JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 47 (2018) no. 1, pp. 26–41
ictp.acad.ro/jnaat

NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR HA-CONVEX FUNCTIONS

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Abstract. Some new inequalities of Hermite-Hadamard type for HA-convex functions defined on positive intervals are given.

MSC 2010. 26D15; 25D10.

Keywords. Convex functions, Integral inequalities, HA-Convex functions.

1. INTRODUCTION

Following [1] (see also [41]) we say that the function $f : I \subset R \setminus \{0\} \to \mathbb{R}$ is *HA-convex* or *harmonically convex* if

$$f\left(\tfrac{xy}{tx+(1-t)y}\right) \leq (1-t)\, f(x) + tf(y) \tag{1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1) is reversed, then f is said to be *HA-concave* or *harmonically concave*.

In order to avoid any confusion with the class of *AH-convex* functions, namely the functions satisfying the condition

$$f((1-t)\, x + ty) \leq \tfrac{f(x)f(y)}{(1-t)f(y)+tf(x)}, \tag{2}$$

we call the class of functions satisfying (1) as *HA-convex functions*.

If $I \subset (0, \infty)$ and f is convex and nondecreasing function then f is HA-convex and if f is HA-convex and nonincreasing function then f is convex.

The following simple but important fact is as follows:

Criterion 1. *If $[a, b] \subset I \subset (0, \infty)$ and if we consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$, defined by $g(t) = f\left(\frac{1}{t}\right)$, then f is HA-convex on $[a, b]$ if and only if g is convex in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$.*

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For a convex function $h:[c,d]\to\mathbb{R}$, the following inequality is well known in the literature as the Hermite-Hadamard inequality

$$h\left(\tfrac{c+d}{2}\right)\le\tfrac{1}{d-c}\int_c^d h(t)\,dt\le\tfrac{h(c)+h(d)}{2} \tag{3}$$

for any convex function $h:[c,d]\to\mathbb{R}$.

For related results, see [1]–[18], [21]–[26], [27]–[37] and [38]–[49].

If we write the Hermite-Hadamard inequality for the convex function $g(t)=f\left(\frac{1}{t}\right)$ on the closed interval $\left[\frac{1}{b},\frac{1}{a}\right]$, then we have

$$f\left(\tfrac{2ab}{a+b}\right)\le\tfrac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\tfrac{1}{t}\right)dt\le\tfrac{f(b)+f(a)}{2}. \tag{4}$$

Using the change of variable $s=\frac{1}{t}$, we have

$$\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\tfrac{1}{t}\right)dt=\int_a^b \tfrac{f(s)}{s^2}ds$$

and by (4) we get

$$f\left(\tfrac{2ab}{a+b}\right)\le\tfrac{ab}{b-a}\int_a^b \tfrac{f(s)}{s^2}ds\le\tfrac{f(b)+f(a)}{2}. \tag{5}$$

The inequality (5) has been obtained in a different manner in [41] by I. Işcan.

The *identric mean* $I(a,b)$ is defined by

$$I(a,b):=\tfrac{1}{e}\left(\tfrac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a,b):=\tfrac{b-a}{\ln b-\ln a}.$$

In the recent paper [25] we established the following inequalities for *HA*-convex functions:

THEOREM 2. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then*

$$f(L(a,b))\le\tfrac{1}{b-a}\int_a^b f(x)\,dx\le\tfrac{(L(a,b)-a)bf(b)+(b-L(a,b))af(a)}{(b-a)L(a,b)}, \tag{6}$$

and

THEOREM 3. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then*

$$f\left(\tfrac{a+b}{2}\right)\tfrac{a+b}{2}\le\tfrac{1}{b-a}\int_a^b xf(x)\,dx\le\tfrac{bf(b)+af(a)}{2}. \tag{7}$$

Motivated by the above results, we establish in this paper some new inequalities of Hermite-Hadamard type for *HA*-convex functions. Some applications for special means are also given.

2. FURTHER RESULTS

We start with the following characterization of *HA*-convex functions.

THEOREM 4. *Let* $f, h : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be so that* $h(t) = tf(t)$ *for* $t \in [a,b]$. *Then* f *is HA-convex on the interval* $[a,b]$ *if and only if* h *is convex on* $[a,b]$.

Proof. Assume that f is *HA*-convex on the interval $[a,b]$. Then the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$, $g(t) = f\left(\frac{1}{t}\right)$ is convex on $\left[\frac{1}{b}, \frac{1}{a}\right]$. By replacing t with $\frac{1}{t}$ we have $f(t) = g\left(\frac{1}{t}\right)$.

If $\lambda \in [0,1]$ and $x, y \in [a,b]$ then, by the convexity of g on $\left[\frac{1}{b}, \frac{1}{a}\right]$, we have

$$\begin{aligned} h((1-\lambda)x + \lambda y) &= [(1-\lambda)x + \lambda y] f((1-\lambda)x + \lambda y) \\ &= [(1-\lambda)x + \lambda y] g\left(\frac{1}{(1-\lambda)x+\lambda y}\right) \\ &= [(1-\lambda)x + \lambda y] g\left(\frac{(1-\lambda)x\frac{1}{x}+\lambda y\frac{1}{y}}{(1-\lambda)x+\lambda y}\right) \\ &\le [(1-\lambda)x + \lambda y] \frac{(1-\lambda)xg\left(\frac{1}{x}\right)+\lambda yg\left(\frac{1}{y}\right)}{(1-\lambda)x+\lambda y} \\ &= (1-\lambda)xg\left(\frac{1}{x}\right) + \lambda yg\left(\frac{1}{y}\right) \\ &= (1-\lambda)xf(x) + \lambda yf(y) = (1-\lambda)h(x) + \lambda h(y), \end{aligned}$$

which shows that h is convex on $[a,b]$.

We have $f(t) = \frac{h(t)}{t}$ for $t \in [a,b]$. If $\lambda \in [0,1]$ and $x, y \in [a,b]$ then, by the convexity of h on $[a,b]$, we have

$$\begin{aligned} f\left(\frac{xy}{\lambda x+(1-\lambda)y}\right) &= \frac{h\left(\frac{xy}{\lambda x+(1-\lambda)y}\right)}{\frac{xy}{\lambda x+(1-\lambda)y}} \\ &= \frac{\lambda x+(1-\lambda)y}{xy} h\left(\frac{xy}{\lambda x+(1-\lambda)y}\right) \\ &= \frac{\lambda x+(1-\lambda)y}{xy} h\left(\frac{1}{(1-\lambda)\frac{1}{x}+\lambda\frac{1}{y}}\right) \\ &= \frac{\lambda x+(1-\lambda)y}{xy} h\left(\frac{(1-\lambda)\frac{1}{x}x+\lambda\frac{1}{y}y}{(1-\lambda)\frac{1}{x}+\lambda\frac{1}{y}}\right) \\ &\le \frac{\lambda x+(1-\lambda)y}{xy} \frac{(1-\lambda)\frac{1}{x}h(x)+\lambda\frac{1}{y}h(y)}{(1-\lambda)\frac{1}{x}+\lambda\frac{1}{y}} \\ &= (1-\lambda)\frac{1}{x}h(x) + \lambda\frac{1}{y}h(y) = (1-\lambda)f(x) + \lambda f(y), \end{aligned}$$

which shows that f is *HA*-convex on the interval $[a,b]$. □

REMARK 5. If f is HA-convex on the interval $[a,b]$, then by Theorem 4 the function $h(t) = tf(t)$ is convex on $[a,b]$ and by Hermite-Hadamard inequality (3) we get the inequality (7). This gives a direct proof of (7) and it is simpler than in [25]. □

In 1994, [11] (see also [32, p. 22]) we proved the following refinement of Hermite-Hadamard inequality. For a direct proof that is different from the one in [11], see the recent paper [24].

LEMMA 6. *Let $p:[c,d]\to\mathbb{R}$ be a convex function on $[c,d]$. Then for any division $c=y_0<y_1<...<y_{n-1}<y_n=d$ with $n\geq 1$ we have the inequalities*

$$
\begin{aligned}
p\left(\tfrac{c+d}{2}\right) &\leq \tfrac{1}{d-c}\sum_{i=0}^{n-1}\left(y_{i+1}-y_i\right)p\left(\tfrac{y_{i+1}+y_i}{2}\right) \\
&\leq \tfrac{1}{d-c}\int_c^d p\left(y\right)dy \leq \tfrac{1}{d-c}\sum_{i=0}^{n-1}\left(y_{i+1}-y_i\right)\tfrac{p(y_i)+p(y_{i+1})}{2} \\
&\leq \tfrac{1}{2}\left[p\left(c\right)+p\left(d\right)\right].
\end{aligned} \tag{8}
$$

We can state the following result:

THEOREM 7. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then for any division $a=x_0<x_1<...<x_{n-1}<x_n=b$ with $n\geq 1$ we have the inequalities*

$$
\begin{aligned}
\tfrac{a+b}{2}f\left(\tfrac{a+b}{2}\right) &\leq \tfrac{1}{2(b-a)}\sum_{i=0}^{n-1}\left(x_{i+1}^2-x_i^2\right)f\left(\tfrac{x_{i+1}+x_i}{2}\right) \\
&\leq \tfrac{1}{b-a}\int_a^b xf\left(x\right)dx \\
&\leq \tfrac{1}{b-a}\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)\tfrac{x_if(x_i)+x_{i+1}f(x_{i+1})}{2} \\
&\leq \tfrac{1}{2}\left[af\left(a\right)+bf\left(b\right)\right].
\end{aligned} \tag{9}
$$

Follows by Lemma 6 for the convex function $p\left(x\right)=xf\left(x\right)$, $x\in[a,b]$.

If we take $n=2$ and $x\in[a,b]$, then by (9) we have

$$
\begin{aligned}
\tfrac{a+b}{2}f\left(\tfrac{a+b}{2}\right) &\leq \tfrac{1}{2(b-a)}\left[(x^2-a^2)f\left(\tfrac{x+a}{2}\right)+(b^2-x^2)f\left(\tfrac{x+b}{2}\right)\right] \\
&\leq \tfrac{1}{b-a}\int_a^b tf\left(t\right)dt \\
&\leq \tfrac{1}{2(b-a)}\left[\left(b-a\right)xf\left(x\right)+\left(x-a\right)af\left(a\right)+\left(b-x\right)bf\left(b\right)\right] \\
&\leq \tfrac{1}{2}\left[af\left(a\right)+bf\left(b\right)\right].
\end{aligned} \tag{10}
$$

If in this inequality we choose $x=\frac{a+b}{2}$, then we get the inequality

$$
\begin{aligned}
\tfrac{a+b}{2}f\left(\tfrac{a+b}{2}\right) &\leq \tfrac{1}{2(b-a)}\left[\tfrac{b+3a}{4}f\left(\tfrac{b+3a}{4}\right)+\tfrac{a+3b}{4}f\left(\tfrac{a+3b}{4}\right)\right] \\
&\leq \tfrac{1}{b-a}\int_a^b tf\left(t\right)dt \\
&\leq \tfrac{1}{2}\left[\tfrac{a+b}{2}f\left(\tfrac{a+b}{2}\right)+\frac{af(a)+bf(b)}{2}\right]\leq\tfrac{1}{2}\left[af\left(a\right)+bf\left(b\right)\right].
\end{aligned} \tag{11}
$$

If we take in (10) $x=\frac{2ab}{a+b}$, then we get

$$\frac{a+b}{2}f\left(\frac{a+b}{2}\right)\leq\frac{1}{4(a+b)^{2}}\left[a^{2}\left(a+3b\right)f\left(\frac{a(a+3b)}{2(a+b)}\right)+b^{2}\left(3a+b\right)f\left(\frac{b(3a+b)}{2(a+b)}\right)\right] \tag{12}$$
$$\leq\frac{1}{b-a}\int_{a}^{b}tf\left(t\right)dt$$
$$\leq\frac{1}{a+b}\left[abf\left(\frac{2ab}{a+b}\right)+\frac{a^{2}f(a)+b^{2}f(b)}{2}\right]\leq\frac{1}{2}\left[af\left(a\right)+bf\left(b\right)\right].$$

We also have:

THEOREM 8. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then for any division $a=x_0<x_1<...<x_{n-1}<x_n=b$ with $n\geq 1$ we have the inequalities*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\sum_{j=0}^{n-1}\left(\frac{x_{j+1}-x_j}{x_{j+1}x_j}\right)f\left(\frac{2x_{j+1}x_j}{x_{j+1}+x_j}\right) \tag{13}$$
$$\leq\frac{ab}{b-a}\int_{a}^{b}\frac{f(x)}{x^{2}}dx$$
$$\leq\frac{ab}{b-a}\sum_{i=0}^{n-1}\left(\frac{x_{j+1}-x_j}{x_{j+1}x_j}\right)\frac{f(x_j)+f(x_{j+1})}{2}\leq\frac{f(b)+f(a)}{2}.$$

Proof. Consider the convex function $p\left(x\right)=f\left(\frac{1}{x}\right)$ that is convex on the interval $\left[\frac{1}{b},\frac{1}{a}\right]$. The division $a=x_0<x_1<...<x_{n-1}<x_n=b$ with $n\geq 1$ produces the division $y_i=\frac{1}{x_{n-i}}$, $i\in\{0,...,n\}$ of the interval $\left[\frac{1}{b},\frac{1}{a}\right]$.

Using the inequality (8) we get

$$f\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right)\leq\frac{1}{\frac{1}{a}-\frac{1}{b}}\sum_{i=0}^{n-1}\left(t\frac{1}{x_{n-i-1}}-\frac{1}{x_{n-i}}\right)f\left(\frac{1}{\frac{\frac{1}{x_{n-i-1}}+\frac{1}{x_{n-i}}}{2}}\right) \tag{14}$$
$$\leq\frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}}f\left(\frac{1}{t}\right)dt$$
$$\leq\frac{1}{\frac{1}{a}-\frac{1}{b}}\sum_{i=0}^{n-1}\left(\frac{1}{x_{n-i-1}}-\frac{1}{x_{n-i}}\right)\frac{f\left(\frac{1}{\frac{1}{x_{n-i-1}}}\right)+f\left(\frac{1}{\frac{1}{x_{n-i}}}\right)}{2}$$
$$\leq\frac{1}{2}\left[f\left(\frac{1}{\frac{1}{b}}\right)+f\left(\frac{1}{\frac{1}{a}}\right)\right]$$

that is equivalent to

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\sum_{i=0}^{n-1}\left(\frac{x_{n-i}-x_{n-i-1}}{x_{n-i-1}x_{n-i}}\right)f\left(\frac{2x_{n-i-1}x_{n-i}}{x_{n-i}+x_{n-i-1}}\right) \tag{15}$$
$$\leq\frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}}f\left(\frac{1}{t}\right)dt\leq$$

$$\leq \frac{ab}{b-a}\sum_{i=0}^{n-1}\left(\frac{x_{n-i}-x_{n-i-1}}{x_{n-i-1}x_{n-i}}\right)\frac{f(x_{n-i-1})+f(x_{n-i})}{2}$$
$$\leq \frac{1}{2}\left[f(b)+f(a)\right].$$

By re-indexing the sums and taking into account that

$$\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\tfrac{1}{t}\right)dt = \int_a^b \frac{f(x)}{x^2}dx$$

we obtain the desired result (13). □

REMARK 9. If we take $n = 2$ and $x \in [a, b]$, then by (13) we have, after appropriate calculations, that

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{1}{x}\left[\frac{(x-a)bf\left(\frac{2ax}{a+x}\right)+(b-x)af\left(\frac{2xb}{x+b}\right)}{b-a}\right] \tag{16}$$
$$\leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$
$$\leq \frac{1}{2}\left[f(x)+\frac{(x-a)bf(a)+(b-x)af(b)}{x(b-a)}\right]$$
$$\leq \frac{f(b)+f(a)}{2}.$$

If we take in (16) $x = \frac{2ab}{a+b} \in [a, b]$, then we get

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{1}{2}\left[f\left(\frac{4ab}{a+3b}\right)+f\left(\frac{4ab}{3a+b}\right)\right] \tag{17}$$
$$\leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$
$$\leq \frac{1}{2}\left[f\left(\frac{2ab}{a+b}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}.$$

If we take in (16) $x = \frac{a+b}{2} \in [a, b]$, then we get

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{bf\left(\frac{a(a+b)}{3a+b}\right)+af\left(\frac{b(a+b)}{a+3b}\right)}{a+b} \tag{18}$$
$$\leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$
$$\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{bf(a)+af(b)}{a+b}\right] \leq \frac{f(b)+f(a)}{2}.$$

□

3. RELATED RESULTS

We recall some facts on the lateral derivatives of a convex function.

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi(\mathring{I}) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x - a)\varphi(a) \text{ for any } x, a \in I. \tag{19}$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function. If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

Lemma 10. *Let $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a, b]$. Then f has lateral derivatives in every point of (a, b) and*

$$f(t) - f(s) \geq s f'_\pm(s)\left(1 - \tfrac{s}{t}\right) \tag{20}$$

for any $s \in (a, b)$ and $t \in [a, b]$.

Also, we have

$$f(t) - f(a) \geq a f'_+(a)\left(1 - \tfrac{a}{t}\right) \tag{21}$$

and

$$f(t) - f(b) \geq b f'_-(b)\left(1 - \tfrac{b}{t}\right) \tag{22}$$

for any $t \in [a, b]$ provided the lateral derivatives $f'_+(a)$ and $f'_-(b)$ are finite.

Proof. If f is *HA*-convex function on the interval $[a, b]$, then the function $h(t) = t f(t)$ is convex on $[a, b]$, therefore the function f has lateral derivatives in each point of (a, b) and

$$h'_\pm(t) = f(t) + t f'_\pm(t)$$

for any $t \in (a, b)$. Also, if $f'_+(a)$ and $f'_-(b)$ are finite then

$$h'_+(a) = f(a) + a f'_+(a) \text{ and } h'_-(b) = f(b) + b f'_-(b).$$

Writing the gradient inequality for the convex function h, namely

$$h(t) - h(s) \geq h'_\pm(s)(t - s)$$

for any $s \in (a, b)$ and $t \in [a, b]$, we have

$$t f(t) - s f(s) \geq \left[f(s) + s f'_\pm(s)\right](t - s) = f(s)(t - s) + s f'_\pm(s)(t - s)$$

that is equivalent to

$$t f(t) - t f(s) \geq s f'_\pm(s)(t - s)$$

for any $s \in (a, b)$ and $t \in [a, b]$.

Now, by dividing with $t > 0$ we get the desired result (20).

The rest follows by the corresponding properties of convex function h. □

We use the following results obtained by the author in [19] and [20]

Lemma 11. *Let $h:[\alpha,\beta]\to\mathbb{R}$ be a convex function on $[\alpha,\beta]$. Then we have the inequalities*

$$\frac{1}{8}\left[h'_{+}\left(\frac{\alpha+\beta}{2}\right)-h'_{-}\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha)\le\frac{h(\alpha)+h(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}h(t)\,dt \le\frac{1}{8}\left[h'_{-}(\beta)-h'_{+}(\alpha)\right](\beta-\alpha) \tag{23}$$

and

$$\frac{1}{8}\left[h'_{+}\left(\frac{\alpha+\beta}{2}\right)-h'_{-}\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha)\le\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}h(t)\,dt-h\left(\frac{\alpha+\beta}{2}\right) \le\frac{1}{8}\left[h'_{-}(\beta)-h'_{+}(\alpha)\right](\beta-\alpha). \tag{24}$$

The constant $\frac{1}{8}$ is best possible in (23) *and* (24).

The following result holds:

Theorem 12. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then we have*

$$\begin{aligned}&\frac{1}{16}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b^2-a^2)\le\\ &\le\frac{af(a)+bf(b)}{2}-\frac{1}{b-a}\int_a^b tf(t)\,dt\\ &\le\frac{1}{8}\left[f(b)-f(a)\right](b-a)+\frac{1}{8}\left[bf'_{-}(b)-af'_{+}(a)\right](b-a)\end{aligned} \tag{25}$$

and

$$\begin{aligned}&\frac{1}{16}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right]\left(b^2-a^2\right)\le\\ &\le\frac{1}{b-a}\int_a^b tf(t)\,dt-\frac{a+b}{2}f\left(\frac{a+b}{2}\right)\\ &\le\frac{1}{8}\left[f(b)-f(a)\right](b-a)+\frac{1}{8}\left[bf'_{-}(b)-af'_{+}(a)\right](b-a).\end{aligned} \tag{26}$$

Proof. Making use of inequality (23) in Lemma 11 for the convex function $h(t)=tf(t)$ we have

$$\begin{aligned}&\frac{1}{8}\left[\frac{a+b}{2}f'_{+}\left(\frac{a+b}{2}\right)-\frac{a+b}{2}f'_{-}\left(\frac{a+b}{2}\right)\right](b-a)\le\\ &\le\frac{af(a)+bf(b)}{2}-\frac{1}{b-a}\int_a^b tf(t)\,dt\\ &\le\frac{1}{8}\left[f(b)+bf'_{-}(b)-f(a)-af'_{+}(a)\right](b-a),\end{aligned}$$

which proves the inequality (25).

The inequality (26) follows by (24). □

Corollary 13. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a differentiable HA-convex function on the interval $[a,b]$. Then we have*

$$\begin{aligned}0&\le\frac{af(a)+bf(b)}{2}-\frac{1}{b-a}\int_a^b tf(t)\,dt\\ &\le\frac{1}{8}\left[f(b)-f(a)\right](b-a)+\frac{1}{8}\left[bf'_{-}(b)-af'_{+}(a)\right](b-a)\end{aligned} \tag{27}$$

and

$$0 \le \frac{1}{b-a}\int_a^b tf(t)\,dt - \frac{a+b}{2} f\left(\frac{a+b}{2}\right) \le \frac{1}{8}[f(b)-f(a)](b-a) + \frac{1}{8}[bf'_-(b) - af'_+(a)](b-a). \tag{28}$$

We remark that from (27) we have

$$\frac{(3a+b)f(a)+(a+3b)f(b)}{8} - \frac{1}{8}[bf'_-(b) - af'_+(a)](b-a) \le \le \frac{1}{b-a}\int_a^b tf(t)\,dt \le \frac{af(a)+bf(b)}{2} \tag{29}$$

and from (28) we have

$$\frac{a+b}{2} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b tf(t)\,dt \le \frac{a+b}{2} f\left(\frac{a+b}{2}\right) + \frac{1}{8}[f(b)-f(a)](b-a) + \frac{1}{8}[bf'_-(b) - af'_+(a)](b-a). \tag{30}$$

The *identric mean* $I(a,b)$ is defined by

$$I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

The following result also holds:

THEOREM 14. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a,b]$.*

(i) If $bf(b) - af(a) \ne \int_a^b f(s)\,ds$ and

$$\alpha_f := \frac{\int_a^b s^2 f'(s)ds}{\int_a^b sf'(s)ds} = \frac{b^2 f(b) - a^2 f(a) - 2\int_a^b sf(s)ds}{bf(b) - af(a) - \int_a^b f(s)ds} \in [a,b] \tag{31}$$

then

$$f(\alpha_f) \ge \frac{1}{b-a}\int_a^b f(s)\,ds. \tag{32}$$

(ii) If $f(b) \ne f(a)$ and

$$\beta_f = \frac{\int_a^b sf'(s)ds}{\int_a^b f'(s)ds} = \frac{bf(b) - af(a) - \int_a^b f(s)ds}{f(b)-f(a)} \in [a,b] \tag{33}$$

then

$$f(\beta_f) \ge \frac{1}{\ln b - \ln a}\int_a^b f(s)\,ds. \tag{34}$$

(iii) If $af(b) \ne bf(a)$ and

$$\gamma_f := \frac{(f(b)-f(a))ab}{af(b)-bf(a)} \in [a,b] \tag{35}$$

then

$$f(\gamma_f) \geq \tfrac{2ab}{b-a} \int_a^b \tfrac{f(s)}{s^2} ds. \tag{36}$$

Proof. We know that if $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ is an HA-convex function on the interval $[a, b]$ then the functions is differentiable except for at most countably many points. Then, from (20) we have

$$f(t) - f(s) \geq sf'(s)\left(1 - \tfrac{s}{t}\right) \tag{37}$$

for any $t \in [a, b]$ and almost every $s \in (a, b)$.

(i) If we take the Lebesgue integral mean in (37), then we get

$$f(t) - \tfrac{1}{b-a} \int_a^b f(s)\, ds \geq \tfrac{1}{b-a} \int_a^b sf'(s)\, ds - \tfrac{1}{t} \tfrac{1}{b-a} \int_a^b s^2 f'(s)\, ds \tag{38}$$

for any $t \in [a, b]$.

If we take $t = \alpha_f$ in (38) then we get the desired inequality (32).

(ii) If we divide the inequality (37) by s then we get

$$\tfrac{1}{s} f(t) - \tfrac{f(s)}{s} \geq f'(s) - \tfrac{1}{t} sf'(s) \tag{39}$$

for any $t \in [a, b]$ and almost every $s \in (a, b)$.

If we take the Lebesgue integral mean in (39), then we get

$$f(t) \tfrac{1}{b-a} \int_a^b \tfrac{1}{s} ds - \tfrac{1}{b-a} \int_a^b \tfrac{f(s)}{s} ds \tfrac{1}{b-a} \int_a^b f'(s)\, ds - \tfrac{1}{t} \tfrac{1}{b-a} \int_a^b sf'(s)\, ds$$

that is equivalent to

$$\tfrac{f(t)}{L(a,b)} - \tfrac{1}{b-a} \int_a^b \tfrac{f(s)}{s} ds \geq \tfrac{f(b)-f(a)}{b-a} - \tfrac{1}{t} \tfrac{bf(b)-af(a)-\int_a^b f(s)ds}{b-a} \tag{40}$$

for any $t \in [a, b]$

If we take $t = \beta_f$ in (40) then we get the desired result (34).

(iii) If we divide the inequality (37) by s^2 then we get

$$\tfrac{1}{s^2} f(t) - \tfrac{f(s)}{s^2} \geq \tfrac{f'(s)}{s} - \tfrac{1}{t} f'(s) \tag{41}$$

for any $t \in [a, b]$ and almost every $s \in (a, b)$.

If we take the Lebesgue integral mean in (41), then we get

$$f(t) \tfrac{1}{b-a} \int_a^b \tfrac{1}{s^2} ds - \tfrac{1}{b-a} \int_a^b \tfrac{f(s)}{s^2} ds \geq \tfrac{1}{b-a} \int_a^b \tfrac{f'(s)}{s} ds - \tfrac{1}{t} \tfrac{1}{b-a} \int_a^b f'(s)\, ds,$$

which is equivalent to

$$f(t) \tfrac{1}{ab} - \tfrac{1}{b-a} \int_a^b \tfrac{f(s)}{s^2} ds \geq \tfrac{1}{b-a} \left[\tfrac{f(b)}{b} - \tfrac{f(a)}{a} + \int_a^b \tfrac{f(s)}{s^2} ds \right] - \tfrac{1}{t} \tfrac{f(b)-f(a)}{b-a}$$

or, to

$$f(t) \tfrac{1}{ab} - \tfrac{2}{b-a} \int_a^b \tfrac{f(s)}{s^2} ds \geq \tfrac{1}{b-a} \tfrac{af(b)-bf(a)}{ba} - \tfrac{1}{t} \tfrac{f(b)-f(a)}{b-a}.$$

□

Remark 15. We observe that a sufficient condition for (31) and (33) to hold is that f is increasing on $[a,b]$. If $f(a)<0<f(b)$, then the inequality (35) also holds.

We also have the following result:

Theorem 16. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then we have*

$$f\left(\tfrac{a+b}{2}\right)\le\frac{1}{\ln b-\ln a}\int_a^b\frac{f(t)}{a+b-t}dt\le\frac{af(a)+bf(b)}{a+b}. \tag{42}$$

Proof. Since the function $h(t)=tf(t)$ is convex, then we have

$$\frac{x+y}{2}f\left(\frac{x+y}{2}\right)\le\frac{xf(x)+yf(y)}{2}$$

for any $x,y\in[a,b]$.

If we divide this inequality by $xy>0$ we get

$$\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right)f\left(\frac{x+y}{2}\right)\le\frac{1}{2}\left(\frac{f(x)}{y}+\frac{f(y)}{x}\right), \tag{43}$$

for any $x,y\in[a,b]$.

If we replace x by $(1-t)a+tb$ and y by $ta+(1-t)b$ in (43), then we get

$$\frac{1}{2}\left(\frac{1}{(1-t)a+tb}+\frac{1}{ta+(1-t)b}\right)f\left(\frac{a+b}{2}\right)\le\frac{1}{2}\left(\frac{f((1-t)a+tb)}{ta+(1-t)b}+\frac{f(ta+(1-t)b)}{(1-t)a+tb}\right), \tag{44}$$

for any $t\in[0,1]$.

Integrating (44) on $[0,1]$ over t we get

$$\begin{aligned}&\frac{1}{2}\left(\int_0^1\frac{1}{(1-t)a+tb}dt+\int_0^1\frac{1}{ta+(1-t)b}dt\right)f\left(\frac{a+b}{2}\right)\le\\&\le\frac{1}{2}\left(\int_0^1\frac{f((1-t)a+tb)}{ta+(1-t)b}dt+\int_0^1\frac{f(ta+(1-t)b)}{(1-t)a+tb}dt\right).\end{aligned} \tag{45}$$

Observe that, by the appropriate change of variable,

$$\int_0^1\frac{1}{(1-t)a+tb}dt=\int_0^1\frac{1}{ta+(1-t)b}dt=\frac{1}{b-a}\int_a^b\frac{du}{u}=\frac{\ln b-\ln a}{b-a}$$

and

$$\int_0^1\frac{f((1-t)a+tb)}{ta+(1-t)b}dt=\int_0^1\frac{f(ta+(1-t)b)}{(1-t)a+tb}=\frac{1}{b-a}\int_a^b\frac{f(u)}{a+b-u}du$$

and by (45) we get the first inequality in (42).

From the convexity of h we also have

$$((1-t)a+tb)f((1-t)a+tb)\le(1-t)af(a)+tbf(b)$$

and

$$(ta+(1-t)b)f(ta+(1-t)b)\le taf(a)+(1-t)bf(b)$$

for any $t\in[0,1]$.

Add these inequalities to get

$$((1-t)a+tb)f((1-t)a+tb)+(ta+(1-t)b)f(ta+(1-t)b)\le$$
$$\le af(a)+bf(b)$$

for any $t\in[0,1]$.

If we divide this inequality by $((1-t)a+tb)(ta+(1-t)b)$, then we get

$$\frac{f((1-t)a+tb)}{ta+(1-t)b}+\frac{f(ta+(1-t)b)}{(1-t)a+tb}\le\frac{af(a)+bf(b)}{((1-t)a+tb)(ta+(1-t)b)} \tag{46}$$

for any $t\in[0,1]$.

If we integrate the inequality (46) over t on $[0,1]$, then we obtain

$$\int_0^1\frac{f((1-t)a+tb)}{ta+(1-t)b}dt+\int_0^1\frac{f(ta+(1-t)b)}{(1-t)a+tb}dt\le \tag{47}$$
$$\le[af(a)+bf(b)]\int_0^1\frac{dt}{((1-t)a+tb)(ta+(1-t)b)}.$$

Since

$$\int_0^1\frac{dt}{((1-t)a+tb)(ta+(1-t)b)}=\frac{1}{b-a}\int_a^b\frac{du}{u(a+b-u)}$$

and

$$\frac{1}{u(a+b-u)}=\frac{1}{a+b}\left(\frac{1}{u}+\frac{1}{a+b-u}\right),$$

then

$$\int_a^b\frac{du}{u(a+b-u)}=\frac{1}{a+b}\int_a^b\left(\frac{1}{u}+\frac{1}{a+b-u}\right)du=\frac{2}{a+b}(\ln b-\ln a).$$

By (47) we then have

$$\frac{2}{b-a}\int_a^b\frac{f(u)}{a+b-u}du\le2\left[\frac{af(a)+bf(b)}{a+b}\right]\frac{\ln b-\ln a}{b-a},$$

which proves the second inequality in (42). □

4. APPLICATIONS

We consider the *arithmetic mean* $A(a,b)=\frac{a+b}{2}$, the *geometric mean* $G(a,b)=\sqrt{ab}$ and *harmonic mean* $H(a,b)=\frac{2ab}{a+b}$ for the positive numbers $a,b>0$.

If we use the inequalities (13) for the HA-convex function $f(t)=t$ on the interval $[a,b]\subset(0,\infty)$ then for any division $a=x_0<x_1<...<x_{n-1}<x_n=b$ with $n\ge1$ we have the inequalities

$$\frac{2ab}{a+b}\le\frac{2ab}{b-a}\sum_{j=0}^{n-1}\frac{x_{j+1}-x_j}{x_{j+1}+x_j}\le\frac{G^2(a,b)}{L(a,b)}\le\frac{ab}{2(b-a)}\sum_{i=0}^{n-1}\frac{x_{j+1}^2-x_j^2}{x_{j+1}x_j}\le A(a,b). \tag{48}$$

In particular, we have

$$H(a,b)\le2ab\left(\frac{1}{a+3b}+\frac{1}{3a+b}\right)\le\frac{G^2(a,b)}{L(a,b)}\le\frac{H(a,b)+A(a,b)}{2}\,(\le A(a,b)). \tag{49}$$

Consider the function $f:(0,\infty)\to\mathbb{R}$, $f(t)=\frac{\ln t}{t}$. Observe that $g(t)=f\left(\frac{1}{t}\right)=-t\ln t$, which shows that f is HA-concave on $(0,\infty)$.

If we write the inequality (11) for the HA-concave function $f(t)=\frac{\ln t}{t}$ on $(0,\infty)$, then we have for any division $a=x_0<x_1<...<x_{n-1}<x_n=b$ with $n\geq 1$ that

(50)
$$A(a,b)\geq \prod_{i=0}^{n-1}\left(\frac{x_{i+1}+x_i}{2}\right)^{\frac{x_{i+1}-x_i}{b-a}}\geq I(a,b)\geq \prod_{i=0}^{n-1}\left(x_ix_{i+1}\right)^{\frac{x_{i+1}-x_i}{2(b-a)}}\geq G(a,b).$$

In particular, we have

(51)
$$A(a,b)\geq \left(\frac{b+3a}{4}\right)^{\frac{1}{2(b-a)}}\left(\frac{a+3b}{4}\right)^{\frac{1}{2(b-a)}}$$
$$\geq I(a,b)\geq \sqrt{A(a,b)G(a,b)}\,(\geq G(a,b)).$$

The interested reader may apply the above inequalities for other HA-convex functions such as $f(t)=\frac{h(t)}{t}$, $t>0$ with h any convex function on an interval $I\subset(0,\infty)$ etc. The details are omitted.

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Received by the editors: April 11, 2017. Accepted: September 11, 2017. Published online: August 6, 2018.