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APPROXIMATION OF CONTINUOUS FUNCTIONS ON HEXAGONAL DOMAINS

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Abstract. Some approximation properties of hexagonal Fourier series are investigated. The order of approximation by Nörlund means of hexagonal Fourier series is estimated in terms of modulus of continuity.

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1. INTRODUCTION

Let $C_{2\pi}$ be the Banach space of 2π-periodic continuous functions on the real line, equipped with the norm

$$\|f\|_{C_{2\pi}} := \sup_{0\leq x\leq 2\pi} |f(x)| .$$

The modulus of continuity of a function $f \in C_{2\pi}$ is defined by

$$\omega(f,\delta) := \sup_{0<|h|\leq\delta} \|f - T_h(f)\|_{C_{2\pi}} , \quad (\delta > 0) ,$$

where $T_h(f)(x) := f(x+h)$. For $0 < \alpha \leq 1$, we denote by $H^{\alpha}_{2\pi}$ the Hölder class of functions $f \in C_{2\pi}$ such that $\omega(f,\delta) \ll \delta^{\alpha}$, where $A \ll B$ means that there exists a constant $K > 0$ such that $A \leq KB$ holds.

Approximation of functions belonging the space $C_{2\pi}$ by trigonometric polynomials is one of the most important topics in approximation theory and it has a very rich history. Especially, the order of approximation of functions in $H^{\alpha}_{2\pi}$ classes was studied by several mathematicians. Linear summation methods of Fourier series are mostly used tools in these studies.

Let $f \in C_{2\pi}$ has the Fourier series

$$f(x) \sim \sum_{k=-\infty}^{\infty} \widehat{f}_k e^{ikx}, \tag{1}$$

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with partial sums

$$S_n(f)(x) := \sum_{k=-n}^{n} \widehat{f}_k e^{ikx}, \quad (n = 0, 1, \ldots).$$

We denote by $(\sigma_n(f))$ the sequence of Fejér means of (1), *i.e.*,

$$\sigma_n(f)(x) = \tfrac{1}{n+1}\sum_{k=0}^{n} S_k(f)(x).$$

In 1912, S.N. Bernstein obtained the following estimate for the approximation order by Fejér means.

Theorem A. [2]. *Let $f \in H^{\alpha}_{2\pi}$ $(0 < \alpha \leq 1)$. Then the estimate*

$$\|f - \sigma_n(f)\|_{C_{2\pi}} \ll \begin{cases} \frac{1}{n^{\alpha}}, & \alpha < 1 \\ \frac{\log n}{n}, & \alpha = 1 \end{cases} \tag{2}$$

holds for $n \geq 2$.

S.B. Stechkin extended Bernstein's result as follows.

Theorem B. [14]. *Let $f \in C_{2\pi}$. Then the estimate*

$$\|f - \sigma_n(f)\|_{C_{2\pi}} \ll \tfrac{1}{n+1}\sum_{k=0}^{n} \omega(f, \tfrac{1}{k+1}) \tag{3}$$

holds for every natural number n.

Let $p = (p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers and let $P_n = \sum\limits_{k=0}^{n} p_k$. Nörlund means of the series (1) with respect to the sequence p are defined by

$$N_n(p; f)(x) = \tfrac{1}{P_n}\sum_{k=0}^{n} p_{n-k} S_k(f)(x).$$

It is known that Nörlund summability method is regular if and only if $p_n/P_n \to 0$ as $n \to \infty$ [8, p. 64]. It is clear that $N_n(p; f)$ coincides with $\sigma_n(f)$ in the special case $p_n = 1$ $(n = 0, 1, \ldots)$.

In 1976, A.S.B. Holland, B. Sahney and J. Tzimbalario obtained a more general result than Theorem B.

Theorem C. [9]. *Let $p = (p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers such that $np_n \ll P_n$. Then for every $f \in C_{2\pi}$, the inequality*

$$\|f - N_n(p; f)\|_{C_{2\pi}} \ll \tfrac{1}{P_n}\sum_{k=1}^{n} \tfrac{1}{k} P_k \omega(f, \tfrac{1}{k}) \tag{4}$$

holds.

It is clear that in the case $p_n = 1$ $(n = 1, 2, \ldots)$ (4) reduces to (3). Theorem C also extends a result of B. Sahney and D.S. Goel [13] which states that

$$\|f - N_n(p; f)\|_{C_{2\pi}} \ll \tfrac{1}{P_n}\sum_{k=1}^{n} \tfrac{P_k}{k^{1+\alpha}} \tag{5}$$

for $f \in H_{2\pi}^{\alpha}$, where (p_n) is a non-increasing sequence of positive real numbers.

These theorems can be found in the survey [10]. Also, we refer to the monographs [1], [3], [4], [16] and [18] for more information and results about trigonometric approximation theory.

Approximation problems on cubes of the $d-$dimensional Euclidean space $\mathbb{R}^d$ are studied by assuming that the functions are 2π-periodic in each of their variables (see, for example [16, Sections 5.3 and 6.3] and [18, vol. II, ch. XVII]). But, in the case of non-tensor product domain, for example for hexagonal domains in the Euclidean plane $\mathbb{R}^2$, another definition of periodicity is needed. For such domains the most useful periodicity is the periodicity with respect to the lattices.

Let A be a non-singular $d \times d$ matrix. The discrete subgroup $A\mathbb{Z}^d = \{Ak : k \in \mathbb{Z}^d\}$ of the Euclidean space $\mathbb{R}^d$ is called the lattice generated by A, and the matrix A is called the generator matrix of this lattice. The lattice $A^{-tr}\mathbb{Z}^d$, where A^{-tr} is the transpose of the inverse matrix A^{-1}, is called the dual lattice of $A\mathbb{Z}^d$. A bounded set $\Omega \subset \mathbb{R}^d$ is said to tile $\mathbb{R}^d$ with the lattice $A\mathbb{Z}^d$ if

$$\sum_{\alpha \in A\mathbb{Z}^d} \chi_\Omega (x + \alpha) = 1$$

holds almost everwhere, that is, for almost every $x \in \mathbb{R}^d$ there exists exactly one $\alpha \in A\mathbb{Z}^d$ such that $x + \alpha \in \Omega$. In this case the set Ω is called a spectral set for the lattice $A\mathbb{Z}^d$. One suppose that the spectral set Ω contains 0 as an interior point and tiles $\mathbb{R}^d$ with the lattice $A\mathbb{Z}^d$ without overlapping and without gap, *i.e.*,

$$\sum_{k \in \mathbb{Z}^d} \chi_\Omega (x + Ak) = 1$$

for all $x \in \mathbb{R}^d$ and $\Omega + Ak$ and $\Omega + Aj$ are disjoint if $k \neq j$. For example we can take $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ for the standard lattice $\mathbb{Z}^d$ (the lattice generated by the identity matrix).

Let Ω be the spectral set of the lattice $A\mathbb{Z}^d$. $L^2(\Omega)$ becomes a Hilbert space with respect to the inner product

$$\langle f, g \rangle_\Omega := \frac{1}{|\Omega|} \int_\Omega f(x) \overline{g(x)} dx,$$

where $|\Omega|$ is the $d-$dimensional Lebesgue measure of Ω. A theorem of Fuglede states that the set $\left\{e^{2\pi i \langle \alpha, x \rangle} : \alpha \in A^{-tr}\mathbb{Z}^d\right\}$ is an orthonormal basis of the Hilbert space $L^2(\Omega)$, where $\langle \alpha, x \rangle$ is the usual Euclidean inner product of α and x [5]. According to this theorem, Fourier series and approximation on the spectral set of the lattice $A\mathbb{Z}^d$ can be studied by using the exponentials $e^{2\pi i \langle \alpha, x \rangle}$ $\left(\alpha \in A^{-tr}\mathbb{Z}^d\right)$.

A function f is said to be periodic with respect to the lattice $A\mathbb{Z}^d$ if

$$f(x + Ak) = f(x)$$

for all $k \in \mathbb{Z}^d$.

If we consider the standard lattice $\mathbb{Z}^d$ and its spectral set $\left[-\frac{1}{2}, \frac{1}{2}\right)^d$, Fourier series with respect to this lattice coincide with usual multiple Fourier series of functions of $d-$variables.

We refer to [11] for more detailed information about Fourier analysis on lattices.

2. HEXAGONAL FOURIER SERIES

In the Euclidean plane $\mathbb{R}^2$, besides the standard lattice $\mathbb{Z}^2$ and the rectangular domain $\left[-\frac{1}{2}, \frac{1}{2}\right)^2$, the simplest lattice is the hexagon lattice and the simplest spectral set is the regular hexagon.

The generator matrix and the spectral set of the hexagonal lattice $H\mathbb{Z}^2$ are given by

$$H = \begin{bmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{bmatrix}$$

and

$$\Omega_H = \left\{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_2, \tfrac{\sqrt{3}}{2}x_1 \pm \tfrac{1}{2}x_2 < 1\right\}.$$

It is more convenient to use the homogeneous coordinates (t_1, t_2, t_3) that satisfy $t_1 + t_2 + t_3 = 0$. If we define

$$t_1 := -\tfrac{x_2}{2} + \tfrac{\sqrt{3}x_1}{2},\ t_2 := x_2,\ t_3 := -\tfrac{x_2}{2} - \tfrac{\sqrt{3}x_1}{2}, \tag{6}$$

the hexagon Ω_H becomes

$$\Omega = \left\{(t_1, t_2, t_3) \in \mathbb{R}^3 : -1 \leq t_1, t_2, -t_3 < 1,\ t_1 + t_2 + t_3 = 0\right\}.$$

We use bold letters $\mathbf{t}$ for homogeneous coordinates and we denote by $\mathbb{R}^3_H$ the plane $t_1 + t_2 + t_3 = 0$, that is

$$\mathbb{R}^3_H = \left\{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3 : t_1 + t_2 + t_3 = 0\right\}.$$

Also we use the notation $\mathbb{Z}^3_H$ for the set of points in $\mathbb{R}^3_H$ with integer components, that is $\mathbb{Z}^3_H = \mathbb{Z}^3 \cap \mathbb{R}^3_H$.

It follows from (6) that the Jacobian determinant of the change of variables $x = (x_1, x_2) \to \mathbf{t} = (t_1, t_2, t_3)$ is $dx_1 dx_2 = \frac{2\sqrt{3}}{3} dt_1 dt_2$.

In the homogeneous coordinates, the inner product on $L^2(\Omega)$ becomes

$$\langle f, g\rangle_H = \frac{1}{|\Omega|}\int_\Omega f(\mathbf{t})\,\overline{g(\mathbf{t})}d\mathbf{t},$$

where $|\Omega|$ denotes the area of Ω, and the orthonormal basis of $L^2(\Omega)$ becomes

$$\left\{\phi_{\mathbf{j}}(\mathbf{t}) = e^{\frac{2\pi i}{3}\langle \mathbf{j}, \mathbf{t}\rangle} : \mathbf{j} \in \mathbb{Z}^3_H,\ \mathbf{t} \in \mathbb{R}^3_H\right\}.$$

Also, a function f is periodic with respect to the hexagonal lattice (or H-periodic) if and only if $f(\mathbf{t}) = f(\mathbf{t}+\mathbf{s})$ whenever $\mathbf{s} \equiv \mathbf{0} \pmod 3$, where $\mathbf{t} \equiv \mathbf{s} \pmod 3$ defined as

$$t_1 - s_1 \equiv t_2 - s_2 \equiv t_3 - s_3 \pmod 3 .$$

It is clear that the functions $\phi_{\mathbf{j}}(\mathbf{t})$ are $H-$periodic. If the function f is $H-$periodic then

$$\int_{\Omega} f(\mathbf{t}+\mathbf{s})\, d\mathbf{t} = \int_{\Omega} f(\mathbf{t})\, d\mathbf{t}, \quad \left(\mathbf{s} \in \mathbb{R}_H^3\right).$$

For every natural number n, we define a subset of $\mathbb{Z}_H^3$ by

$$\mathbb{H}_n := \left\{\mathbf{j} = (j_1, j_2, j_3) \in \mathbb{Z}_H^3 : -n \leq j_1, j_2, j_3 \leq n\right\}.$$

Note that, $\mathbb{H}_n$ consists of all points with integer components inside the hexagon $n\overline{\Omega}$. Members of the set

$$\mathcal{H}_n := \operatorname{span}\left\{\phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{H}_n\right\}, \quad (n \in \mathbb{N})$$

are called hexagonal trigonometric polynomials. It is clear that the dimension of $\mathcal{H}_n$ is $\#\mathbb{H}_n = 3n^2 + 3n + 1$.

The hexagonal Fourier series of an $H-$periodic function $f \in L^1(\Omega)$ is

$$f(\mathbf{t}) \sim \sum_{\mathbf{j} \in \mathbb{Z}_H^3} \widehat{f}_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}), \tag{7}$$

where

$$\widehat{f}_{\mathbf{j}} = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t})\, e^{-\frac{2\pi i}{3}\langle \mathbf{j}, \mathbf{t}\rangle} d\mathbf{t}, \quad \left(\mathbf{j} \in \mathbb{Z}_H^3\right).$$

The nth partial sum of the series (7) is defined by

$$S_n(f)(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}_n} \widehat{f}_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}), \quad (n \in \mathbb{N}).$$

The partial sums have the integral representation

$$S_n(f)(\mathbf{t}) = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t}-\mathbf{s})\, D_n(\mathbf{s})\, d\mathbf{s}, \tag{8}$$

where

$$D_n(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}_n} \phi_{\mathbf{j}}(\mathbf{t})$$

is the Dirichlet kernel of order n.

It is known that ([15], [11]) the Dirichlet kernel can be expressed as

$$D_n(\mathbf{t}) = \Theta_n(\mathbf{t}) - \Theta_{n-1}(\mathbf{t}), \quad (n \in \mathbb{N}), \tag{9}$$

where

$$\Theta_n(\mathbf{t}) := \frac{\sin\frac{(n+1)(t_1-t_2)\pi}{3} \sin\frac{(n+1)(t_2-t_3)\pi}{3} \sin\frac{(n+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3} \sin\frac{(t_2-t_3)\pi}{3} \sin\frac{(t_3-t_1)\pi}{3}} \tag{10}$$

for $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3_H$.

More general information about hexagonal Fourier series can be found in [11] and [17].

3. MAIN RESULT

We denote by $C_H(\overline{\Omega})$ the set of complex valued $H-$periodic continuous functions defined on $\mathbb{R}^3_H$. $C_H(\overline{\Omega})$ becomes a Banach space with respect to the uniform norm

$$\|f\|_{C_H(\overline{\Omega})} = \sup\left\{|f(\mathbf{t})| : \mathbf{t} \in \overline{\Omega}\right\}.$$

The modulus of continuity of the function $f \in C_H(\overline{\Omega})$ is defined by

$$\omega_H(f, \delta) := \sup_{0<\|\mathbf{h}\|\leq\delta} \|f - T_{\mathbf{h}}(f)\|_{C_H(\overline{\Omega})},$$

where $T_{\mathbf{h}}(f)(\mathbf{t}) = f(\mathbf{t} + \mathbf{h})$ and

$$\|\mathbf{h}\| := \max\{|h_1|, |h_2|, |h_3|\}$$

for $\mathbf{h} = (h_1, h_2, h_3) \in \mathbb{R}^3_H$. It is known that [17] the modulus of continuity is a non-decreasing function and satisfies

$$\omega_H(f, \lambda\delta) \leq (1 + \lambda)\,\omega_H(f, \delta) \tag{11}$$

for $\lambda > 0$.

For $0 < \alpha \leq 1$, we define the Hölder class $H^\alpha(\overline{\Omega})$ of $H-$periodic continuous functions as

$$H^\alpha(\overline{\Omega}) := \left\{f \in C_H(\overline{\Omega}) : \omega_H(f, \delta) \ll \delta^\alpha,\ \delta > 0\right\}.$$

The Fejér means of the series (7) are defined by

$$\sigma_n(f)(\mathbf{t}) = \tfrac{1}{n+1}\sum_{k=0}^{n} S_k(f)(\mathbf{t}).$$

The following analogue of Theorem A for hexagonal Fourier series was proved in [6].

Theorem D. *Let* $f \in H^\alpha\left(\overline{\Omega}\right)$ $(0 < \alpha \leq 1)$. *Then the estimate*

$$\|f - \sigma_n(f)\|_{C_H(\overline{\Omega})} \ll \begin{cases} \frac{1}{n^\alpha}, & \alpha < 1 \\ \frac{(\log n)^2}{n}, & \alpha = 1 \end{cases} \tag{12}$$

holds for $n \geq 2$.

Let $p = (p_n)_{n=0}^\infty$ be a sequence of positive real numbers and $(N_n(p; f))$ be the sequence of Nörlund means of the series (7) with respect to the sequence p, that is

$$N_n(p; f)(\mathbf{t}) = \tfrac{1}{P_n}\sum_{k=0}^{n} p_{n-k} S_k(f)(\mathbf{t}),\ \ (n \in \mathbb{N}). \tag{13}$$

By considering (8), we get

$$N_n(p;f)(\mathbf{t}) = \tfrac{1}{|\Omega|}\int\limits_{\Omega} f(\mathbf{t}-\mathbf{s})\,F_n(p;\mathbf{s})\,d\mathbf{s}, \tag{14}$$

where

$$F_n(p;\mathbf{t}) := \tfrac{1}{P_n}\sum_{k=0}^{n} p_{n-k}D_k(\mathbf{t}).$$

The aim of this work is to prove an analogue of Theorem C for hexagonal Fourier series. The main result is the following.

THEOREM 1. *Let $p=(p_n)$ be a non-increasing sequence of positive real numbers. Then the estimate*

$$\|f - N_n(p;f)\|_{C_H(\overline{\Omega})} \ll \tfrac{1}{P_n}\log\left(\tfrac{P_n}{p_n}\right)\sum_{k=1}^{n}\tfrac{1}{k}P_k\omega_H(f,\tfrac{1}{k}) \tag{15}$$

holds for every $f\in C_H(\overline{\Omega})$ and for every natural number n.

Proof. By (14), definition of $\omega_H(f,\cdot)$, (13) and (10) we have

(16)

$$|f(\mathbf{t}) - N_n(p;f)(\mathbf{t})| \ll \tfrac{1}{P_n}\int_{\Omega}\omega_H(f,\|\mathbf{s}\|)\left|p_n+\sum_{k=1}^{n}p_{n-k}(\Theta_k(\mathbf{s})-\Theta_{k-1}(\mathbf{s}))\right|d\mathbf{s}.$$

Since the function

$$\mathbf{t}\to\omega_H(f,\|\mathbf{t}\|)\left|p_n+\sum_{k=1}^{n}p_{n-k}(\Theta_k(\mathbf{t})-\Theta_{k-1}(\mathbf{t}))\right|$$

is symmetric with respect to variables t_1, t_2 and t_3, where $\mathbf{t}=(t_1,t_2,t_3)\in\Omega$, it is sufficient to estimate the integral

$$I_n := \int_{\Delta}\omega_H(f,\|\mathbf{t}\|)\left|p_n+\sum_{k=1}^{n}p_{n-k}(\Theta_k(\mathbf{t})-\Theta_{k-1}(\mathbf{t}))\right|d\mathbf{t},$$

where

$$\begin{aligned}\Delta &:= \left\{\mathbf{t}=(t_1,t_2,t_3)\in\mathbb{R}^3_H : 0\le t_1,t_2,-t_3\le 1\right\}\\ &= \{(t_1,t_2): t_1\ge 0,\ t_2\ge 0,\ t_1+t_2\le 1\},\end{aligned}$$

which is one of the six equilateral triangles in $\overline{\Omega}$. By considering the formula

(10), we obtain

$$
\begin{aligned}
I_n =& \int\limits_{\Delta} \omega_H\left(f, \|\mathbf{t}\|\right) \left| p_n + \sum_{k=1}^{n} p_{n-k}\left(\Theta_k\left(\mathbf{t}\right) - \Theta_{k-1}\left(\mathbf{t}\right)\right)\right| d\mathbf{t} \\
=& \int\limits_{\Delta} \omega_H\left(f, t_1+t_2\right) \left| p_n + \sum_{k=1}^{n} p_{n-k}\Big(\frac{\sin\frac{(k+1)(t_1-t_2)\pi}{3}\sin\frac{(k+1)(t_2-t_3)\pi}{3}\sin\frac{(k+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}} \right. \\
& \left. - \frac{\sin\frac{k(t_1-t_2)\pi}{3}\sin\frac{k(t_2-t_3)\pi}{3}\sin\frac{k(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}}\Big)\right| d\mathbf{t}.
\end{aligned}
$$

If we use the change of variables

$$
s_1 := \tfrac{t_1-t_3}{3} = \tfrac{2t_1+t_2}{3},\ \ s_2 := \tfrac{t_2-t_3}{3} = \tfrac{t_1+2t_2}{3}
$$

as in [17], we get

$$
\begin{aligned}
I_n =& 3\int\limits_{\widetilde{\Delta}} \omega_H\left(f, s_1+s_2\right) \left| p_n + \sum_{k=1}^{n} p_{n-k}\Big(\frac{\sin((k+1)(s_1-s_2)\pi)\sin((k+1)s_2\pi)\sin((k+1)(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)} \right. \\
& \left. - \frac{\sin(k(s_1-s_2)\pi)\sin(ks_2\pi)\sin(k(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)}\Big)\right| ds_1 ds_2,
\end{aligned}
$$

where $\widetilde{\Delta}$ is the image of Δ in the plane, that is

$$
\widetilde{\Delta} := \left\{(s_1, s_2) : 0 \le s_1 \le 2s_2,\ 0 \le s_2 \le 2s_1,\ s_1+s_2 \le 1\right\}.
$$

Since the integrated function is symmetric with respect to s_1 and s_2, we have

$$
\begin{aligned}
I_n =& 6\int\limits_{\Delta^*} \left(s_1+s_2\right)^{\alpha} \left| p_n + \sum_{k=1}^{n} p_{n-k}\Big(\frac{\sin((k+1)(s_1-s_2)\pi)\sin((k+1)s_2\pi)\sin((k+1)(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)} \right. \\
& \left. - \frac{\sin(k(s_1-s_2)\pi)\sin(ks_2\pi)\sin(k(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)}\Big)\right| ds_1 ds_2,
\end{aligned}
$$

where Δ^* is the half of $\widetilde{\Delta}$:

$$
\Delta^* := \left\{(s_1, s_2) \in \widetilde{\Delta} : s_1 \le s_2\right\} = \left\{(s_1, s_2) : s_1 \le s_2 \le 2s_1,\ s_1+s_2 \le 1\right\}.
$$

The change of variables

$$
s_1 := \tfrac{u_1-u_2}{2},\ \ s_2 := \tfrac{u_1+u_2}{2}
$$

transforms the triangle Δ^* to the triangle

$$
\Gamma := \left\{(u_1, u_2) : 0 \le u_2 \le \tfrac{u_1}{3},\ 0 \le u_1 \le 1\right\},
$$

hence we have

$$
I_n = 3\int\limits_{\Gamma} \omega_H\left(f, u_1\right) \left| p_n + \sum_{k=1}^{n} p_{n-k} D_k^*\left(u_1, u_2\right)\right| du_1 du_2,
$$

where

$$D_k^*(u_1,u_2) := \frac{\sin((k+1)u_2\pi)\sin\left((k+1)\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\left(\frac{u_1-u_2}{2}\pi\right)\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)} - \frac{\sin(ku_2\pi)\sin\left(k\frac{u_1+u_2}{2}\pi\right)\sin\left(k\left(\frac{u_1-u_2}{2}\pi\right)\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}.$$

By elementary trigonometric identities, we obtain

$$D_k^*(u_1,u_2) = D_{k,1}^*(u_1,u_2) + D_{k,2}^*(u_1,u_2) + D_{k,3}^*(u_1,u_2), \tag{17}$$

where

$$D_{k,1}^*(u_1,u_2) := 2\cos\left((k+\tfrac{1}{2})u_2\pi\right)\frac{\sin\left(\frac{1}{2}u_2\pi\right)\sin\left((k+1)\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)},$$

$$D_{k,2}^*(u_1,u_2) := 2\cos\left((k+\tfrac{1}{2})\frac{u_1+u_2}{2}\pi\right)\frac{\sin(ku_2\pi)\sin\left(\frac{1}{2}\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)},$$

and

$$D_{k,3}^*(u_1,u_2) := 2\cos\left((k+\tfrac{1}{2})\frac{u_1-u_2}{2}\pi\right)\frac{\sin(ku_2\pi)\sin\left(k\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{1}{2}\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}.$$

Since

$$\sin 2x + \sin 2y + \sin 2z = -4\sin x\sin y\sin z$$

for $x+y+z=0$, we also get the expression

$$D_k^*(u_1,u_2) = H_{k,1}(u_1,u_2) + H_{k,2}(u_1,u_2) + H_{k,3}(u_1,u_2), \tag{18}$$

where

$$H_{k,1}(u_1,u_2) := \frac{1}{2}\frac{\cos((2k+1)u_2\pi)}{\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)},$$

$$H_{k,2}(u_1,u_2) := -\frac{1}{2}\frac{\cos\left((2k+1)\frac{u_1+u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1-u_2}{2}\pi\right)},$$

$$H_{k,3}(u_1,u_2) := \frac{1}{2}\frac{\cos\left((2k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)}.$$

By considering the fact $(n+1)p_n \ll P_n$ and by (11) we get

$$\begin{aligned}\int_\Gamma p_n\omega_H(f,u_1)\,du_1du_2 &\le p_n\omega_H(f,1) \ll \frac{P_n}{n}\omega_H(f,1)\\ &= \frac{P_n}{n}\omega_H(f,n\tfrac{1}{n}) \ll \frac{P_n}{n}n\omega_H(f,\tfrac{1}{n})\\ &= \sum_{k=1}^{n}\frac{1}{n}P_n\omega_H(f,\tfrac{1}{n}) \le \sum_{k=1}^{n}\frac{1}{k}P_k\omega_H(f,\tfrac{1}{k}),\end{aligned}$$

since the sequence (P_n/n) non-increasing and $\omega_H(f,\cdot)$ is non-decreasing. Hence,

$$I_n \ll I_n^* + \sum_{k=1}^{n}\frac{1}{k}P_k\omega_H(f,\tfrac{1}{k}), \tag{19}$$

where

$$I_n^* := \int_{\Gamma} \omega_H(f,u_1) \left| \sum_{k=1}^{n} p_{n-k} D_k^*(u_1,u_2) \right| du_1 du_2.$$

If we partition the triangle Γ as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 := \left\{ (u_1,u_2) \in \Gamma : u_1 \le \tfrac{1}{n+1} \right\},$$
$$\Gamma_2 := \left\{ (u_1,u_2) \in \Gamma : u_1 \ge \tfrac{1}{n+1},\ u_2 \le \tfrac{1}{3(n+1)} \right\},$$
$$\Gamma_3 := \left\{ (u_1,u_2) \in \Gamma : u_1 \ge \tfrac{1}{n+1},\ u_2 \ge \tfrac{1}{3(n+1)} \right\},$$

we have

$$I_n^* = I_{n,1}^* + I_{n,2}^* + I_{n,3}^*,$$

where

$$I_{n,j}^* := \int_{\Gamma_j} \omega_H(f,u_1) \left| \sum_{k=1}^{n} p_{n-k} D_k^*(u_1,u_2) \right| du_1 du_2, \quad (j = 1,2,3).$$

We shall need the well known inequalities

(20) $$\left| \frac{\sin nt}{\sin t} \right| \le n, \quad (n \in \mathbb{N}),$$

and

(21) $$\sin t \ge \tfrac{2}{\pi} t, \quad \left(0 \le t \le \tfrac{\pi}{2}\right)$$

to estimate integrals $I_{n,1}^*, I_{n,2}^*$ and $I_{n,3}^*$.

By (17) and (20) we obtain

$$\begin{aligned}
I_{n,1}^* &= \int_{\Gamma_1} \omega_H(f,u_1) \left| \sum_{k=1}^{n} p_{n-k} D_k^*(u_1,u_2) \right| du_1 du_2 \\
&\le \int_{\Gamma_1} \omega_H(f,u_1) \left(\sum_{k=1}^{n} (k+1)^2 p_{n-k} \right) du_1 du_2 \\
&\le (n+1)^2 P_n \int_{\Gamma_1} \omega_H(f,u_1)\, du_1 du_2 \\
&= (n+1)^2 P_n \int_0^{1/(3(n+1))} \int_{3u_2}^{1/(n+1)} \omega_H(f,u_1)\, du_1 du_2 \\
&\le (n+1)^2 P_n \omega_H\left(f, \tfrac{1}{n+1}\right) \int_0^{1/(3(n+1))} \int_{3u_2}^{1/(n+1)} du_1 du_2 \\
&\le P_n \omega_H\left(f, \tfrac{1}{n}\right) = \sum_{k=1}^{n} \tfrac{1}{n} P_n \omega_H(f, \tfrac{1}{n}).
\end{aligned}$$

Since the sequence (P_n/n) is non-increasing we get

$$I_{n,1}^{*} \leq \sum_{k=1}^{n} \tfrac{1}{k} P_k \omega_H(f, \tfrac{1}{k}). \tag{22}$$

We write the rectangle Γ_2 as $\Gamma_2 = \Gamma_2' \cup \Gamma_2''$, where

$$\Gamma_2' := \left\{ (u_1, u_2) \in \Gamma_2 : u_2 \leq \tfrac{p_n}{3(n+1)P_n} \right\}$$

and

$$\Gamma_2'' := \left\{ (u_1, u_2) \in \Gamma_2 : u_2 \geq \tfrac{p_n}{3(n+1)P_n} \right\}$$

to estimate $I_{n,2}^{*}$.

By (21) we obtain

$$\begin{aligned}
&\int_{\Gamma_2'} \omega_H(f, u_1) \left| \sum_{k=1}^{n} p_{n-k} D_{k,1}^{*}(u_1, u_2) \right| du_1 du_2 \leq \\
\leq\ & \int_0^{\frac{p_n}{3(n+1)P_n}} \int_{\frac{1}{n+1}}^{1} \omega_H(f, u_1) \left(\sum_{k=1}^{n} p_{n-k} \left| D_{k,1}^{*}(u_1, u_2) \right| \right) du_1 du_2 \\
\ll\ & P_n \int_0^{\frac{p_n}{3(n+1)P_n}} \int_{\frac{1}{n+1}}^{1} \tfrac{\omega_H(f,u_1)}{u_1^2} du_1 du_2 = \tfrac{p_n}{3(n+1)} \int_{\frac{1}{n+1}}^{1} \tfrac{\omega_H(f,u_1)}{u_1^2} du_1 \\
=\ & \tfrac{p_n}{3(n+1)} \int_1^{n+1} \omega_H(f, \tfrac{1}{t}) dt = \tfrac{p_n}{3(n+1)} \sum_{k=1}^{n} \left(\int_k^{k+1} \omega_H(f, \tfrac{1}{t}) dt \right) \\
\leq\ & \tfrac{p_n}{n+1} \sum_{k=1}^{n} \omega_H(f, \tfrac{1}{k}) \leq \sum_{k=1}^{n} \tfrac{1}{k} P_k \omega_H(f, \tfrac{1}{k}).
\end{aligned}$$

For $j = 2, 3$, by (20) and (21),

$$\begin{aligned}
&\int_{\Gamma_2'} \omega_H(f, u_1) \left| \sum_{k=1}^{n} p_{n-k} D_{k,j}^{*}(u_1, u_2) \right| du_1 du_2 \leq \\
\leq\ & \int_{\frac{1}{n+1}}^{1} \int_0^{\frac{p_n}{3(n+1)P_n}} \omega_H(f, u_1) \left(\sum_{k=1}^{n} p_{n-k} \left| D_{k,j}^{*}(u_1, u_2) \right| \right) du_2 du_1 \\
\ll\ & \int_{\frac{1}{n+1}}^{1} \int_0^{\frac{p_n}{3(n+1)P_n}} \tfrac{\omega_H(f,u_1)}{u_1} \left(\sum_{k=1}^{n} k p_{n-k} \right) du_2 du_1 \leq
\end{aligned}$$

$$\begin{aligned}
&\leq \quad nP_n \int\limits_{\frac{1}{n+1}}^{1} \int\limits_{0}^{\frac{p_n}{3(n+1)P_n}} \tfrac{\omega_H(f,u_1)}{u_1} du_2 du_1 \leq p_n \int\limits_{\frac{1}{n+1}}^{1} \tfrac{\omega_H(f,u_1)}{u_1} du_1 \\
&= \quad p_n \int\limits_{1}^{n+1} \tfrac{\omega_H(f,1/t)}{t} dt = \sum_{k=1}^{n} \left(\int\limits_{k}^{k+1} \tfrac{\omega_H(f,1/t)}{t} dt \right) \\
&\leq \quad p_n \sum_{k=1}^{n} \tfrac{1}{k} \omega_H(f, \tfrac{1}{k}) = \sum_{k=1}^{n} \tfrac{1}{k} p_n \omega_H(f, \tfrac{1}{k}) \leq \sum_{k=1}^{n} \tfrac{1}{k} P_k \omega_H(f, \tfrac{1}{k}).
\end{aligned}$$

Hence we get

$$\int\limits_{\Gamma_2'} \omega_H(f, u_1) \left| \sum_{k=1}^{n} p_{n-k} D_k^*(u_1, u_2) \right| du_1 du_2 \ll \sum_{k=1}^{n} \tfrac{1}{k} P_k \omega_H(f, \tfrac{1}{k}). \tag{23}$$

To estimate the integrals $I_{n,3}^*$ and

$$\int\limits_{\Gamma_2''} \omega_H(f, u_1) \left| \sum_{k=1}^{n} p_{n-k} D_k^*(u_1, u_2) \right| du_1 du_2$$

we shall use the expression (18) of $D_k^*(u_1, u_2)$.

Lemma 5. 11 of [12] yields

$$\left| \sum_{k=1}^{n} p_{n-k} \cos((2k+1) u_2 \pi) \right| \ll P(\tfrac{1}{2\pi u_2})$$

and

$$\left| \sum_{k=1}^{n} p_{n-k} \cos\left((2k+1) \tfrac{u_1 - u_2}{2} \pi\right) \right| \ll P\left(\tfrac{1}{(u_1-u_2)\pi}\right)$$

for $(u_1, u_2) \in \Gamma_2'' \cup \Gamma_3$, where $P(t) := P_{[t]}$. By Lemmas 5. 11 and 5. 10 of [12], the fact

$$\sin \tfrac{u_1 \pi}{2} \leq \tfrac{2}{\sqrt{3}} \sin\left(\tfrac{u_1+u_2}{2}\pi\right),$$

and (21), we get

$$\left| \sum_{k=1}^{n} p_{n-k} \cos\left((2k+1) \tfrac{u_1+u_2}{2} \pi\right) \right| \ll P(\tfrac{1}{u_1 \pi})$$

for $(u_1, u_2) \in \Gamma_2'' \cup \Gamma_3$. Hence by considering these inequalities and (21) we obtain

$$\left| \sum_{k=1}^{n} p_{n-k} H_{k,1}(u_1, u_2) \right| \ll \tfrac{1}{u_1^2} P(\tfrac{1}{2\pi u_2}) \tag{24}$$

and

$$\left| \sum_{k=1}^{n} p_{n-k} H_{k,j}(u_1, u_2) \right| \ll \tfrac{1}{u_1 u_2} P(\tfrac{3}{2\pi u_1}) \quad (j = 2, 3) \tag{25}$$

for $(u_1, u_2) \in \Gamma_2'' \cup \Gamma_3$.

By (21) we obtain

$$\begin{aligned}
&\int\limits_{\Gamma_2''} \omega_H(f, u_1) \left| \sum_{k=1}^{n} p_{n-k} H_{k,1}(u_1, u_2) \right| du_1 du_2 \le \\
\le \quad & \int\limits_{\frac{1}{n+1}}^{1} \int\limits_{\frac{p_n}{3(n+1)P_n}}^{\frac{1}{3(n+1)}} \omega_H(f, u_1) \left(\sum_{k=1}^{n} p_{n-k} \left| H_{k,1}(u_1, u_2) \right| \right) du_2 du_1 \\
\le \quad & P_n \int\limits_{\frac{1}{n+1}}^{1} \int\limits_{\frac{p_n}{3(n+1)P_n}}^{\frac{1}{3(n+1)}} \frac{\omega_H(f, u_1)}{u_1^2} du_2 du_1 \le \frac{P_n}{n+1} \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_H(f, u_1)}{u_1^2} du_1 \\
= \quad & \frac{P_n}{n+1} \int\limits_{1}^{n+1} \omega_H(f, \tfrac{1}{t}) dt \le \frac{P_n}{n+1} \sum_{k=1}^{n} \omega_H(f, \tfrac{1}{k}) \\
\le \quad & \sum_{k=1}^{n} \frac{1}{k} P_k \omega_H(f, \tfrac{1}{k}).
\end{aligned}$$

For $j = 2, 3$ by (25) we get

$$\begin{aligned}
&\int\limits_{\Gamma_2''} \omega_H(f, u_1) \left| \sum_{k=1}^{n} p_{n-k} H_{k,j}(u_1, u_2) \right| du_1 du_2 \ll \\
\ll \quad & \int\limits_{\frac{1}{n+1}}^{1} \int\limits_{\frac{p_n}{3(n+1)P_n}}^{\frac{1}{3(n+1)}} \frac{\omega_H(f, u_1)}{u_1 u_2} P\left(\frac{3}{2\pi u_1} \right) du_2 du_1 \\
= \quad & \log\left(\frac{P_n}{p_n} \right) \int\limits_{\frac{1}{n+1}}^{1} \frac{\omega_H(f, u_1)}{u_1} P\left(\frac{3}{2\pi u_1} \right) du_1 \\
= \quad & \log\left(\frac{P_n}{p_n} \right) \int\limits_{\frac{3}{2\pi}}^{\frac{3}{2\pi}(n+1)} \omega_H\left(f, \frac{3}{2\pi t}\right) \frac{P(t)}{t} dt \\
= \quad & \log\left(\frac{P_n}{p_n} \right) \sum_{k=1}^{n} \left(\int\limits_{\frac{3}{2\pi}k}^{\frac{3}{2\pi}(k+1)} \omega_H\left(f, \frac{3}{2\pi t}\right) \frac{P(t)}{t} dt \right) \le
\end{aligned}$$

$$\leq \quad \log\left(\tfrac{P_n}{p_n}\right)\sum_{k=1}^{n}\tfrac{\omega_H\left(f,\frac{1}{k}\right)}{k}P\left(\tfrac{3}{2\pi}\left(k+1\right)\right)$$

$$\ll \quad \log\left(\tfrac{P_n}{p_n}\right)\sum_{k=1}^{n}\tfrac{1}{k}P_k\omega_H\big(f,\tfrac{1}{k}\big).$$

Thus, (23) and this inequality give

$$\int\limits_{\Gamma_2''}\omega_H\left(f,u_1\right)\left|\sum_{k=1}^{n}p_{n-k}D_k^*\left(u_1,u_2\right)\right|du_1du_2 \ll \log\left(\tfrac{P_n}{p_n}\right)\sum_{k=1}^{n}\tfrac{1}{k}P_k\omega_H\big(f,\tfrac{1}{k}\big),$$

and hence

$$I_{n,2}^* \ll \log\left(\tfrac{P_n}{p_n}\right)\sum_{k=1}^{n}\tfrac{P_k\omega_H(f,1/k)}{k}. \tag{26}$$

By (24) and by the inequality

$$\tfrac{\omega_H(f,\delta_2)}{\delta_2} \leq 2\tfrac{\omega_H(f,\delta_1)}{\delta_1} \quad (\delta_1 < \delta_2)$$

which is easily obtained from (11),

$$\int\limits_{\Gamma_3}\omega_H\left(f,u_1\right)\left|\sum_{k=1}^{n}p_{n-k}H_{k,1}\left(u_1,u_2\right)\right|du_1du_2 \ll$$

$$\ll \int\limits_{\frac{1}{3(n+1)}}^{\frac{1}{3}}\int\limits_{3u_2}^{1}\tfrac{\omega_H(f,u_1)}{u_1^2}P\big(\tfrac{1}{2\pi u_2}\big)du_1du_2$$

$$\ll \int\limits_{\frac{1}{3(n+1)}}^{\frac{1}{3}}\int\limits_{3u_2}^{1}\tfrac{\omega_H(f,3u_2)}{u_1u_2}P\big(\tfrac{1}{2\pi u_2}\big)du_1du_2$$

$$= \int\limits_{\frac{1}{3(n+1)}}^{\frac{1}{3}}\tfrac{\omega_H(f,3u_2)}{u_2}P\big(\tfrac{1}{2\pi u_2}\big)\log\big(\tfrac{1}{3u_2}\big)du_2$$

$$\leq \quad \log\left(n+1\right)\int\limits_{\frac{1}{3(n+1)}}^{\frac{1}{3}}\tfrac{\omega_H(f,3u_2)}{u_2}P\big(\tfrac{1}{2\pi u_2}\big)du_2$$

$$= \quad \log\left(n+1\right)\int\limits_{\frac{3}{2\pi}}^{\frac{3}{2\pi}(n+1)}\omega_H\big(f,\tfrac{3}{2\pi t}\big)\tfrac{P(t)}{t}dt$$

$$\ll \quad \log\left(\tfrac{P_n}{p_n}\right)\sum_{k=1}^{n}\tfrac{1}{k}P_k\omega_H\big(f,\tfrac{1}{k}\big).$$

By (25), for $j=2,3$,

$$\begin{aligned}
&\int\limits_{\Gamma_3}\omega_H\left(f,u_1\right)\left|\sum_{k=1}^{n}p_{n-k}H_{k,j}\left(u_1,u_2\right)\right|du_1du_2\leq\\
&\leq \int\limits_{\frac{1}{n+1}}^{1}\int\limits_{\frac{1}{3(n+1)}}^{\frac{u_1}{3}}\tfrac{\omega_H(f,u_1)}{u_1u_2}P\left(\tfrac{3}{2\pi u_1}\right)du_2du_1\\
&= \int\limits_{\frac{1}{n+1}}^{1}\tfrac{\omega_H(f,u_1)}{u_1}P\left(\tfrac{3}{2\pi u_1}\right)\log\left(\left(n+1\right)u_1\right)du_1\\
&\leq \log\left(n+1\right)\int\limits_{\frac{1}{n+1}}^{1}\tfrac{\omega_H(f,u_1)}{u_1}P\left(\tfrac{3}{2\pi u_1}\right)du_1\\
&\ll \log\left(\tfrac{P_n}{p_n}\right)\sum_{k=1}^{n}\tfrac{1}{k}P_k\omega_H\left(f,\tfrac{1}{k}\right).
\end{aligned}$$

Thus,

$$I_{n,3}^{*}\ll\log\left(\tfrac{P_n}{p_n}\right)\sum_{k=1}^{n}\tfrac{1}{k}P_k\omega_H\left(f,\tfrac{1}{k}\right). \tag{27}$$

Combining (16), (19), (22), (26) and (27) give (15). □

4. CONCLUSIONS

Conclusion 1. For $f\in H^{\alpha}\left(\overline{\Omega}\right)$ $\left(0<\alpha\leq1\right)$, Theorem 1 yields the following analogue of (5):

$$\left\|f-N_n\left(p;f\right)\right\|_{C_H\left(\overline{\Omega}\right)}\ll\tfrac{1}{P_n}\log\left(\tfrac{P_n}{p_n}\right)\sum_{k=1}^{n}\tfrac{P_k}{k^{1+\alpha}}.$$

Note that this estimate was obtained directly in [7].

Conclusion 2. In the case $p_n=1$, $\left(n=0,1,...\right)$, (15) reduces to

$$\left\|f-\sigma_n\left(f\right)\right\|_{C_H\left(\overline{\Omega}\right)}\ll\tfrac{\log n}{n+1}\sum_{k=1}^{n}\omega_H\left(f,\tfrac{1}{k}\right),$$

which is the analogue of (3) for hexagonal Fourier series.

Conclusion 3. In the case $p_n=1$, $\left(n=0,1,...\right)$ and $f\in H^{\alpha}\left(\overline{\Omega}\right)$ $\left(0<\alpha\leq1\right)$, (15) gives

$$\left\|f-\sigma_n\left(f\right)\right\|_{C_H\left(\overline{\Omega}\right)}\ll\begin{cases}\frac{\log n}{n^{\alpha}}, & 0<\alpha<1\\ \frac{(\log n)^2}{n}, & \alpha=1.\end{cases}$$

This estimate yields the same approximation order with (12) in the case $\alpha=1$.

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REFERENCES

[1] G. Alexits, *Convergence Problems of Orthogonal Series*, Pergamon Press, New York-Oxford-Paris, 1961.
[2] S.N. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par les polynômes de degré donné*, Mem. Cl. Sci. Acad. Roy. Belg., **4** (1912), pp. 1–103.
[3] P.L. Butzer, R.J. Nessel, *Fourier analysis and Approximation*, Academic Press, New York-London, 1971.
[4] R.A. DeVore, G.G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, 1993.
[5] B. Fuglede, *Commuting self-adjoint partial differential operators and a group theoretic problem*, J. Funct. Anal., **16** (1974), pp. 101–121.
[6] A. Guven, *Approximation by $(C, 1)$ and Abel-Poisson means of Fourier series on hexagonal domains*, Math. Inequal. Appl., **16** (2013), pp. 175–191.
[7] A. Guven, *Approximation by Nörlund means of hexagonal Fourier series*, Anal. Theory Appl., **33** (2017), pp. 384–400.
[8] G.H. Hardy, *Divergent Series*, Clarendon Press, Oxford, 1949.
[9] A.S.B. Holland, B.N. Sahney, J. Tzimbalario, *On degree of approximation of a class of functions by means of Fourier series*, Acta Sci. Math., **38** (1976), pp. 69–72.
[10] A.S.B. Holland, *A survey of degree of approximation of continuous functions*, SIAM Rev., **23** (1981), pp. 344–379.
[11] H. Li, J. Sun, Y. Xu, *Discrete Fourier analysis, cubature and interpolation on a hexagon and a triangle*, SIAM J. Numer. Anal., **46** (2008), pp. 1653–1681.
[12] L. Mcfadden, *Absolute Nörlund summability*, Duke Math. J., **9** (1942), pp. 168–207.
[13] B.N. Sahney, D.S. Goel, *On the degree of approximation of continuous functions*, Ranchii Univ. Math. J., **4** (1973), pp. 50–53.
[14] S.B. Stechkin, *The approximation of periodic functions by Fejér sums*, Trudy Mat. Inst. Steklov., **62** (1961), pp. 48–60 (in Russian).
[15] J. Sun, *Multivariate Fourier series over a class of non tensor-product partition domains*, J. Comput. Math., **21** (2003), pp. 53–62.
[16] A.F. Timan, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press, New York, 1963.
[17] Y. Xu, *Fourier series and approximation on hexagonal and triangular domains*, Constr. Approx., **31** (2010), pp. 115–138.
[18] A. Zygmund, *Trigonometric Series*, 2nd ed., vol. I, Cambridge Univ. Press, New York, 1959.

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