = \alpha_i$, $\hat{\beta}_j = \beta_j$, $\hat{\gamma}_k = \gamma_k$, $\hat{\delta}_l = \delta_l$ and $b_{ijkl} = 0$, (b_{ijkl} is a boolean variable equal to 1 if x_{ijkl} has already been determined and 0 if not yet),

$E = \{(i, j, k, l),$ such that $b_{ijkl} = 0\}$.

Iteration:
While E is non empty do

- Choose an 4-tuple $(\bar{i}, \bar{j}, \bar{k}, \bar{l}) \in E$, such that $c_{\bar{i}\bar{j}\bar{k}\bar{l}} = \min_{(i,j,k,l) \in E} c_{ijkl}$, (see Remark 6)
- Take $x_{\bar{i}\bar{j}\bar{k}\bar{l}} = \min(\hat{\alpha}_{\bar{i}}, \hat{\beta}_{\bar{j}}, \hat{\gamma}_{\bar{k}}, \hat{\delta}_{\bar{l}},\ d_{\bar{i}\bar{j}\bar{k}\bar{l}})$, and $b_{\bar{i}\bar{j}\bar{k}\bar{l}} = 1$, (i.e., $x_{\bar{i}\bar{j}\bar{k}\bar{l}}$ is determined),
- Update $\hat{\alpha}_{\bar{i}}, \hat{\beta}_{\bar{j}}, \hat{\gamma}_{\bar{k}}$, and $\hat{\delta}_{\bar{l}}$ by the procedure (**P1**) below.

Step 2:

a) Determine ξ as shown in the procedure (**P2**) below.
If $\xi = 0$, then $x = (x_{ijkl})$ is an **initial basic feasible solution** for the problem T_{4C}, we denote it by $x^{(0)}$. **Go to Phase 2.**

Else, construct the problem $T_{4C}(\tilde{M})$ (as shown in the procedure (**P3**) below) and determine for it an initial basic feasible solution $\overline{x}^{(0)}$ as in step 1 by taking $\bar{x}^{(0)}_{m+1,n+1,p+1,q+1} = 0$. Note that $\overline{x}^{(0)} = (\overline{x}^{(0)}_{ijkl})$, with $i = 1, \dots, m+1$, $j = 1, \dots, n+1$, $k = 1, \dots, p+1$ and $l = 1, \dots, q+1$. If $\overline{x}^{(0)}$ is **optimal** then the problem T_{4C} **is not solvable. Stop.**

b) *Improvement of a basic feasible solution for* $T_{4C}(\tilde{M})$.
Initialization: $r = 1$, $\xi > \mathbf{0}$ is known,
1) Determine $\overline{x}^{(r)}$ as in Phase 2.
2) If $\bar{x}^{(r)}_{m+1,n+1,p+1,q+1} = \xi$, then $x^{(r)} = (x^{(r)}_{ijkl})$ with $i = 1, \dots, m$, $j = 1, \dots, n$, $k = 1, \dots, p$, and $l = 1, \dots, q$, is an **initial basic feasible solution** for the problem (T_{4C}). **Go to Phase 2.**
3) If $\overline{x}^{(r)}$ is optimal (Phase 2), then **the problem** T_{4C} **is not solvable. Stop.** 4) Do $r = r + 1$ and repeat 1), to 3).

Next, we describe the second phase.

Phase 2. (*Research of an optimal solution for* T_{4C})

When Phase 2 starts, we know an initial basic feasible solution $x^{(0)}$. Take $r = 0$.

a) Determine the set $I^{(r)}$ of the interesting 4-tuple (i, j, k, l), (see Remark 7).

b) For all $(i, j, k, l) \in I^{(r)}$, solve the linear system

$$u_i^{(r)} + v_j^{(r)} + w_k^{(r)} + t_l^{(r)} = c_{ijkl}.$$

c) For all $(i, j, k, l) \notin I^{(r)}$ determine

$$\triangle^{(r)}_{ijkl} = c_{ijkl} - (u_i^{(r)} + v_j^{(r)} + w_k^{(r)} + t_l^{(r)}).$$

d) Apply the procedure described in (**P4)** below.

If the optimality condition holds then the **feasible solution** $x^{(r)}$ is **optimal. stop.**

Else determine a vector $P_{i_0j_0k_0l_0}$ entering at the base, it corresponds to $\triangle^{(r)}_{i_0j_0k_0l_0}$.

e) Construct a cycle $\mu^{(r)}$ via the procedure described in (**P5**) and determine a new feasible solution as the procedure (**P6**) shows.

f) Do $r = r + 1$ and repeat a), to e) until the optimality condition holds.

REMARK 6. If there are several elements corresponding to the minimum of c_{ijkl}, we choose one, for instance the first found in the transportation table by going from the left to the right. □

REMARK 7. If the feasible solution is degenerate, i.e., the number of columns of A_x is strictly less than rank(A), we complete A_x with additional columns so that we obtain a matrix having rank(A) linearly independent columns. Next $I^{(r)}$ can be determined. Thus will be done in the procedure (**P7**). □

Also, the algorithm makes appeal to the following procedures.

(P1) - Updating of $\hat{\alpha}_{\bar{i}}, \hat{\beta}_{\bar{j}}, \hat{\gamma}_{\bar{k}}$, and $\hat{\delta}_{\bar{l}}$.

1) $\hat{\alpha}_{\bar{i}} = \hat{\alpha}_{\bar{i}} - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$,

if $\hat{\alpha}_{\bar{i}} = 0$ then take $x_{\bar{i}jkl} = 0$ for all $(j,k,l) \neq (\bar{j},\bar{k},\bar{l})$ and $b_{\bar{i}jkl} = 1$ for all (j,k,l),

2) $\hat{\beta}_{\bar{j}} = \hat{\beta}_{\bar{j}} - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$,

if $\hat{\beta}_{\bar{j}} = 0$ then take $x_{i\bar{j}kl} = 0$ for all $(i,k,l) \neq (\bar{i},\bar{k},\bar{l})$ and $b_{i\bar{j}kl} = 1$ for all (i,k,l),

3) $\hat{\gamma}_{\bar{k}} = \hat{\gamma}_{\bar{k}} - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$,

if $\hat{\gamma}_{\bar{k}} = 0$ then take $x_{ij\bar{k}l} = 0$ for all $(i,j,l) \neq (\bar{i},\bar{j},\bar{l})$ and $b_{ij\bar{k}l} = 1$ for all (i,j,l),

4) $\hat{\delta}_{\bar{l}} = \hat{\delta}_{\bar{l}} - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$,

if $\hat{\delta}_{\bar{l}} = 0$ then take $x_{ijk\bar{l}} = 0$ for all $(i,j,k) \neq (\bar{i},\bar{j},\bar{k})$ and $b_{ijk\bar{l}} = 1$ for all (i,j,k).

(P2) - Determination of ξ.

$$\xi = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \sum_{k=1}^{p} e_k = \sum_{l=1}^{q} f_l, \quad \text{such that:}$$

$$a_i = \alpha_i - \sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} \quad \text{with} \quad i = 1,...,m,$$

$$b_j = \beta_j - \sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} \quad \text{with} \quad j = 1,...,n,$$

$$e_k = \gamma_k - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} \quad \text{with} \quad k = 1,...,p,$$

$$f_l = \delta_l - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} \quad \text{with} \quad l = 1,...,q.$$

Note that the numbers a_i, b_j, e_k and f_l are nonnegative.

(P3) - Construction of the problem $T_{4C}(\tilde{M})$.

The problem $T_{4C}(\tilde{M})$ is obtained from problem T_{4C} by adding four fictitious points with indices $m+1$, $n+1$, $p+1$, and $q+1$ such that: $c_{m+1,n+1,p+1,q+1} = 0$, and $c_{m+1,jkl} = c_{i,n+1,kl} = c_{ij,p+1,l} = c_{ijk,q+1} = \tilde{M}$, where $\tilde{M}$ is a very large number and there are no limitation on the capacities for the paths involving a fictitious point.

(P4) - Determination of a vector $P_{i_0j_0k_0l_0}$ entering at the base or confirming that the feasible solution $x^{(r)}$ is optimal.

Take $\Gamma_0^{(r)}$ and $\Gamma_d^{(r)}$ as two tables such that

$$\Gamma_0^{(r)} = \left\{ \triangle_{ijkl}^{(r)} \quad \text{such that} \quad x_{ijkl}^{(r)} = 0 \right\},$$
$$\Gamma_d^{(r)} = \left\{ \triangle_{ijkl}^{(r)} \quad \text{such that} \quad x_{ijkl}^{(r)} = d_{ijkl} \right\},$$

and elements $\triangle_{ijkl}^{(r)}$ are represented as variables x_{ijkl} in the transportation table.

By going from the left to the right in $\Gamma_0^{(r)}$, choose

$$\triangle_{i_0j_0k_0l_0}^{(r)} \text{ as the first element } \triangle_{ijkl}^{(r)} < 0 \text{ found,}$$

if all elements of $\Gamma_0^{(r)}$ are nonnegative then choose in $\Gamma_d^{(r)}$ similarly,

$$\triangle_{i_0j_0k_0l_0}^{(r)} \text{ as the first element } \triangle_{ijkl}^{(r)} > 0 \text{ found.}$$

If all elements of $\Gamma_d^{(r)}$ are non positive, then **the feasible solution $x^{(r)}$ is optimal. Stop.**

(P5) - Determination of a cycle.

A cycle $\mu^{(r)}$ containing some interesting 4-tuple (i, j, k, l) and the non interesting 4-tuple (i_0, j_0, k_0, l_0) corresponding to $\triangle_{i_0j_0k_0l_0}^{(r)}$ is determined by solving the linear system $\sum\limits_{(i,j,k,l)\in I^{(r)}} \alpha_{ijkl}^{(r)} P_{ijkl} = -P_{i_0j_0k_0l_0}$

The non null solutions $\alpha_{ijkl}^{(r)}$ are called coefficients of the cycle $\mu^{(r)}$.

(P6) - Determination of a new feasible solution.

Take

$$\sigma^{(r)} = \left\{ (i, j, k, l) \text{ such that } (i, j, k, l) \text{ is a case of the cycle } \mu^{(r)} \right\}$$
$$\sigma^{(r)-} = \left\{ (i, j, k, l) \text{ such that } (i, j, k, l) \in \sigma^{(r)}, \text{ with } \alpha_{ijkl}^{(r)} < 0 \right\}$$
$$\sigma^{(r)+} = \left\{ (i, j, k, l) \text{ such that } (i, j, k, l) \in \sigma^{(r)}, \text{ with } \alpha_{ijkl}^{(r)} > 0 \right\}$$

If $\triangle^{(r)}_{i_0j_0k_0l_0} \in \Gamma^{(r)}_0$, determine

$$\theta_1^{(r)} = \min_{(i,j,k,l)\in\sigma^{(r)-}} \left(x^{(r)}_{ijkl} \middle/ -\alpha^{(r)}_{ijkl} \right),$$

$$\theta_2^{(r)} = \min_{(i,j,k,l)\in\sigma^{(r)+}} \left((d_{ijkl} - x^{(r)}_{ijkl}) \middle/ \alpha^{(r)}_{ijkl} \right),$$

$$\theta^{(r)} = \min(\theta_1^{(r)}, \theta_2^{(r)}).$$

Next, take

$$x^{(r+1)} = \left\{ x^{(r)}_{ijkl},\ (i,j,k,l) \notin \sigma^{(r)} \right\} \cup \left\{ x^{(r)}_{ijkl} + \alpha^{(r)}_{ijkl}\theta^{(r)},\ (i,j,k,l) \in \sigma^{(r)} \right\}.$$

Else ($\triangle^{(r)}_{i_0j_0k_0l_0} \in \Gamma^{(r)}_d$)**,** determine

$$\theta_1^{(r)} = \min_{(i,j,k,l)\in\sigma^{(r)+}} \left(x^{(r)}_{ijkl} \middle/ \alpha^{(r)}_{ijkl} \right),$$

$$\theta_2^{(r)} = \min_{(i,j,k,l)\in\sigma^{(r)-}} \left((d_{ijkl} - x^{(r)}_{ijkl}) \middle/ -\alpha^{(r)}_{ijkl} \right),$$

$$\theta^{(r)} = \min(\theta_1^{(r)}, \theta_2^{(r)}).$$

Next, take

$$x^{(r+1)} = \left\{ x^{(r)}_{ijkl},\ (i,j,k,l) \notin \sigma^{(r)} \right\} \cup \left\{ x^{(r)}_{ijkl} - \alpha^{(r)}_{ijkl}\theta^{(r)},\ (i,j,k,l) \in \sigma^{(r)} \right\}.$$

(P7) - Determination of a set $I^{(r)}$ in a degenerate case.

1. Take
   - E_b the set of vectors corresponding to variables $x^{(r)}_{ijkl}$ verifying
     $$0 < x^{(r)}_{ijkl} < d_{ijkl}$$
     and N_b is its element number.
   - E_h the set of vectors corresponding to variables $x^{(r)}_{ijkl}$ verifying
     $$x^{(r)}_{ijkl} = 0 \text{ or } x^{(r)}_{ijkl} = d_{ijkl}$$
     and N_h is its element number. We are $N_h = N - N_b$, with $N = mnpq$.
   - E_s any subset with $s = \text{rank}(A) - N_b$ elements of E_h, for example the first s elements in E_h by going from the left to the right in the transportation table.
2. At the beginning of this procedure, we know a subset E_s of E_h.

   i) If the set$(E_s \cup E_b)$ is a free base, then a set $I^{(r)}$ is determined. **Stop**.

   ii) Replace the first (the second, the third,..., or the s^{th}) element in E_s by the $(s+1)^{th}$ element in E_h, or take any other subset E_s of E_h, and repeat **i)** until a set $I^{(r)}$ will be determined. **Stop**.

4.3. The convergence of Algorithm 2. Assume that the problem T_{4C} is non degenerate. Observe that if $x^{(r)}$ and $x^{(r+1)}$ are two successive feasible solutions of the problem T_{4C} with $Z^{(r)}$ and $Z^{(r+1)}$ are respectively the corresponding objective values then

$$Z^{(r+1)} = Z^{(r)} - (-1)^{\eta}\theta^{(r)}\triangle_{i_0j_0k_0l_0}^{(r+1)}$$

with

$$\eta = \begin{cases} 1 & \text{if} \quad x_{i_0j_0k_0l_0}^{(r)} = 0, \\ 0 & \text{if} \quad x_{i_0j_0k_0l_0}^{(r)} = d_{i_0j_0k_0l_0}. \end{cases}$$

So $Z^{(r+1)} < Z^{(r)}$.

Hence the Algorithm 2 guaranties that the same base never appear in two distinct iterations and since the number of the visited vertices is necessarily finite (at most C_N^{M-3}), then it converges finitely.

5. COMPUTATIONAL RESULTS

The implementation of the two algorithms is running on a Pentium IV with a Microsoft Windows Environment, they are totally written with Borland Delphi. Table 2 summarizes the results obtained for a few numerical experiments.

Ex. no.	Dimension of the problem	Number of iterations		Optimal value (Z^*)		Time in seconds	
		Alg. 1	Alg. 2	Alg. 1	Alg. 2	Alg. 1	Alg. 2
1	22×32	22	6	41.6025	41.6025	0.078	0.016
2	46×72	26	8	1317.79	1317.79	0.391	0.031
3	121×216	38	10	369.536	369.536	0.172	0.078
4	149×270	46	11	22.25	22.25	0.282	0.125
5	167×300	55	16	40	40	0.375	0.172
6	198×360	67	17	10.25	10.25	0.609	0.266
7	257×480	81	26	10.4375	10.4375	2.281	0.329
8	318×600	54	15	38.75	38.75	1.282	0.485
9	378×720	102	35	95.0625	95.0625	3.609	0.687
10	469×900	178	55	11.3125	11.3125	6.437	1.000
11	560×1080	257	83	46.25	46.25	10.438	1.406
12	741×1440	383	117	48.75	48.75	22.890	8.672

Table 2. Comparison between Algorithm 1 and Algorithm 2.

6. CONCLUSION AND FUTURE RESEARCHES

In this paper, we presented two exact simplicial methods for solving the capacited 4-index transportation problem. The numerical experiments showed that Algorithm 2 is more efficient than Algorithm 1 with respect to the number of iterations and also to the computational time. Moreover, as we have successfully done in some experiments, the arrangements made on Algorithm 2 have appropriately treated the degeneracy problems and led to a considerable reduction of the computational results.

Finally, we point out that our obtained results are independent of the number of indices of the problem, so we can extend Algorithm 2 for solving capacitated transportation problems with the number of indices that are greater than four.

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