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FOURIER SERIES APPROXIMATION FOR THE CAUCHY SINGULAR INTEGRAL EQUATION

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Abstract. Using the Fourier series as a projection in the Galerkin method, we approach the solution of the Cauchy singular integral equation. This study is carried in L^2. Numerical examples are developped to show the effectiveness of this method.

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1. INTRODUCTION

Numerical solution methods of integral equations play a very important role in various scientific fields. With the advantage of numerical calculation machines, including computers, these methods have now become an essential tool for investigation in various fundamental problems of our assimilation of scientific phenomena that are difficult, i.e. impossible to solve in the past.

Integral equations with singular kernel represent a great numerical challenge. In addition, if the approximation is performed in a space with weak property as L^2, it accentuates the difficulty.

In this article, we are interested in the following problem: given $f \in L^2(0,1)$, find a function $u \in L^2(0,1)$ such that

$$u = Cu + f, \tag{1}$$

where, C is the Cauchy operator defined by:

$$\text{For all } s \in \left]0,1\right[,\ Cu(s) = \lim_{\varepsilon \to 0} \left(\int_0^{s-\varepsilon} \tfrac{u(t)}{s-t} dt + \int_{s+\varepsilon}^1 \tfrac{u(t)}{s-t} dt \right).$$

C is an integral operator with singular kernel. But, in [1] the authors show that C is a bounded operator on $L^2(0,1)$ to itself with $\|C\| \leq \pi$. In addition, they show that it is a skew-hermitian operator, *i.e.*, $C^* = -C$. Which gives that $(I-C)^{-1}$ exists and that $\| (I-C)^{-1} \| \leq 1$ [1].

Then, for all given $f \in L^2(0,1)$, the solution u of (1) exists and is unique.

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Our goal is to construct an approximate solution for (1), using Galerkin method. This method is well studied and very known [2, 3, 4], but for compact operator only. Recently, this method is developed for the Cauchy operator using the piecewise constant projection[5, 6]. Our aim is to use Fourier series as projection to make the Galerkin method more efficient.

2. NUMERICAL STUDY

Let $\{\pi_n\}_{n\geq 0}$ be the set of operators defined on $L^2(0,1)$ to itself by: For all $x \in L^2(0,1)$, $n \geq 0$:

$$\pi_n x(t) = \sum_{k=0}^{n} a_k(x) \cos(k\pi t), \ t \in [0,1],$$

where,

$$\begin{aligned} a_0(x) &= \int_0^1 x(t)\, dt, \\ a_k(x) &= 2\int_0^1 x(t) \cos(k\pi t)\, dt, \quad k \geq 1. \end{aligned}$$

It is clear that for all $n \geq 0$, π_n is a projection and $\|\pi_n\| \leq 1$. π_n is the Fourier series, truncated at n, of x extended over $\mathbb{R}$ in the following sense:

$$\begin{aligned} x(t) &:= x(2-t), \ t \in [1,2], \\ x \quad &\text{is} \quad \text{2-periodic over } \mathbb{R}. \end{aligned}$$

We choose this approach to weaken the effect of the Gibbs phenomenon [7], [8].

Proposition 1. *For all $n \geq 0$, π_n is a selfadjoint operator and*

$$\lim_{n\to\infty} \|(I - \pi_n)\, x\|_{L^2(0,1)} = 0.$$

Proof. For all $n \geq 0$, for all $x, y \in L^2(0,1)$, we have,

$$\begin{aligned} \langle \pi_n x, y\rangle = &\int_0^1 \pi_n x(t) y(t) dt \sum_{k=0}^{n} a_k(x) \int_0^1 \cos(k\pi t) y(t) dt \\ =& a_0(x)a_0(y) + \tfrac{1}{2}\sum_{k=0}^{n} a_k(x)a_k(y) = \sum_{k=0}^{n}\left(\int_0^1 \cos(k\pi t)x(t)dt\right) a_k(y) \\ =& \int_0^1 x(t)\pi_n y(t) dt = \langle x, \pi_n y\rangle. \end{aligned}$$

For the pointwise convergence of π_n to the identity I, notice that using the previous extension of x, we obtain [7], [8]

$$\|(I-\pi_n)\, x\|_{L^2(0,1)} = \tfrac{1}{2}\,\|(I-\pi_n)\, x\|_{L^2(-1,1)} \to_{n\to\infty} 0.$$

□

Galerkin's method is to approximate C by $C_n = \pi_n C \pi_n$, for $n \geq 0$.

Theorem 2. *For all* $n \geq 1$, $C_n^* = -C_n$, $I - C_n$ *is invertible and*

$$\| (I - C_n)^{-1} \| \leq 1.$$

Proof. We have,

$$\begin{aligned} C_n^* &= (\pi_n C \pi_n)^* = \pi_n^* C^* \pi_n^* \\ &= \pi_n (-C) \pi_n = -C_n. \end{aligned}$$

In the same way as in [1], we define the operators $A_n = iC_n$ for all $n \geq 0$. Then, A_n is selfadjoint and its spectrum is real. Therefore,

$$\mathrm{sp}(C_n) \subset \{ i\alpha \; : \; \alpha \in \mathbb{R} \}, \; n \geq 0.$$

This means that $(I - C_n)$ is invertible. But, for all $x \in L^2(0,1)$,

$$\mathrm{Re}\left(\langle (I - C_n)x, x \rangle \right) = \frac{1}{2} \left(\langle (I - C_n)x, x \rangle + \overline{\langle (I - C_n)x, x \rangle} \right) = \langle x, x \rangle.$$

Hence,

$$\|x\|_{L^2(0,1)}^2 \leq |\langle (I - C_n)x, x \rangle| \leq \|(I - C_n)x\|_{L^2(0,1)} \|x\|_{L^2(0,1)},$$

and $\| (I - C_n)^{-1} \| \leq 1$. □

The last theorem ensures, for all $n \geq 0$ and all $f \in L^2(0,1)$, the existence and the unicity of u_n the solution of the following approximate equation:

$$u_n = C_n u_n + f.$$

However, the following theorem ensures the convergence of u_n to u.

Theorem 3. *For all* $n \geq 0$,

$$\|u - u_n\|_{L^2(0,1)} \leq \pi \|(I - \pi_n) u\|_{L^2(0,1)} + \|(I - \pi_n) Cu\|_{L^2(0,1)}.$$

Proof. We have,

$$\begin{aligned} u - u_n &= Cu - C_n u_n \\ &= Cu - C_n u + C_n (u - u_n), \\ (I - C_n)(u - u_n) &= Cu - C_n u. \end{aligned}$$

Then,

$$\begin{aligned} \|u - u_n\|_{L^2(0,1)} &\leq \| (I - C_n)^{-1} \| \, \|(Cu - C_n u)\|_{L^2(0,1)} \\ &\leq \|(Cu - C_n u)\|_{L^2(0,1)}. \end{aligned}$$

But,

$$\begin{aligned} \|Cu - C_n u\|_{L^2(0,1)} &= \|Cu - \pi_n C \pi_n u\|_{L^2(0,1)} \\ &= \|Cu - \pi_n Cu + \pi_n Cu - \pi_n C \pi_n u\|_{L^2(0,1)} \\ &= \|(I - \pi_n) Cu + \pi_n C (I - \pi_n) u\|_{L^2(0,1)} \\ &\leq \|(I - \pi_n) Cu\|_{L^2(0,1)} + \|\pi_n C (I - \pi_n) u\|_{L^2(0,1)}. \end{aligned}$$

We use $\|\pi_n C\| \leq \pi$ to conclude. □

3. NUMERICAL STRUCTURE

We have for: For all $n \geq 0$,

$$\begin{aligned} u_n &= \pi_n C \pi_n u_n + f, \\ &= \pi_n C \left(\sum_{k=0}^{n} a_k (u_n) \cos (k\pi \cdot) \right) + f, \\ &= \sum_{k=1}^{n} a_k (u_n) \pi_n \varphi_k + f \end{aligned} \tag{2}$$

where

$$\varphi_k = C (\cos (k\pi \cdot)) , \ k \geq 0.$$

We have

$$\varphi_0(s) = \lim_{\varepsilon \to 0} \left(\int_0^{s-\varepsilon} \tfrac{1}{s-t} dt + \int_{s+\varepsilon}^{1} \tfrac{1}{s-t} dt \right) = \ln \left(\tfrac{s}{1-s} \right).$$

For $k \geq 1$,

$$\begin{aligned} \varphi_k(s) &= \lim_{\varepsilon \to 0} \left(\int_0^{s-\varepsilon} \tfrac{\cos(k\pi t)}{s-t} dt + \int_{s+\varepsilon}^{1} \tfrac{\cos(k\pi t)}{s-t} dt \right) \\ &= \lim_{\varepsilon \to 0} \Big(\sin (k\pi s) \left[\mathrm{Si} (k\pi (-\varepsilon)) - \mathrm{Si} (k\pi (\varepsilon)) \right] \\ &\quad + \sin (k\pi s) \left[\mathrm{Si} (k\pi (1-s)) - \mathrm{Si} (k\pi (-\varepsilon)) \right] \\ &\quad - \cos (k\pi s) \left[\mathrm{Ci} (k\pi (-\varepsilon)) - \mathrm{Ci} (k\pi (\varepsilon)) \right] \\ &\quad - \cos (k\pi s) \left[\mathrm{Ci} (k\pi (1-s)) - \mathrm{Ci} (k\pi (-\varepsilon)) \right] \Big), \end{aligned}$$

where, Ci is the cosine integral function and Si is the sine integral function. Using the fact that,

$$\begin{aligned} \mathrm{Ci} (-x) &= \mathrm{Ci} (x) + i\pi, \\ \mathrm{Si} (-x) &= -\mathrm{Si} (x) , \ \mathrm{Si} (0) = 0, \end{aligned}$$

we get,

$$\begin{aligned} \varphi_k(s) &= \sin (k\pi s) \left[\mathrm{Si} (k\pi (1-s)) + \mathrm{Si} (k\pi s) \right] \\ &\quad - \cos (k\pi s) \left[\mathrm{Ci} (k\pi (1-s)) + \mathrm{Ci} (k\pi s) \right] , \ k \geq 1 \end{aligned}$$

Multiplying (2) by $\cos(i\pi s), \ 0 \leq i \leq n$ and integrating over $(0, 1)$, we get the following system:

$$\begin{pmatrix} a_0(u_n) \\ \vdots \\ a_n(u_n) \end{pmatrix} = M \begin{pmatrix} a_0(u_n) \\ \vdots \\ a_n(u_n) \end{pmatrix} + \begin{pmatrix} a_0(f) \\ \vdots \\ a_n(f) \end{pmatrix},$$

where, $(a_i(u_n))_{0\leq i\leq n}$ and $(a_i(f))_{0\leq i\leq n}$ are respectively the Fourier series coefficients of u_n and f and,

$$M\left(i,k\right)=\left(2-\delta\left(i\right)\right)\int_0^1\varphi_k\left(s\right)\cos\left(i\pi s\right)ds,$$

where, $\delta\left(0\right)=1$ and $\delta\left(k\right)=0$ if $k\neq0$. Once, the previous system solved, we construct u_n using the following formula:

$$\text{For all, } s\in[0,1]\,,\ u_n(s)=\sum_{k=0}^{n}a_k\left(u_n\right)\cos\left(k\pi s\right).$$

The coefficients $(a_i(f))_{1\leq i\leq n}$ and those of the matrices M are approximated using the numerical integration method, mid-point, with the subdivision: $M\geq 2,\ h=\frac{1}{M},\ t_{p+\frac{1}{2}}=\left(p+\frac{1}{2}\right)h$, for $0\leq p\leq M-1$.

4. NUMERICAL EXAMPLE

To show the effectiveness of our approximation method, we will compare it with the one developed in [5], [6]. The latter consists in using the piecewise constant projection denoted $\{\tilde{\pi}_n\}_{n\geq2}$. In what follows, $u_{n,f}$ denotes the approximate solution calculated by our method and $u_{n,pc}$ denotes the one calculated by the method developed in [5], [6]. The theoretical error bounds obtained for the two methods are denoted by:

$$\begin{aligned}TEBF_n &:= \pi\left\|\left(I-\pi_n\right)u\right\|_{L^2(0,1)}+\left\|\left(I-\pi_n\right)Cu\right\|_{L^2(0,1)},\\ TEBPC_n &:= \pi\left\|\left(I-\tilde{\pi}_n\right)u\right\|_{L^2(0,1)}+\left\|\left(I-\tilde{\pi}_n\right)Cu\right\|_{L^2(0,1)},\end{aligned}$$

where, $\|\cdot\|_{L^2(0,1)}$ is approximated using the mid-point method described below with $M=1000$.

EXAMPLE 4. If we take

$$f\left(s\right)=s+1+s\ln\left(\tfrac{1-s}{s}\right),$$

then the equation (1) admits the following unique solution:

$$u\left(s\right)=s.$$

n	M	$\|u-u_{n,f}\|_{L^2(0,1)}$	$TEBF_n$	$\|u-u_{n,pc}\|_{L^2(0,1)}$	$TEBPC_n$
10	200	$7.41\cdot10^{-2}$	$2.30\cdot10^{-1}$	$2.06\cdot10^{-1}$	$4.55\cdot10^{-1}$
50	300	$2.85\cdot10^{-2}$	$8.89\cdot10^{-2}$	$7.49\cdot10^{-2}$	$1.65\cdot10^{-1}$
100	500	$1.79\cdot10^{-2}$	$5.60\cdot10^{-2}$	$4.82\cdot10^{-2}$	$1.06\cdot10^{-1}$
200	1000	$1.03\cdot10^{-2}$	$3.21\cdot10^{-2}$	$2.99\cdot10^{-2}$	$6.62\cdot10^{-2}$
300	1000	$6.83\cdot10^{-3}$	$2.13\cdot10^{-2}$	$2.16\cdot10^{-2}$	$4.79\cdot10^{-2}$
400	1000	$4.71\cdot10^{-3}$	$1.46\cdot10^{-2}$	$1.67\cdot10^{-2}$	$3.71\cdot10^{-2}$

Table 1. Fourier projection vs. piecewise constant projection for Ex. 4.

Example 5. If we take

$$f(s) = s^{-\frac{1}{4}} - s^{-\frac{1}{4}} \left(\ln \left(\frac{1+s^{\frac{1}{4}}}{1-s^{\frac{1}{4}}} \right) - 2\tan^{-1}\left(s^{-\frac{1}{4}} \right) \right),$$

then the equation (1) admits the following unique solution:

$$u(s) = s^{-\frac{1}{4}}.$$

n	M	$\|u - u_{n,f}\|_{L^2(0,1)}$	$TEBF_n$	$\|u - u_{n,pc}\|_{L^2(0,1)}$	$TEBPC_n$
10	200	$3.89 \cdot 10^{-1}$	$1.21 \cdot 10^{+0}$	$7.41 \cdot 10^{-1}$	$1.64 \cdot 10^{+0}$
50	300	$2.09 \cdot 10^{-1}$	$6.52 \cdot 10^{-1}$	$4.23 \cdot 10^{-1}$	$9.36 \cdot 10^{-1}$
100	500	$1.47 \cdot 10^{-1}$	$4.59 \cdot 10^{-1}$	$3.17 \cdot 10^{-1}$	$7.01 \cdot 10^{-1}$
200	1000	$9.33 \cdot 10^{-2}$	$2.90 \cdot 10^{-1}$	$2.23 \cdot 10^{-1}$	$4.94 \cdot 10^{-1}$
300	1000	$6.48 \cdot 10^{-2}$	$2.01 \cdot 10^{-1}$	$1.70 \cdot 10^{-1}$	$3.78 \cdot 10^{-1}$
400	1000	$4.60 \cdot 10^{-2}$	$1.43 \cdot 10^{-1}$	$1.52 \cdot 10^{-1}$	$3.33 \cdot 10^{-1}$

Table 2. Fourier projection vs. piecewise constant projection for Ex. 5.

5. CONCLUSION

We have constructed a Galerkin-type approximation method for the Cauchy singular integral equation, using the Fourier series as projection in $L^2(0,1)$.

Based on the fact that the Cauchy operator is skew-hermitian and that the Fourier series is a selfadjoint projection, we show the convergence of our method.

The numerical examples developed show that, regarding the approximation error, our method based on Fourier series is more efficient than the procedure that uses piecewise constant functions.

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