CAPUTO FRACTIONAL APPROXIMATION
BY SUBLINEAR OPERATORS

GEORGE A. ANASTASSIOU

Abstract. Here we consider the approximation of functions by sublinear positive operators with applications to a big variety of Max-Product operators under Caputo fractional differentiability. Our study is based on our general fractional results about positive sublinear operators. We produce Jackson type inequalities under simple initial conditions. So our approach is quantitative by producing inequalities with their right hand sides involving the modulus of continuity of fractional derivative of the function under approximation.

MSC 2010. 26A33, 41A17, 41A25, 41A36.

Keywords. positive sublinear operators, Max-product operators, modulus of continuity, Caputo fractional derivative.

1. INTRODUCTION

The main motivation here is the monograph by B. Bede, L. Coroianu and S. Gal [5, 2016].

Let \( N \in \mathbb{N} \), the well-known Bernstein polynomials [11] are positive linear operators, defined by the formula

\[
B_N(f)(x) = \sum_{k=0}^{N} \binom{N}{k} x^k (1-x)^{N-k} f\left(\frac{k}{N}\right), \quad x \in [0,1], \ f \in C([0,1]).
\]

T. Popoviciu in [12] (1935), proved for \( f \in C([0,1]) \) that

\[
|B_N(f)(x) - f(x)| \leq \frac{5}{4} \omega_1 \left(f, \frac{1}{\sqrt{N}}\right), \quad \forall x \in [0,1],
\]

where

\[
\omega_1(f,\delta) = \sup_{x,y \in [a,b]: |x-y| \leq \delta} |f(x) - f(y)|, \quad \delta > 0,
\]

is the first modulus of continuity, here \([a,b] = [0,1]\).

G.G. Lorentz in [11, p. 21] (1986), proved for \( f \in C^1([0,1]) \) that

\[
|B_N(f)(x) - f(x)| \leq \frac{3}{4\sqrt{N}} \omega_1 \left(f', \frac{1}{\sqrt{N}}\right), \quad \forall x \in [0,1],
\]

*Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A, e-mail: ganastss@memphis.edu.
In [5, p. 1], the authors introduced the basic Max-product Bernstein operators,
\[(1.5) \quad B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x)}{\bigvee_{k=0}^{N} p_{N,k}(x)} f(\frac{k}{N}), \quad N \in \mathbb{N},\]
where \(\bigvee\) stands for maximum, and \(p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}\) and \(f : [0,1] \to \mathbb{R}_+ = [0, \infty)\).

These are nonlinear and piecewise rational operators.

The authors in [5] studied similar such nonlinear operators such as: the Max-product Favard-Szász-Mirakjan operators and their truncated version, the Max-product Baskakov operators and their truncated version, also many other similar specific operators. The study in [5] is based on presented there general theory of sublinear operators. These Max-product operators tend to converge faster to the on hand function.

So we mention from [5, p. 30], that for \(f : [0,1] \to \mathbb{R}_+\) continuous, we have
\[(1.6) \quad \left| B_N^{(M)}(f)(x) - f(x) \right| \leq 12 \omega_1 \left( f, \frac{1}{\sqrt{N+1}} \right), \quad \text{for all } N \in \mathbb{N}, \ x \in [0,1],\]
Also from [5, p. 36], we mention that for \(f : [0,1] \to \mathbb{R}_+\) being concave function we get that
\[(1.7) \quad \left| B_N^{(M)}(f)(x) - f(x) \right| \leq 2 \omega_1 \left( f, \frac{1}{N} \right), \quad \text{for all } x \in [0,1],\]
a much faster convergence.

In this article we expand the study in [5] by considering Caputo fractional smoothness of functions. So our inequalities are with respect to \(\omega_1(D^\alpha f, \delta), \ \delta > 0,\) where \(D^\alpha f\) with \(\alpha > 0\) is the Caputo fractional derivative.

2. MAIN RESULTS

We need

**Definition 2.1.** Let \(\nu \geq 0, \ n = [\nu] \ (\lceil \cdot \rceil \ is \ the \ ceiling \ of \ the \ number), \ f \in AC^n([a,b]) \ (space \ of \ functions \ f \ with \ f^{(n-1)} \in AC([a,b]), \ absolutely \ continuous \ functions). \ We \ call \ left \ Caputo \ fractional \ derivative \ (see \ [7, p. 49], \ [10], \ [13]) \ the \ function
\[(2.1) \quad D^\nu_{sa} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t) \ dt, \ \forall x \in [a,b],\]
where \(\Gamma\) is the gamma function \(\Gamma (v) = \int_0^\infty e^{-t} t^{v-1} dt, \ v > 0.\)

We set \(D^0_{sa} f(x) = f(x), \ \forall x \in [a,b].\)

**Lemma 2.2.** [2] Let \(\nu > 0, \ \nu \notin \mathbb{N}, \ n = [\nu], \ f \in C^{n-1}([a,b]) \ and \ f^{(n)} \in L_\infty ([a,b]). \ Then \ D^\nu_{sa} f (a) = 0.\)

We need
Definition 2.3. (see also [1], [8], [10]) Let \( f \in AC^m ([a,b]) \), \( m = \lceil \alpha \rceil \), \( \alpha > 0 \). The right Caputo fractional derivative of order \( \alpha > 0 \) is given by

\[
D_\alpha^b f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta) \, d\zeta, \quad \forall x \in [a,b].
\]

We set \( D_0^b f(x) = f(x) \).

Lemma 2.4. [2] Let \( f \in C^{m-1} ([a,b]) \), \( f^{(m)} \in L_\infty ([a,b]) \), \( m = \lceil \alpha \rceil \), \( \alpha > 0 \), \( \alpha \not\in \mathbb{N} \). Then \( D_\alpha^b f(b) = 0 \).

Convention 2.5. We assume that

\[
D_{x_0}^a f(x) = 0, \quad \text{for } x < x_0,
\]

and

\[
D_{x_0}^\alpha f(x) = 0, \quad \text{for } x > x_0,
\]

for all \( x, x_0 \in [a,b] \).

We mention

Proposition 2.6. [2] Let \( f \in C^n ([a,b]) \), \( n = \lceil \nu \rceil \), \( \nu > 0 \). Then \( D_\nu^a f(x) \) is continuous in \( x \in [a,b] \).

Proposition 2.7. [2] Let \( f \in C^m ([a,b]) \), \( m = \lceil \alpha \rceil \), \( \alpha > 0 \). Then \( D_\alpha^b f(x) \) is continuous in \( x \in [a,b] \).

The modulus of continuity \( \omega_1 (f, \delta) \) is defined the same way for bounded functions, see [1.3], and it is finite.

We make

Remark 2.8. [2] Let \( f \in C^{m-1} ([a,b]) \), \( f^{(n)} \in L_\infty ([a,b]) \), \( n = \lceil \nu \rceil \), \( \nu > 0 \), \( \nu \not\in \mathbb{N} \). Then

\[
\omega_1 (D_\nu^a f, \delta) \leq \frac{2\|f^{(n)}\|_\infty}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}.
\]

Similarly, let \( f \in C^{m-1} ([a,b]) \), \( f^{(m)} \in L_\infty ([a,b]) \), \( m = \lceil \alpha \rceil \), \( \alpha > 0 \), \( \alpha \not\in \mathbb{N} \), then

\[
\omega_1 (D_\alpha^b f, \delta) \leq \frac{2\|f^{(m)}\|_\infty}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}.
\]

That is \( \omega_1 (D_\nu^a f, \delta), \omega_1 (D_\alpha^b f, \delta) \) are finite.

Clearly, above \( D_\nu^a f \) and \( D_\alpha^b f \) are bounded, from

\[
|D_\nu^a f(x)| \leq \frac{\|f^{(n)}\|_\infty}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}, \quad \forall x \in [a,b],
\]

see [2].

We need
\textbf{Definition 2.9.} Let $D^\alpha_{x_0} f$ denote any of $D^\alpha_{x_0} - f$, $D^\alpha_{x_0} f$, and $\delta > 0$. We set

\begin{equation}
\omega_1(D^\alpha_{x_0} f, \delta) := \max \left\{ \omega_1(D^\alpha_{x_0} - f, \delta)_{[a,x_0]}, \omega_1(D^\alpha_{x_0} f, \delta)_{[x_0,b]} \right\},
\end{equation}

where $x_0 \in [a,b]$. Here the moduli of continuity are considered over $[a,x_0]$ and $[x_0,b]$, respectively.

We need

\textbf{Theorem 2.10.} Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = [\alpha]$, $x_0 \in [a,b] \subset \mathbb{R}$, $f \in AC^m([a,b], \mathbb{R})$ (i.e. $f^{(m-1)} \in AC([a,b])$, absolutely continuous functions on $[a,b]$), and $f^{(m)} \in L_\infty([a,b])$. Furthermore we assume that $f^{(k)}(x_0) = 0$, $k = 1,...,m-1$. Then

\begin{equation}
|f(x) - f(x_0)| \leq \frac{\omega_1(D^\alpha_{x_0} f, \delta)}{1(\alpha+1)} \left[ |x - x_0|^{\alpha} + \frac{|x - x_0|^{\alpha+1}}{(\alpha+1)!} \right], \quad \delta > 0,
\end{equation}

for all $a \leq x \leq b$.

If $0 < \alpha < 1$, then we do not need initial conditions.

\textit{Proof.} From [7, p. 54], we get by left Caputo Taylor formula that

\begin{equation}
f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{1(\alpha)} \int_{x_0}^{x} (x - z)^{\alpha-1} D^\alpha_{x_0} f(z) dz,
\end{equation}

for all $x_0 \leq x \leq b$.

Also from [1], using the right Caputo fractional Taylor formula we get

\begin{equation}
f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{1(\alpha)} \int_{x}^{x_0} (z - x)^{\alpha-1} D^\alpha_{x_0} f(z) dz,
\end{equation}

for all $a \leq x \leq x_0$.

By the assumption $f^{(k)}(x_0) = 0$, $k = 1,...,m-1$, we get

\begin{equation}
f(x) - f(x_0) = \frac{1}{1(\alpha)} \int_{x_0}^{x} (x - z)^{\alpha-1} D^\alpha_{x_0} f(z) dz,
\end{equation}

for all $x_0 \leq x \leq b$.

And it holds

\begin{equation}
f(x) - f(x_0) = \frac{1}{1(\alpha)} \int_{x}^{x_0} (z - x)^{\alpha-1} D^\alpha_{x_0} f(z) dz,
\end{equation}

for all $a \leq x \leq x_0$.

Notice that when $0 < \alpha < 1$, then $m = 1$, and \eqref{2.12} and \eqref{2.13} are valid without initial conditions.

Since $D^\alpha_{x_0} f(x_0) = D^\alpha_{x_0} f(x_0) = 0$, we get

\begin{equation}
f(x) - f(x_0) = \frac{1}{1(\alpha)} \int_{x_0}^{x} (x - z)^{\alpha-1} ((D^\alpha_{x_0} f)(z) - D^\alpha_{x_0} f(x)) dz,
\end{equation}

$x_0 \leq x \leq b$, and

\begin{equation}
f(x) - f(x_0) = \frac{1}{1(\alpha)} \int_{x}^{x_0} (z - x)^{\alpha-1} (D^\alpha_{x_0} f(z) - D^\alpha_{x_0} f(x)) dz,
\end{equation}

$x_0 \leq x \leq b$. \hfill \Box
We have proved that \((x_0 \leq x \leq b)\)
\[
|f(x) - f(x_0)| \leq \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-z)^{\alpha-1} |(D_{x=z}^{\alpha} f)(z) - D_{x=x_0}^{\alpha} f(x_0)| \, dz
\]
(2.16)
\[
\leq \frac{\omega_1(D_{x=x_0}^{\alpha} f, \delta_1)_{[x_0,b]}}{\Gamma(\alpha)} \int_{x_0}^{x} (x-z)^{\alpha-1} \left(1 + \frac{(x-z_0)}{\delta_1}\right) \, dz
\]
\[
\leq \frac{\omega_1(D_{x=x_0}^{\alpha} f, \delta_1)_{[x_0,b]}}{\Gamma(\alpha)} \left[ (x-z_0)^{\alpha} + \frac{1}{\delta_1} \int_{x_0}^{x} (x-z)^{\alpha-1} (z-x_0)^{\alpha+1} \, dz \right]
\]
(2.17)
\[
\leq \frac{\omega_1(D_{x=x_0}^{\alpha} f, \delta_1)_{[x_0,b]}}{\Gamma(\alpha)} \left[ (x-z_0)^{\alpha} + \frac{1}{\delta_1} \frac{1}{(\alpha+1)!} (x-x_0)^{\alpha+1} \right]
\]

We have proved that
\[
|f(x) - f(x_0)| \leq \frac{\omega_1(D_{x=x_0}^{\alpha} f, \delta_1)_{[x_0,b]}}{\Gamma(\alpha+1)} \left[ (x-x_0)^{\alpha} + \frac{(x-x_0)^{\alpha+1}}{(\alpha+1)!} \right],
\]
(2.18)
\[
\delta_1 > 0, \text{ and } x_0 \leq x \leq b.
\]
Similarly acting, we get \((a \leq x \leq x_0)\)
\[
|f(x) - f(x_0)| \leq \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (z-x)^{\alpha-1} |D_{x}^{\alpha} f(z) - D_{x=x_0}^{\alpha} f(x_0)| \, dz
\]
(2.19)
\[
\leq \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (z-x)^{\alpha-1} \omega_1(D_{x}^{\alpha} f, \delta_2)_{[a,x_0]} \, dz
\]
\[
\leq \frac{\omega_1(D_{x}^{\alpha} f, \delta_2)_{[a,x_0]}}{\Gamma(\alpha)} \left[ \int_{x}^{x_0} (z-x)^{\alpha-1} \left(1 + \frac{z-a}{\delta_2}\right) \, dz \right]
\]
\[
\leq \frac{\omega_1(D_{x}^{\alpha} f, \delta_2)_{[a,x_0]}}{\Gamma(\alpha)} \left[ (x-a)^{\alpha} + \frac{1}{\delta_2} \int_{x}^{x_0} (x-z)^{\alpha+1} (z-x)^{\alpha-1} \, dz \right]
\]
(2.20)
\[
\leq \frac{\omega_1(D_{x}^{\alpha} f, \delta_2)_{[a,x_0]}}{\Gamma(\alpha)} \left[ (x-a)^{\alpha} + \frac{(x-a)^{\alpha+1}}{(\alpha+1)!} \right]
\]
\[
\leq \frac{\omega_1(D_{x}^{\alpha} f, \delta_2)_{[a,x_0]}}{\Gamma(\alpha+1)} \left[ (x-x_0)^{\alpha} + \frac{(x-x_0)^{\alpha+1}}{(\alpha+1)!} \right].
\]
We have proved that

\[
|f(x) - f(x_0)| \leq \frac{\omega_1(D_{\alpha N}^* f, \delta^2(x-x_0))}{1(\alpha+1)} \left| (x_0 - x)^\alpha + (x_0-x)^{\alpha+1} \right|,
\]

\(\delta_2 > 0\), and \((a \leq x \leq x_0)\). Choosing \(\delta = \delta_1 = \delta_2 > 0\), by \([2.18]\) and \([2.21]\), we get \([2.9]\).

We need

**Definition 2.11.** Here \(C_+([a,b]) := \{f : [a,b] \rightarrow \mathbb{R}_+, f \text{ continuous}\}\). Let \(L_N : C_+([a,b]) \rightarrow C_+([a,b])\), operators, \(\forall N \in \mathbb{N}\), such that

(i)

\(L_N(\alpha f) = \alpha L_N(f), \forall \alpha \geq 0, \forall f \in C_+([a,b])\),

(ii) if \(f, g \in C_+([a,b]): f \leq g\), then

\(L_N(f) \leq L_N(g), \forall N \in \mathbb{N}\),

(iii)

\(L_N(f + g) \leq L_N(f) + L_N(g), \forall f, g \in C_+([a,b]).\)

We call \(\{L_N\}_{N \in \mathbb{N}}\) positive sublinear operators.

We need a Hölder’s type inequality, see next:

**Theorem 2.12.** (see [3]) Let \(L : C_+([a,b]) \rightarrow C_+([a,b])\), be a positive sublinear operator and \(f, g \in C_+([a,b]),\) furthermore let \(p, q > 1: \frac{1}{p} + \frac{1}{q} = 1\). Assume that \(L((f(\cdot))^p)(s_\alpha), L((g(\cdot))^q)(s_\alpha) > 0\) for some \(s_\alpha \in [a,b]\). Then

\[
L(f(\cdot)g(\cdot))(s_\alpha) \leq (L((f(\cdot))^p)(s_\alpha))^\frac{1}{p} (L((g(\cdot))^q)(s_\alpha))^\frac{1}{q}.
\]

We make

**Remark 2.13.** By [3], p. 17, we get: let \(f, g \in C_+([a,b]),\) then

\[
|L_N(f)(x) - L_N(g)(x)| \leq L_N(|f - g|(x)), \forall x \in [a,b].
\]

Furthermore, we also have that

\[
|L_N(f)(x) - f(x)| \leq L_N(|f(\cdot) - f(\cdot)(x)))(x) + |f(x)| L_N(\epsilon_0)(x) - 1, \forall x \in [a,b]; \epsilon_0(t) = 1.
\]

From now on we assume that \(L_N(1) = 1\). Hence it holds

\[
|L_N(f)(x) - f(x)| \leq L_N(|f(\cdot) - f(\cdot)(x)|(x)), \forall x \in [a,b].
\]

Using Theorem 2.10 and \([2.9]\) with \([2.28]\) we get:

\[
|L_N(f)(x_0) - f(x_0)| \leq \frac{\omega_1(D_{\alpha N}^* f, \delta)}{1(\alpha+1)} \left[ L_N(|f - x_0|^\alpha)(x_0) + \frac{L_N(|f - x_0|^\alpha)(x_0)}{\alpha+1}\delta \right], \delta > 0.
\]
We have proved

**Theorem 2.14.** Let \( \alpha > 0, \alpha \notin \mathbb{N}, m = [\alpha], x_0 \in [a,b] \subset \mathbb{R}, f \in AC^m([a,b],\mathbb{R}_+), \) and \( f^{(m)} \in L_\infty([a,b]). \) Furthermore we assume that \( f^{(k)} (x_0) = 0, k = 1, \ldots, m - 1. \) Let \( L_N : C_+([a,b]) \to C_+([a,b]), \forall N \in \mathbb{N}, \) be positive sublinear operators, such that \( L_N (1) = 1, \forall N \in \mathbb{N}. \) Then

\[
(2.30)
\]

\[
|L_N (f) (x_0) - f (x_0)| \leq \frac{\omega_1(D_{x_0}^\alpha f, \delta)}{\Gamma(\alpha + 1)} \cdot \left[ L_N \left( |\cdot - x_0|^\alpha \right) (x_0) + \frac{L_N \left( |\cdot - x_0|^{\alpha + 1} \right) (x_0)}{\Gamma(\alpha + 1)} \right]
\]

\( \delta > 0, \forall N \in \mathbb{N}. \)

In particular \((2.30)\) is true for \( \alpha > 1, \alpha \notin \mathbb{N}. \)

**Corollary 2.15.** Let \( 0 < \alpha < 1, x_0 \in [a,b] \subset \mathbb{R}, f \in AC ([a,b],\mathbb{R}_+), \) and \( f' \in L_\infty ([a,b]). \) Let \( L_N : C_+([a,b]) \to C_+([a,b]), \forall N \in \mathbb{N}, \) be positive sublinear operators, such that \( L_N (1) = 1, \forall N \in \mathbb{N}. \) Then \((2.30)\) is valid.

We give

**Theorem 2.16.** Let \( 0 < \alpha < 1, x_0 \in [a,b] \subset \mathbb{R}, f \in AC ([a,b],\mathbb{R}_+), \) and \( f' \in L_\infty ([a,b]). \) Let \( L_N \) from \( C_+([a,b]) \) into itself be positive sublinear operators, such that \( L_N (1) = 1, \forall N \in \mathbb{N}. \) Assume that \( L_N \left( |\cdot - x_0|^{\alpha + 1} \right) (x_0) > 0, \forall N \in \mathbb{N}. \) Then

\[
|L_N (f) (x_0) - f (x_0)| \leq \frac{\omega_1(D_{x_0}^\alpha f, L_N \left( |\cdot - x_0|^{\alpha + 1} \right) (x_0))}{\Gamma(\alpha + 2)} \left( L_N \left( |\cdot - x_0|^{\alpha + 1} \right) (x_0) \right)^\frac{\alpha}{\alpha + 1}.
\]

**Proof.** By Theorem 2.12 see \((2.25)\), we get

\[
(2.31)
\]

\[
L_N \left( |\cdot - x_0|^\alpha \right) (x_0) \leq \left( L_N \left( |\cdot - x_0|^{\alpha + 1} \right) (x_0) \right)^\frac{\alpha}{\alpha + 1}.
\]

Choose

\[
(2.32)
\]

\[
\delta := \left( L_N \left( |\cdot - x_0|^{\alpha + 1} \right) (x_0) \right)^\frac{1}{\alpha + 1} > 0,
\]

i.e. \( \delta^{\alpha + 1} = L_N \left( |\cdot - x_0|^{\alpha + 1} \right) (x_0). \)

By \((2.30)\) we obtain

\[
|L_N (f) (x_0) - f (x_0)| \leq \frac{1}{\Gamma(\alpha + 1)} \omega_1 \left( D_{x_0}^\alpha f, \left( L_N \left( |\cdot - x_0|^{\alpha + 1} \right) (x_0) \right)^\frac{1}{\alpha + 1} \right) \cdot \left( L_N \left( |\cdot - x_0|^{\alpha + 1} \right) (x_0) \right)^\frac{\alpha}{\alpha + 1} + \frac{1}{\Gamma(\alpha + 1)} \left( L_N \left( |\cdot - x_0|^{\alpha + 1} \right) (x_0) \right)^\frac{\alpha}{\alpha + 1} =
\]
\[
\begin{align*}
\omega_1 \left( \frac{D_{\alpha \epsilon}^\alpha f \left( L_N \left( |x - x_0|^{\alpha+1} \right) (x_0) \right)^{1+\gamma}}{\Gamma(\alpha+1)} \right) \\
\cdot \left( L_N \left( |x - x_0|^{\alpha+1} \right) (x_0) \right)^{\frac{\alpha}{\alpha+1}} \left[ 1 + \frac{1}{\alpha+1} \right]
\end{align*}
\]

(2.34) \[
= \omega_1 \left( \frac{D_{\alpha \epsilon}^\alpha f \left( L_N \left( |x - x_0|^{\alpha+1} \right) (x_0) \right)^{1+\gamma}}{\Gamma(\alpha+1)} \right) \left( L_N \left( |x - x_0|^{\alpha+1} \right) (x_0) \right)^{\frac{\alpha}{\alpha+1}} \left( \frac{\alpha+2}{\alpha+1} \right)
\]

\[
= \frac{(\alpha+2)\omega_1 \left( D_{\alpha \epsilon}^\alpha f \left( L_N \left( |x - x_0|^{\alpha+1} \right) (x_0) \right)^{1+\gamma} \right)}{\Gamma(\alpha+2)} \left( L_N \left( |x - x_0|^{\alpha+1} \right) (x_0) \right)^{\frac{\alpha}{\alpha+1}},
\]

proving (2.31).

3. APPLICATIONS

I) Case \( 0 < \alpha < 1 \).

Here we apply Theorem 2.16 to well known Max-product operators.

We make

Remark 3.1. The Max-product Bernstein operators \( B_N^{(M)} (f) (x) \) are defined by (1.5), see also [5, p. 10]; here \( f : [0, 1] \to \mathbb{R}_+ \) is a continuous function.

We have \( B_N^{(M)} (1) = 1 \), and

(3.1) \[
B_N^{(M)} (|x - x|) (x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall \; x \in [0, 1], \; \forall \; N \in \mathbb{N},
\]

see [5, p. 31].

\( B_N^{(M)} \) are positive sublinear operators and thus they possess the monotonicity property, also since \( |x - x| \leq 1 \), then \( |x - x|^\beta \leq 1, \; \forall \; x \in [0, 1], \; \forall \; \beta > 0 \).

Therefore it holds

(3.2) \[
B_N^{(M)} \left( |x - x|^{1+\beta} \right) (x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall \; x \in [0, 1], \; \forall \; N \in \mathbb{N}, \; \forall \; \beta > 0.
\]

Furthermore, clearly it holds that

(3.3) \[
B_N^{(M)} \left( |x - x|^{1+\beta} \right) (x) > 0, \quad \forall \; N \in \mathbb{N}, \; \forall \; \beta \geq 0 \text{ and any } x \in (0, 1).
\]

The operator \( B_N^{(M)} \) maps \( C_+ ([0, 1]) \) into itself.

We present

Theorem 3.2. Let \( 0 < \alpha < 1 \), any \( x \in (0, 1) \), \( f \in AC ([0, 1], \mathbb{R}_+) \), and \( f' \in L_\infty ([0, 1]) \). Then

(3.4) \[
\left| B_N^{(M)} (f) (x) - f (x) \right| \leq \frac{(\alpha+2)\omega_1 \left( D_{\alpha \epsilon}^\alpha f \left( \frac{6}{\sqrt{N+1}} \right)^{1+\gamma} \right)}{\Gamma(\alpha+2)} \left( \frac{6}{\sqrt{N+1}} \right)^{\frac{\alpha}{\alpha+1}},
\]

\( \forall \; N \in \mathbb{N}. \)

As \( N \to +\infty \), we get \( B_N^{(M)} (f) (x) \to f (x) \), any \( x \in (0, 1) \).

Proof. By Theorem 2.16

We continue with
Remark 3.3. The truncated Favard-Szász-Mirakjan operators are given by

$$T_N^{(M)}(f)(x) = \frac{\sum_{k=0}^{N} s_{N,k}(x)f(\frac{k}{N})}{\sum_{k=0}^{N} s_{N,k}(x)}, \quad x \in [0,1], \; N \in \mathbb{N}, \; f \in C_+([0,1]),$$

where

$$s_{N,k}(x) = \frac{(Nx)^k}{k!},$$

see also [5, p. 11].

By [5, p. 178-179], we get that

$$\left| T_N^{(M)}(\cdot - x) \right|(x) \leq \frac{3}{\sqrt{N}}, \quad \forall \; x \in [0,1], \; \forall \; N \in \mathbb{N}. \quad (3.6)$$

Clearly it holds

$$\left| T_N^{(M)}(\cdot - x)^{1+\beta} \right|(x) \leq \frac{3}{\sqrt{N}}, \quad \forall \; x \in [0,1], \; \forall \; N \in \mathbb{N}, \; \forall \; \beta > 0. \quad (3.7)$$

The operators $T_N^{(M)}$ are positive sublinear operators mapping $C_+([0,1])$ into itself, with $T_N^{(M)}(1) = 1$.

Furthermore it holds

$$T_N^{(M)}(\cdot - x^\lambda)(x) = \frac{\sum_{k=0}^{N} (Nx)^k \frac{k}{N-x}^\lambda}{\sum_{k=0}^{N} (Nx)^k \frac{k}{N-x}^\lambda} > 0, \quad \forall \; x \in (0,1], \; \forall \; \lambda \geq 1, \; \forall \; N \in \mathbb{N}. \quad (3.8)$$

We make

**Theorem 3.4.** Let $0 < \alpha < 1$, any $x \in (0,1]$, $f \in AC([0,1],\mathbb{R}_+)$, and $f' \in L_\infty((0,1])$. Then

$$\left| T_N^{(M)}(f)(x) - f(x) \right| \leq \frac{(\alpha+2)\omega_1 D_\infty^\alpha \left( \frac{1}{n} \right)^{\frac{1}{\alpha+1}}}{\Gamma(\alpha+2)} \left( \frac{3}{\sqrt{N}} \right)^{\frac{\alpha}{\alpha+1}}, \quad \forall \; N \in \mathbb{N}. \quad (3.9)$$

As $N \to +\infty$, we get $T_N^{(M)}(f)(x) \to f(x)$, for any $x \in (0,1]$.

**Proof.** By Theorem 2.16 \square

We give

**Remark 3.5.** Next we study the truncated Max-product Baskakov operators (see [5, p. 11])

$$U_N^{(M)}(f)(x) = \frac{\sum_{k=0}^{N} b_{N,k}(x)f(\frac{k}{N})}{\sum_{k=0}^{N} b_{N,k}(x)}, \quad x \in [0,1], \; f \in C_+([0,1]), \; N \in \mathbb{N}, \quad (3.10)$$

where

$$b_{N,k}(x) = \frac{(N+k-1)}{k} \frac{x^k}{(1+x)^{\alpha+n}}. \quad (3.11)$$
From [5, pp. 217-218], we get \((x \in [0,1])\)

\[
U^{(M)}_N (|x|) (x) \leq \frac{2\sqrt{3} (\sqrt{2}+2)}{\sqrt{N+1}}, \quad N \geq 2, \; N \in \mathbb{N}.
\]

Let \(\lambda \geq 1\), clearly then it holds

\[
U^{(M)}_N (|x|^\lambda) (x) \leq \frac{2\sqrt{3} (\sqrt{2}+2)}{\sqrt{N+1}}, \quad \forall \; N \geq 2, \; N \in \mathbb{N}.
\]

Also it holds \(U^{(M)}_N (1) = 1\), and \(U^{(M)}_N\) are positive sublinear operators from \(C_+ ([0,1])\) into itself. Furthermore it holds

\[
U^{(M)}_N (|x|^\lambda) (x) > 0, \quad \forall \; x \in (0,1], \; \forall \; \lambda \geq 1, \; \forall \; N \in \mathbb{N}.
\]

We give

**Theorem 3.6.** Let \(0 < \alpha < 1\), any \(x \in (0,1]\), \(f \in AC ([0,1], \mathbb{R}_+)\), and \(f' \in L_\infty ([0,1])\). Then

\[
\left| U^{(M)}_N (f) (x) - f (x) \right| \leq \frac{(\alpha+2) \omega_1 \left( D^\alpha_+ f \left( \frac{2\sqrt{3} (\sqrt{2}+2)}{\sqrt{N+1}} \right) \right)^{\frac{1}{\alpha+1}} \cdot \left( \frac{2\sqrt{3} (\sqrt{2}+2)}{\sqrt{N+1}} \right)^{\frac{\alpha}{\alpha+1}}}{1 \cdot (\alpha+2)} ,
\]

\(\forall \; N \geq 2, \; N \in \mathbb{N}.

As \(N \to +\infty\), we get \(U^{(M)}_N (f) (x) \to f (x)\), for any \(x \in (0,1]\).

**Proof.** By Theorem 2.16

We continue with

**Remark 3.7.** Here we study the Max-product Meyer-König and Zeller operators (see [5, p. 11]) defined by

\[
Z^{(M)}_N (f) (x) = \frac{\int_{k=0}^{\infty} s_{N,k} (x) \cdot f \left( \frac{k}{N+k} \right)}{\int_{k=0}^{\infty} s_{N,k} (x)} , \quad \forall N \in \mathbb{N}, \; f \in C_+ ([0,1]),
\]

\(s_{N,k} (x) = (\frac{N+k}{k}) \cdot x^k , \; x \in [0,1].\)

By [5, p. 253], we get that

\[
Z^{(M)}_N (|x|) (x) \leq \frac{8(1+\sqrt{5})}{3} \sqrt{\frac{1-x}{N}} , \quad \forall \; x \in [0,1], \; \forall \; N \geq 4, \; N \in \mathbb{N}.
\]

As before we get that (for \(\lambda \geq 1\))

\[
Z^{(M)}_N (|x|^\lambda) (x) \leq \frac{8(1+\sqrt{5})}{3} \sqrt{\frac{1-x}{N}} := \rho (x) ,
\]

\(\forall \; x \in [0,1], \; N \geq 4, \; N \in \mathbb{N}.

Also it holds $Z^M_N(1) = 1$, and $Z^M_N$ are positive sublinear operators from $C_+([0,1])$ into itself. Also it holds
\begin{equation}
Z^M_N(|\cdot - x|^\lambda)(x) > 0, \quad \forall \ x \in (0,1), \ \forall \ \lambda \geq 1, \ \forall \ N \in \mathbb{N}.
\end{equation}

We give
\begin{equation}
\text{Theorem 3.8. Let } 0 < \alpha < 1, \text{ any } x \in (0,1), \ f \in AC([0,1],\mathbb{R}_+), \text{ and } f' \in L_\infty([0,1]). \text{ Then}
\end{equation}
\begin{equation}
|Z^M_N(f)(x) - f(x)| \leq \frac{(\alpha + 2) \omega_1(D_\alpha f)(\rho(x))^{\alpha + 1}}{\Gamma(\alpha + 2)(\rho(x))^{\alpha + 1}}
\end{equation}
\forall \ N \geq 4, \ N \in \mathbb{N}.
As $N \to +\infty$, we get $Z^M_N(f)(x) \to f(x)$, for any $x \in (0,1)$.
\begin{proof}
By Theorem 2.16.
\end{proof}

We continue with
\begin{equation}
\text{Remark 3.9. Here we deal with the Max-product truncated sampling operators (see [5, p. 13]) defined by}
\end{equation}
\begin{equation}
W^M_N(f)(x) = \bigvee_{k=0}^{N} \frac{\sin(Nx - k\pi)}{N_x - k\pi} f \left( \frac{k\pi}{N} \right),
\end{equation}
and
\begin{equation}
K^M_N(f)(x) = \bigvee_{k=0}^{N} \frac{\sin^2(Nx - k\pi)}{(N_x - k\pi)^2} f \left( \frac{k\pi}{N} \right),
\end{equation}
\forall \ x \in [0,\pi], \ f : [0, \pi] \to \mathbb{R}_+, \text{ a continuous function.}

Following [5] p. 343, and making the convention $\frac{\sin(0)}{0} = 1$ and denoting $s_{N,k}(x) = \frac{\sin(Nx - k\pi)}{N_x - k\pi}$, we get that $s_{N,k}\left(\frac{k\pi}{N}\right) = 1$, and $s_{N,k}\left(\frac{j\pi}{N}\right) = 0$, if $k \neq j$, furthermore $W^M_N(f)\left(\frac{j\pi}{N}\right) = f\left(\frac{j\pi}{N}\right)$, for all $j \in \{0, ..., N\}$.

Clearly $W^M_N(f)$ is a well-defined function for all $x \in [0,\pi]$, and it is continuous on $[0,\pi]$, also $W^M_N(1) = 1$.

By [5] p. 344, $W^M_N$ are positive sublinear operators.

Call $I^+_N(x) = \{k \in \{0,1,...,N\} : s_{N,k}(x) > 0\}$, and set $x_{N,k} := \frac{k\pi}{N}$, $k \in \{0,1,...,N\}$.

We see that
\begin{equation}
W^M_N(f)(x) = \frac{\bigvee_{k \in I^+_N(x)} s_{N,k}(x) f(x_{N,k})}{\bigvee_{k \in I^+_N(x)} s_{N,k}(x)}.
\end{equation}
By [5, p. 346], we have
\[
W_N^{(M)} (| - x|) (x) \leq \frac{\pi^{\lambda - 1}}{2N}, \quad \forall \ N \in \mathbb{N}, \forall \ x \in [0, \pi].
\]
Notice also \( |x_{N,k} - x| \leq \pi, \forall \ x \in [0, \pi] \).

Therefore (\( \lambda \geq 1 \)) it holds
\[
W_N^{(M)} (| - x|^\lambda) (x) \leq \frac{\pi^{\lambda - 1} \pi}{2N}, \quad \forall \ x \in [0, \pi], \forall \ N \in \mathbb{N}.
\]
If \( x \in \left( \frac{j\pi}{N}, \frac{(j+1)\pi}{N} \right) \), with \( j \in \{0, 1, ..., N\} \), we obtain \( nx - j\pi \in (0, \pi) \) and thus \( s_{N,j} (x) = \frac{\sin(Nx - j\pi)}{Nx - j\pi} > 0 \), see [5] pp. 343-344.

Consequently it holds (\( \lambda \geq 1 \))
\[
W_N^{(M)} (| - x|^\lambda) (x) = \frac{\bigvee_{k \in I_N^\alpha(x)} s_{N,k}(x) |x_{N,k} - x|^\lambda}{\bigvee_{k \in I_N^\alpha(x)} s_{N,k}(x)} > 0, \quad \forall \ x \in [0, \pi],
\]
such that \( x \neq x_{N,k} \), for any \( k \in \{0, 1, ..., N\} \).

We give

**Theorem 3.10.** Let \( 0 < \alpha < 1 \), any \( x \in [0, \pi] \) be such that \( x \neq \frac{k\pi}{N}, k \in \{0, 1, ..., N\}, \forall \ N \in \mathbb{N}; f \in AC ([0, \pi], \mathbb{R}_+) \), and \( f' \in L_\infty ([0, \pi]) \). Then
\[
|W_N^{(M)} (f) (x) - f (x)| \leq \frac{(\alpha+2)\omega_1}{1(\alpha+2)} \frac{\left(\frac{\sin^2(Nx - k\pi)}{N} f\left(\frac{k\pi}{N}\right)\right)^{\frac{1}{\pi}}}{\left(\frac{\pi^{\lambda - 1}}{2N}\right)^{\frac{\alpha}{\pi + 1}}}, \quad \forall \ N \in \mathbb{N}.
\]

As \( N \to +\infty \), we get \( W_N^{(M)} (f) (x) \to f (x) \).

**Proof.** By Theorem 2.10

We make

**Remark 3.11.** Here we continue with the Max-product truncated sampling operators (see [5, p. 13]) defined by
\[
K_N^{(M)} (f) (x) = \frac{\sqrt{N}}{\sum_{k=0}^{N} \sin^2(Nx - k\pi) f\left(\frac{k\pi}{N}\right)}
\]
\( \forall \ x \in [0, \pi], f : [0, \pi] \to \mathbb{R}_+ \), a continuous function.

Following [5, p. 350], and making the convention \( \frac{\sin(0)}{0} = 1 \) and denoting \( s_{N,k} (x) = \frac{\sin^2(Nx - k\pi)}{(Nx - k\pi)^2} \), we get that \( s_{N,k} \left( \frac{k\pi}{N} \right) = 1 \), and \( s_{N,k} \left( \frac{j\pi}{N} \right) = 0 \), if \( k \neq j \), furthermore \( K_N^{(M)} (f) \left( \frac{k\pi}{N} \right) = f \left( \frac{k\pi}{N} \right) \), for all \( j \in \{0, ..., N\} \).

Since \( s_{N,j} \left( \frac{j\pi}{N} \right) = 1 \) it follows that \( \bigvee_{k=0}^{N} s_{N,k} \left( \frac{j\pi}{N} \right) \geq 1 > 0 \), for all \( j \in \{0, 1, ..., N\} \). Hence \( K_N^{(M)} (f) \) is well-defined function for all \( x \in [0, \pi] \), and it
is continuous on $[0, \pi]$, also $K_N^{(M)}(1) = 1$. By \[5\] p. 350, $K_N^{(M)}$ are positive sublinear operators.

Denote $x_{N,k} := \frac{k\pi}{N}$, $k \in \{0, 1, \ldots, N\}$.

By \[3\] p. 352, we have
\[
K_N^{(M)}(|\cdot - x|)(x) \leq \frac{\pi}{2N}, \quad \forall \, N \in \mathbb{N}, \quad \forall \, x \in [0, \pi].
\]
Notice also $|x_{N,k} - x| \leq \pi$, $\forall \, x \in [0, \pi]$.

Therefore ($\lambda \geq 1$) it holds
\[
K_N^{(M)}(|\cdot - x|^\lambda)(x) \leq \left(\frac{\pi}{2N}\right)^{\lambda}, \quad \forall \, x \in [0, \pi], \quad \forall \, N \in \mathbb{N}.
\]

If $x \in \left(\frac{j\pi}{N}, \frac{(j+1)\pi}{N}\right)$, with $j \in \{0, 1, \ldots, N\}$, we obtain $nx - j\pi \in (0, \pi)$ and thus $s_{N,j}(x) = \frac{\sin^2(Nx - j\pi)}{(Nx - j\pi)^2} > 0$, see \[5\] pp. 350.

Consequently it holds ($\lambda \geq 1$)
\[
K_N^{(M)}(|\cdot - x|^\lambda)(x) = \frac{\sum_{k=0}^{N} s_{N,k}(x)|x_{N,k} - x|^\lambda}{\sum_{k=0}^{N} s_{N,k}(x)} > 0, \quad \forall \, x \in [0, \pi],
\]
such that $x \neq x_{N,k}$, for any $k \in \{0, 1, \ldots, N\}$.

We give

**Theorem 3.12.** Let $0 < \alpha < 1$, $x \in [0, \pi]$ be such that $x \neq \frac{k\pi}{N}$, $k \in \{0, 1, \ldots, N\}$, $\forall \, N \in \mathbb{N}$; $f \in AC([0, \pi], \mathbb{R}_+)$, and $f' \in L_\infty([0, \pi])$. Then
\[
|K_N^{(M)}(f)(x) - f(x)| \leq \frac{(\alpha + 2)\omega_1 \left(D_\alpha^\ast f \left(\frac{x^{\alpha+1}}{2N}\right)\right) + \frac{1}{1(\alpha+2)}}{\left(\frac{\alpha+1}{2N}\right)^{\frac{\alpha+1}{\alpha+1}}} \cdot \frac{3^{\alpha+1}}{\sqrt{\pi}} = \frac{10 \sqrt{\pi} \omega_1 \left(D_\alpha^\ast f \left(\frac{3N}{\sqrt{\pi}}\right)\right)}{3\sqrt{\pi} \sqrt{N+1}}, \quad \forall \, N \in \mathbb{N}.
\]

**Proof.** By Theorem 2.16. \hfill \Box

When $\alpha = \frac{1}{2}$ we get:

**Corollary 3.13.** Let $f \in AC([0, 1], \mathbb{R}_+)$, $f' \in L_\infty([0, 1])$. Then
\[
|B_N^{(M)}(f)(x) - f(x)| \leq \frac{10 \sqrt{\pi} \omega_1 \left(D_{1/2}^\ast f \left(3\pi\right)\right)}{3\sqrt{\pi} \sqrt{x}} = \frac{10 \sqrt{\pi} \omega_1 \left(D_{1/2}^\ast f \left(\frac{3\pi}{\sqrt{N+1}}\right)\right)}{3\sqrt{\pi} \sqrt{N+1}}, \quad \forall \, x \in (0, 1), \quad \forall \, N \in \mathbb{N}.
\]

**Proof.** By Theorem 3.2. \hfill \Box

Due to lack of space we avoid to give other applications when $\alpha = \frac{1}{2}$ from the other Max-product operators.

II) Case $\alpha > 1$, $\alpha \notin \mathbb{N}$.

Here we apply Theorem 2.14 to well known Max-product operators.

We present
Theorem 3.14. Let $\alpha > 1$, $\alpha \notin \mathbb{N}$, $m = [\alpha]$, $x \in [0, 1]$, $f \in AC^m([0, 1], \mathbb{R})$, and $f^{(m)} \in L_\infty([0, 1])$. Furthermore we assume that $f^{(k)}(x) = 0$, $k = 1, \ldots, m - 1$. Then

\begin{equation}
\left| B_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1 \left( D_\alpha f \left( \frac{6}{\sqrt{N+1}} \right) \right)^{\frac{1}{\alpha+1}}}{1^{(\alpha+1)}} \cdot \left[ \frac{6}{\sqrt{N+1}} + \frac{1}{(\alpha+1)} \left( \frac{6}{\sqrt{N+1}} \right)^{\frac{\alpha}{\alpha+1}} \right],
\end{equation}

$\forall N \in \mathbb{N}$.

We get $\lim_{N \to +\infty} B_N^{(M)}(f)(x) = f(x)$.

Proof. Applying (2.30) for $B_N^{(M)}$ and using (3.2), we get

\begin{equation}
\left| B_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1(D_\alpha f, \delta)}{1^{(\alpha+1)}} \cdot \left[ \frac{6}{\sqrt{N+1}} + \frac{1}{(\alpha+1)} \left( \frac{6}{\sqrt{N+1}} \right)^{\frac{\alpha}{\alpha+1}} \right].
\end{equation}

Choose $\delta = \left( \frac{6}{\sqrt{N+1}} \right)^{\frac{1}{\alpha+1}}$, then $\delta^{\alpha+1} = \frac{6}{\sqrt{N+1}}$, and apply it to (3.35). Clearly we derive (3.34). $\square$

We continue with

Theorem 3.15. Same assumptions as in Theorem 3.14. Then

\begin{equation}
\left| T_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1 \left( D_\alpha f \left( \frac{3}{\sqrt{N+1}} \right) \right)^{\frac{1}{\alpha+1}}}{1^{(\alpha+1)}} \cdot \left[ \frac{3}{\sqrt{N+1}} + \frac{1}{(\alpha+1)} \left( \frac{3}{\sqrt{N+1}} \right)^{\frac{\alpha}{\alpha+1}} \right],
\end{equation}

$\forall N \in \mathbb{N}$.

We get $\lim_{N \to +\infty} T_N^{(M)}(f)(x) = f(x)$.

Proof. Use of Theorem 2.14 similar to the proof of Theorem 3.14. $\square$

We give

Theorem 3.16. Same assumptions as in Theorem 3.14. Then

\begin{equation}
\left| U_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1 \left( D_\alpha f \left( \frac{2\sqrt{2}\sqrt{2+2}}{\sqrt{N+1}} \right) \right)^{\frac{1}{\alpha+1}}}{1^{(\alpha+1)}} \cdot \left[ \frac{2\sqrt{2}\sqrt{2+2}}{\sqrt{N+1}} + \frac{1}{(\alpha+1)} \left( \frac{2\sqrt{2}\sqrt{2+2}}{\sqrt{N+1}} \right)^{\frac{\alpha}{\alpha+1}} \right],
\end{equation}

$\forall N \in \mathbb{N}$, $N \geq 2$.

We get $\lim_{N \to +\infty} U_N^{(M)}(f)(x) = f(x)$.

Proof. Use of Theorem 2.14 similar to the proof of Theorem 3.14. $\square$
Theorem 3.17. Same assumptions as in Theorem 3.14. Then
(3.38)
$$\left|Z_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1(D_2 f, (\rho(x))^{\frac{1}{\alpha+1}})}{1(\alpha+1)} \cdot \left[\rho(x) + \frac{1}{(\alpha+1)}(\rho(x))^{\frac{\alpha}{\alpha+1}}\right],$$
∀ \(N \in \mathbb{N}, N \geq 4\).

We get \(\lim_{N \to +\infty} Z_N^{(M)}(f)(x) = f(x),\) where \(\rho(x)\) is as in (3.18).

Proof. Use of Theorem 2.14 similar to the proof of Theorem 3.14. \(\square\)

We continue with

Theorem 3.18. Let \(\alpha > 1, \alpha \notin \mathbb{N}, m = [\alpha], x \in [0, \pi] \subset \mathbb{R}, f \in AC^m([0, \pi], \mathbb{R}_+),\) and \(f^{(m)} \in L_\infty([0, \pi])\). Furthermore we assume that \(f^{(k)}(x) = 0, k = 1, ..., m - 1.\) Then
(3.39)
$$\left|W_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1(D_2 f, (\frac{\alpha+1}{2N})^{\frac{1}{\alpha+1}})}{1(\alpha+1)} \cdot \left[\frac{\pi^\alpha}{2N} + \frac{1}{(\alpha+1)}(\frac{\pi^{\alpha+1}}{2N})^{\frac{\alpha}{\alpha+1}}\right],$$
∀ \(N \in \mathbb{N}.

We have that \(\lim_{N \to +\infty} W_N^{(M)}(f)(x) = f(x).\)

Proof. Applying (2.30) for \(W_N^{(M)}\) and using (3.25), we get
(3.40)
$$\left|W_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1(D_2 f, \delta)}{1(\alpha+1)} \left[\frac{\pi^\alpha}{2N} + \frac{\alpha+1}{2N(\alpha+1)}\right].$$

Choose \(\delta = (\frac{\alpha+1}{2N})^{\frac{1}{\alpha+1}}\), i.e. \(\delta^{\alpha+1} = \frac{\alpha+1}{2N}\), and \(\delta^\alpha = (\frac{\alpha+1}{2N})^{\frac{\alpha}{\alpha+1}}\). We use the last into (3.40) and we obtain (3.39). \(\square\)

We also have

Theorem 3.19. Let \(\alpha > 1, \alpha \notin \mathbb{N}, m = [\alpha], x \in [0, \pi] \subset \mathbb{R}, f \in AC^m([0, \pi], \mathbb{R}_+),\) and \(f^{(m)} \in L_\infty([0, \pi])\). Furthermore we assume that \(f^{(k)}(x) = 0, k = 1, ..., m - 1.\) Then
(3.41)
$$\left|K_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1(D_2 f, (\frac{\alpha+1}{2N})^{\frac{1}{\alpha+1}})}{1(\alpha+1)} \cdot \left[\frac{\pi^\alpha}{2N} + \frac{1}{(\alpha+1)}(\frac{\pi^{\alpha+1}}{2N})^{\frac{\alpha}{\alpha+1}}\right],$$
∀ \(N \in \mathbb{N}.

We have that \(\lim_{N \to +\infty} K_N^{(M)}(f)(x) = f(x).\)

Proof. As in Theorem 3.18. \(\square\)

We make
Remark 3.20. We mention the interpolation Hermite-Fejér polynomials on Chebyshev knots of the first kind (see [5, p. 4]): Let \( f : [-1,1] \to \mathbb{R} \) and based on the knots \( x_{N,k} = \cos \left( \frac{2(N+1)(k+1)}{2(N+1)} \pi \right) \in (-1,1) \), \( k \in \{0, ..., N\} \), \(-1 < x_{N,0} < x_{N,1} < ... < x_{N,N} < 1\), which are the roots of the first kind Chebyshev polynomial \( T_{N+1}(x) = \cos \left( (N+1) \arccos x \right) \), we define (see Fejér [9])

\[
H_{2N+1}^2(f)(x) = \sum_{k=0}^{N} h_{N,k}(x) f(x_{N,k}),
\]

where

\[
h_{N,k}(x) = (1 - x \cdot x_{N,k}) \left( \frac{T_{N+1}(x)}{(N+1)(x-x_{N,k})} \right)^2,
\]

the fundamental interpolation polynomials.

The Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind (see [5, p. 12]) are defined by

\[
H_{2N+1}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x)f(x_{N,k})}{\bigvee_{k=0}^{N} h_{N,k}(x)}, \quad \forall \ N \in \mathbb{N},
\]

where \( f : [-1,1] \to \mathbb{R}_+ \) is continuous.

Call

\[
E_N(x) := H_{2N+1}^{(M)}(|\cdot - x|)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x)|x_{N,k}-x|}{\bigvee_{k=0}^{N} h_{N,k}(x)}, \quad x \in [-1,1].
\]

Then by [5, p. 287], we obtain that

\[
E_N(x) \leq \frac{2\pi}{N+1}, \quad \forall \ x \in [-1,1], \ N \in \mathbb{N}.
\]

For \( m > 1 \), we get

\[
H_{2N+1}^{(M)}(|\cdot - x|^m)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x)|x_{N,k}-x|^m}{\bigvee_{k=0}^{N} h_{N,k}(x)} = \frac{\bigvee_{k=0}^{N} h_{N,k}(x)|x_{N,k}-x|^{m-1}}{\bigvee_{k=0}^{N} h_{N,k}(x)} \leq 2^{m-1} \frac{\bigvee_{k=0}^{N} h_{N,k}(x)|x_{N,k}-x|}{\bigvee_{k=0}^{N} h_{N,k}(x)} \leq \frac{2^{m-1}}{N+1}, \quad \forall \ x \in [-1,1], \ N \in \mathbb{N}.
\]

Hence it holds

\[
H_{2N+1}^{(M)}(|\cdot - x|^m)(x) \leq \frac{2^{m-1}}{N+1}, \quad \forall \ x \in [-1,1], \ m > 1, \ \forall \ N \in \mathbb{N}.
\]
Furthermore we have
\begin{equation}
H_{2N+1}^{(M)} (1) (x) = 1, \quad \forall \, x \in [-1,1],
\end{equation}
and \(H_{2N+1}^{(M)}\) maps continuous functions to continuous functions over \([-1,1]\) and for any \(x \in \mathbb{R}\) we have \(\frac{D}{(x) > 0}\)
We also have \(h_{N,k} (x_{N,k}) = 1\), and \(h_{N,k} (x_{N,j}) = 0\), if \(k \neq j\), furthermore it holds
\begin{equation}
H_{2N+1}^{(M)} (f) (x_{N,j}) = f (x_{N,j}), \quad \forall \, j \in \{0,...,N\}, \text{ see } \cite{5}, \text{ p. 282}.
\end{equation}
\(H_{2N+1}^{(M)}\) are positive sublinear operators, \cite{5}, p. 282.

We give

**Theorem 3.21.** Let \(\alpha > 1, \alpha \notin \mathbb{N}, m = [\alpha], x \in [-1,1], f \in AC^m (-1,1), \mathbb{R}_+\), and \(f^{(m)} \in L_\infty (-1,1)\). Furthermore we assume that \(f^{(k)} (x) = 0, k = 1,...,m - 1\). Then
\begin{equation}
\left| H_{2N+1}^{(M)} (f) (x) - f (x) \right| \leq \frac{\omega_x \left( D_{e f} (\frac{2^{\alpha+1}}{N+1}) \right)}{1/(\alpha+1)} \cdot \left[ \frac{2^{\alpha+1}}{N+1} + \frac{1}{(\alpha+1)} \left( \frac{2^{\alpha+1}}{N+1} \right)^{\frac{\alpha}{\alpha+1}} \right],
\end{equation}
\(\forall \, N \in \mathbb{N}\).
Furthermore it holds \(\lim_{N \to +\infty} H_{2N+1}^{(M)} (f) (x) = f (x)\).

**Proof.** By Theorem 2.14, choose \(\delta := \left( \frac{2^{\alpha+1}}{N+1} \right)^{\frac{1}{\alpha+1}}\), use (2.30), (3.48). □

We continue with

**Remark 3.22.** Here we deal with Lagrange interpolation polynomials on Chebyshev knots of second kind plus the endpoints \(\pm 1\) (see \cite{5}, p. 377). These polynomials are linear operators attached to \(f : [-1,1] \to \mathbb{R}\) and to the knots \(x_{N,k} = \cos \left( \frac{N-k}{N-1} \pi \right) \in [-1,1], k = 1,...,N, N \in \mathbb{N}\), which are the roots of \(\omega_N (x) = \sin (N-1) t \sin t, x = \cos t\). Notice that \(x_{N,1} = -1\) and \(x_{N,N} = 1\). Their formula is given by \cite{5}, p. 377
\begin{equation}
L_N (f) (x) = \sum_{k=1}^{N} l_{N,k} (x) f (x_{N,k}),
\end{equation}
where
\begin{equation}
l_{N,k} (x) = \frac{(-1)^{k-1} \omega_N (x)}{(1 + \delta_{k,1} + \delta_{k,N})(N-1)(x-x_{N,k})},
\end{equation}
\(N \geq 2, k = 1,...,N, \text{ and } \omega_N (x) = \prod_{k=1}^{N} (x-x_{N,k})\) and \(\delta_{i,j}\) denotes the Kronecker’s symbol, that is \(\delta_{i,j} = 1, \text{ if } i = j, \text{ and } \delta_{i,j} = 0, \text{ if } i \neq j\).
The Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints ±1, are defined by [5] p. 12

\[ L_N^{(M)}(f)(x) = \frac{\bigvee_{k=1}^{N} l_{N,k}(x) f(x_{N,k})}{\bigvee_{k=1}^{N} l_{N,k}(x)}, \quad x \in [-1,1], \]

where \( f : [-1,1] \rightarrow \mathbb{R}_+ \) continuous.

First we see that \( L_N^{(M)}(f)(x) \) is well defined and continuous for any \( x \in [-1,1] \). Following [5] p. 289, because \( \sum_{k=1}^{N} l_{N,k}(x) = 1, \) \( \forall x \in \mathbb{R} \), for any \( x \) there exists \( k \in \{1,\ldots,N\} : l_{N,k}(x) > 0 \), hence \( \bigvee_{k=1}^{N} l_{N,k}(x) > 0 \). We have that \( l_{N,k}(x_{N,k}) = 1 \), and \( l_{N,k}(x_{N,j}) = 0 \), if \( k \neq j \). Furthermore it holds \( L_N^{(M)}(f)(x_{N,j}) = f(x_{N,j}) \), all \( j \in \{1,\ldots,N\}, \) and \( L_N^{(M)}(1) = 1 \).

Call \( I_N^+(x) = \{k \in \{1,\ldots,N\} : l_{N,k}(x) > 0\} \), then \( I_N^+(x) \neq \emptyset \). So for \( f \in C_+([-1,1]) \) we get

\[ L_N^{(M)}(f)(x) = \frac{\bigvee_{k \in I_N^+(x)} l_{N,k}(x) f(x_{N,k})}{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)} \geq 0. \]

Notice here that \( |x_{N,k} - x| \leq 2, \forall x \in [-1,1] \).

By [5] p. 297, we get that

\[ L_N^{(M)}(|x| - x)(x) = \frac{\bigvee_{k=1}^{N} l_{N,k}(x)|x_{N,k} - x|}{\bigvee_{k=1}^{N} l_{N,k}(x)} = \frac{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)|x_{N,k} - x|}{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)} \leq \frac{\pi^2}{6(N-1)}, \]

\( N \geq 3, \forall x \in (-1,1), \) \( N \) is odd.

We get that \( \langle m > 1 \)

\[ L_N^{(M)}(|x| - x)^m(x) = \frac{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)|x_{N,k} - x|^m}{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)} \leq \frac{2^{m-1} \pi^2}{6(N-1)}, \]

\( N \geq 3 \) odd, \( \forall x \in (-1,1) \).

\( L_N^{(M)} \) are positive sublinear operators, [5] p. 290].

We give

**Theorem 3.23.** Same assumptions as in Theorem 3.21 Then

\[ \left| L_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1 \left( D^\alpha f \right) \left( \frac{2^\alpha \pi^2}{6(N-1)} \right)^{\frac{1}{\alpha+1}}} {\Gamma(\alpha+1)} \left[ \frac{2^{\alpha-1} \pi^2}{6(N-1)} + \frac{1}{\Gamma(\alpha+1)} \left( \frac{2^\alpha \pi^2}{6(N-1)} \right)^{\frac{1}{\alpha+1}} \right], \]
∀ \( N \in \mathbb{N} : N \geq 3 \), odd.

It holds

\[
\lim_{N \to +\infty} L_N^{(M)} (f) (x) = f (x).
\]

**Proof.** By Theorem 2.14, choose \( \delta := \left( \frac{2^{n+2}\pi^2}{6(N-1)} \right)^{\frac{1}{n+1}} \), use of (2.30) and (3.56). At \( \pm 1 \) the left hand side of (3.57) is zero, thus (3.57) is trivially true. \( \square \)

We make

**Remark 3.24.** Let \( f \in C_+ ([−1, 1]), N \geq 4, N \in \mathbb{N}, N \) even.

By [5, p. 298], we get

(3.58)  \[ L_N^{(M)} (|·−x|) (x) \leq \frac{4\pi^2}{3(N−1)} = \frac{2^{n+2}\pi^2}{3(N−1)}, \forall x \in (−1, 1). \]

Hence \( (m > 1) \)

(3.59)  \[ L_N^{(M)} (|·−x|^m) (x) \leq \frac{2^{m+1}\pi^2}{3(N−1)}, \forall x \in (−1, 1). \]

\( \square \)

We present

**Theorem 3.25.** Same assumptions as in Theorem 3.21. Then

(3.60)

\[
\left| L_N^{(M)} (f) (x) − f (x) \right| \leq \frac{\omega_1 \left( D^n f, \left( \frac{2^{n+2}\pi^2}{3(N−1)} \right)^{\frac{1}{n+1}} \right)}{1 (n+1)} \left[ \frac{2^{n+1}\pi^2}{3(N−1)} + \frac{1}{(n+1)} \left( \frac{2^{n+2}\pi^2}{3(N−1)} \right)^{\frac{n}{n+1}} \right],
\]

∀ \( N \in \mathbb{N}, N \geq 4, N \) is even.

It holds

\[
\lim_{N \to +\infty} L_N^{(M)} (f) (x) = f (x).
\]

**Proof.** By Theorem 2.14, choose \( \delta := \left( \frac{2^{n+2}\pi^2}{3(N−1)} \right)^{\frac{1}{n+1}} \), use of (2.30) and (3.59). At \( ±1 \), (3.60) is trivially true. \( \square \)

We need

**Definition 3.26.** ([6, p. 41]). Let \( I \subset \mathbb{R} \) be an interval of finite or infinite length, and \( f : I \to \mathbb{R} \) a bounded or uniformly continuous function. We define the first modulus of continuity

(3.61)  \[ \omega_1 (f, \delta)_I = \sup_{x,y \in I \atop |x−y| \leq \delta} |f (x) − f (y)|, \delta > 0. \]

Clearly, it holds \( \omega_1 (f, \delta)_I < +\infty. \)

We also have

(3.62)  \[ \omega_1 (f, r\delta)_I \leq (r + 1) \omega_1 (f, \delta)_I, \text{ any } r \geq 0. \]
CONVENTION 3.27. Let a real number \( m > 1 \), from now on we assume that \( D_{x_0}^m f \) is either bounded or uniformly continuous function on \( (-\infty, x_0] \), similarly from now on we assume that \( D_{x_0}^m f \) is either bounded or uniformly continuous function on \([x_0, +\infty)\).

We need

DEFINITION 3.28. Let \( D_{x_0}^m f \) (real number \( m > 1 \)) denote any of \( D_{x_0}^m f \), \( D_{x_0}^m f \) and \( \delta > 0 \). We set

\[
\omega_1 (D_{x_0}^m f, \delta) = \max \{ \omega_1 (D_{x_0}^m f, \delta)_{(-\infty, x_0]}, \omega_1 (D_{x_0}^m f, \delta)_{[x_0, +\infty)} \},
\]

where \( x_0 \in \mathbb{R} \). Notice that \( \omega_1 (D_{x_0}^m f, \delta)_{\mathbb{R}} < +\infty \).

We will use

THEOREM 3.29. Let the real number \( m > 0 \), \( m \notin \mathbb{N} \), \( \lambda = [m] \), \( x_0 \in \mathbb{R} \), \( f \in AC^\lambda([a, b], \mathbb{R}_+) \) (i.e. \( f^{(\lambda - 1)} \in AC[a, b] \), absolutely continuous functions on \([a, b] \), \( \forall [a, b] \subset \mathbb{R} \), and \( f^{(\lambda)} \in L_{\infty}(\mathbb{R}) \). Furthermore we assume that \( f^{(k)} (x_0) = 0 \), \( k = 1, \ldots, \lambda - 1 \). The Convention 3.27 is imposed here. Then

\[
|f(x) - f(x_0)| \leq \frac{\omega_1(D_{x_0}^m f, \delta)}{\Gamma(m+1)} \left[ |x - x_0|^m + \frac{|x-x_0|^{m+1}}{(m+1)\delta} \right], \quad \delta > 0,
\]

for all \( x \in \mathbb{R} \).

If \( 0 < m < 1 \), then we do not need initial conditions.

Proof. Similar to Theorem 2.10 \( \square \)

We continue with

REMARK 3.30. Let \( b : \mathbb{R} \to \mathbb{R}_+ \) be a centered (it takes a global maximum at 0) bell-shaped function, with compact support \([-T, T] \), \( T > 0 \) (that is \( b(x) > 0 \) for all \( x \in (-T, T) \)) and \( I = \int_{-T}^T b(x) \, dx > 0 \).

The Cardaliaguet-Euvrard neural network operators are defined by (see [4])

\[
C_{N,\alpha} (f) (x) = \sum_{k=\lfloor -N \rfloor}^{N^2} \frac{f^{(\frac{k}{N})}}{N^{1-\alpha}} b \left( N^{1-\alpha} \left( x - \frac{k}{N} \right) \right),
\]

\( 0 < \alpha < 1, N \in \mathbb{N} \) and typically here \( f : \mathbb{R} \to \mathbb{R} \) is continuous and bounded or uniformly continuous on \( \mathbb{R} \).

\( CB(\mathbb{R}) \) denotes the continuous and bounded function on \( \mathbb{R} \), and

\( CB_+(\mathbb{R}) = \{ f : \mathbb{R} \to [0, \infty); f \in CB(\mathbb{R}) \} \).

The corresponding max-product Cardaliaguet-Euvrard neural network operators will be given by

\[
C_{N,\alpha}^{(M)} (f) (x) = \sum_{k=\lfloor -N \rfloor}^{N^2} \frac{b(N^{1-\alpha}(x - \frac{k}{N}))f^{(\frac{k}{N})}}{\sum_{k=\lfloor -N \rfloor}^{N^2} b(N^{1-\alpha}(x - \frac{k}{N}))},
\]
$x \in \mathbb{R}$, typically here $f \in CB_+(\mathbb{R})$, see also [4].
Next we follow [4].
For any $x \in \mathbb{R}$, denoting
$$J_{T,N} (x) = \left\{ k \in \mathbb{Z}; \; -N^2 \leq k \leq N^2, N^{1-\alpha} \left( x - \frac{k}{N} \right) \in (-T,T) \right\},$$
we can write
$$(3.67) \quad C_{N,\alpha}^{(M)} (f) (x) = \frac{\bigvee_{k \in J_{T,N}(x)} b(N^{1-\alpha}(x-k))f(\frac{x-k}{N})}{\bigvee_{k \in J_{T,N}(x)} b(N^{1-\alpha}(x-k))},$$
$x \in \mathbb{R}$, $N > \max \{ T + |x| , T^{-\frac{1}{\alpha}} \}$, where $J_{T,N} (x) \neq \emptyset$. Indeed, we have
$$\forall k \in J_{T,N}(x) b \left( N^{1-\alpha} \left( x - \frac{k}{N} \right) \right) > 0, \forall x \in \mathbb{R} \text{ and } N > \max \{ T + |x| , T^{-\frac{1}{\alpha}} \}.$$
We have that $C_{N,\alpha}^{(M)} (1) (x) = 1, \forall x \in \mathbb{R}$ and $N > \max \{ T + |x| , T^{-\frac{1}{\alpha}} \}$. 

See in [4] Lemma 2.1, Corollary 2.2 and Remarks.

We need
**THEOREM 3.31.** [4]. Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T,T]$, $T > 0$, $0 < \alpha < 1$ and $C_{N,\alpha}^{(M)}$ be defined as in (3.66).

(i) If $|f(x)| \leq c$ for all $x \in \mathbb{R}$ then $C_{N,\alpha}^{(M)} (f) (x) \leq c$, for all $x \in \mathbb{R}$ and $N > \max \{ T + |x| , T^{-\frac{1}{\alpha}} \}$ and $C_{N,\alpha}^{(M)} (f) (x)$ is continuous at any point $x \in \mathbb{R}$, for all $N > \max \{ T + |x| , T^{-\frac{1}{\alpha}} \}$;

(ii) If $f,g \in CB_+ (\mathbb{R})$ satisfy $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, then $C_{N,\alpha}^{(M)} (f) (x) \leq C_{N,\alpha}^{(M)} (g) (x)$ for all $x \in \mathbb{R}$ and $N > \max \{ T + |x| , T^{-\frac{1}{\alpha}} \}$;

(iii) $C_{N,\alpha}^{(M)} (f + g) (x) \leq C_{N,\alpha}^{(M)} (f) (x) + C_{N,\alpha}^{(M)} (g) (x)$ for all $f,g \in CB_+ (\mathbb{R})$, $x \in \mathbb{R}$ and $N > \max \{ T + |x| , T^{-\frac{1}{\alpha}} \}$;

(iv) For all $f,g \in CB_+ (\mathbb{R})$, $x \in \mathbb{R}$ and $N > \max \{ T + |x| , T^{-\frac{1}{\alpha}} \}$, we have
$$\left| C_{N,\alpha}^{(M)} (f) (x) - C_{N,\alpha}^{(M)} (g) (x) \right| \leq C_{N,\alpha}^{(M)} (|f - g|) (x);$$

(v) $C_{N,\alpha}^{(M)}$ is positive homogeneous, that is $C_{N,\alpha}^{(M)} (\lambda f) (x) = \lambda C_{N,\alpha}^{(M)} (f) (x)$ for all $\lambda \geq 0, x \in \mathbb{R}$, $N > \max \{ T + |x| , T^{-\frac{1}{\alpha}} \}$ and $f \in CB_+ (\mathbb{R})$.

We make
**REMARK 3.32.** We have that
$$(3.68) \quad E_{N,\alpha} (x) := C_{N,\alpha}^{(M)} (| \cdot - x |) (x) = \frac{\bigvee_{k \in J_{T,N}(x)} b(N^{1-\alpha}(x-k))|x-k|}{\bigvee_{k \in J_{T,N}(x)} b(N^{1-\alpha}(x-k))},$$
∀ \( x \in \mathbb{R} \), and \( N > \max \left\{ T + |x|, T^{\frac{1}{\alpha}} \right\} \).

We mention from [4] the following:

**Theorem 3.33.** [4]. Let \( b(x) \) be a centered bell-shaped function, continuous and with compact support \([-T, T] \), \( T > 0 \) and \( 0 < \alpha < 1 \). In addition, suppose that the following requirements are fulfilled:

(i) There exist \( 0 < m_1 \leq M_1 < \infty \) such that \( m_1 (T - x) \leq b(x) \leq M_1 (T - x) \), \( \forall \ x \in [0, T] \);  
(ii) There exist \( 0 < m_2 \leq M_2 < \infty \) such that \( m_2 (x + T) \leq b(x) \leq M_2 (x + T) \), \( \forall \ x \in [-T, 0] \).

Then for all \( f \in CB^+(\mathbb{R}) \), \( x \in \mathbb{R} \) and for all \( N \in \mathbb{N} \) satisfying \( N > \max \left\{ T + |x|, (\frac{2T}{\alpha})^{\frac{1}{\alpha}} \right\} \), we have the estimate

\[
|C_{N,\alpha}^{(M)} (f) (x) - f (x)| \leq c \omega_1 (f, N^{-1})_{\mathbb{R}},
\]

where

\[
c := 2 \left( \max \left\{ \frac{TM_2}{2m_2}, \frac{TM_1}{2m_1} \right\} + 1 \right),
\]

and

\[
\omega_1 (f, \delta)_{\mathbb{R}} := \sup_{x, y \in \mathbb{R} : \ |x - y| \leq \delta} |f (x) - f (y)|.
\]

We make

**Remark 3.34.** In [4], was proved that

\[
E_{N,\alpha} (x) \leq \max \left\{ \frac{TM_2}{2m_2}, \frac{TM_1}{2m_1} \right\} N^{\alpha - 1}, \ \forall \ N > \max \left\{ T + |x|, (\frac{2T}{\alpha})^{\frac{1}{\alpha}} \right\}.
\]

That is

\[
C_{N,\alpha}^{(M)} (| \cdot - x |) (x) \leq \max \left\{ \frac{TM_2}{2m_2}, \frac{TM_1}{2m_1} \right\} N^{\alpha - 1}, \ \forall \ N > \max \left\{ T + |x|, (\frac{2T}{\alpha})^{\frac{1}{\alpha}} \right\}.
\]

From (3.68) we have that \( |x - \frac{k}{N}| \leq \frac{T}{N^{1-\alpha}} \).

Hence \( (m > 1) \) (\( \forall \ x \in \mathbb{R} \) and \( N > \max \left\{ T + |x|, (\frac{2T}{\alpha})^{\frac{1}{\alpha}} \right\} \))

\[
C_{N,\alpha}^{(M)} (| \cdot - x |^m) (x) = \frac{\sum_{k \in J_{T, N}(x)} b(N^{1-\alpha}(x - \frac{k}{N})) |x - \frac{k}{N}|^m}{\sum_{k \in J_{T, N}(x)} b(N^{1-\alpha}(x - \frac{k}{N}))} \leq \left( \frac{T}{N^{1-\alpha}} \right)^{m-1} \max \left\{ \frac{TM_2}{2m_2}, \frac{TM_1}{2m_1} \right\} N^{\alpha - 1},
\]

where

\[
\forall \ N > \max \left\{ T + |x|, (\frac{2T}{\alpha})^{\frac{1}{\alpha}} \right\}.
\]
Then \((m > 1)\) it holds
\[
C_{N,\alpha}^{(M)}(\cdot - x|^{m})(x) \leq T^{m-1} \max \left\{ \frac{T M_2}{2 m_2}, \frac{T M_1}{2 m_1} \right\} \frac{1}{N^{m(1-\alpha)}},
\]
(3.73) \(\forall \; N > \max \left\{ T + |x|, \left( \frac{2}{T} \right)^{\frac{1}{m}} \right\} \).

Call
(3.74) \[ \theta := \max \left\{ \frac{T M_2}{2 m_2}, \frac{T M_1}{2 m_1} \right\} > 0. \]

Consequently \((m > 1)\) we derive
(3.75) \[ C_{N,\alpha}^{(M)}(\cdot - x|^{m})(x) \leq \frac{\theta T^{m-1}}{N^{m(1-\alpha)}}, \quad \forall \; N > \max \left\{ T + |x|, \left( \frac{2}{T} \right)^{\frac{1}{m}} \right\}. \]

We need

**Theorem 3.35.** All here as in Theorem 3.29 where \(x = x_0 \in \mathbb{R}\) is fixed. Let \(b\) be a centered bell-shaped function, continuous and with compact support \([-T, T]\), \(T > 0\), \(0 < \alpha < 1\) and \(C_{N,\alpha}^{(M)}\) be defined as in \(3.66\). Then
\[
\left| C_{N,\alpha}^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1(D_{m}^{\alpha} f, \delta)_{\mathbb{R}}}{1(m+1)} \left[ C_{N,\alpha}^{(M)}(\cdot - x|^{m})(x) + \frac{C_{N,\alpha}^{(M)}(|-x|^{m+1})(x)}{(m+1)\delta} \right],
\]
\(\forall \; N \in \mathbb{N} : N > \max \left\{ T + |x|, T^{-\frac{1}{m}} \right\} \).

**Proof.** By Theorem 3.29 and (3.64) we get
(3.76) \[ |f(\cdot) - f(x)| \leq \frac{\omega_1(D_{m}^{\alpha} f, \delta)_{\mathbb{R}}}{1(m+1)} \left[ |\cdot - x|^{m} + \frac{|\cdot - x|^{m+1}}{(m+1)\delta} \right], \quad \delta > 0, \]
true over \(\mathbb{R}\).

As in Theorem 3.31 and using similar reasoning and \(C_{N,\alpha}^{(M)}(1) = 1\), we get
(3.77) \[ \left| C_{N,\alpha}^{(M)}(f)(x) - f(x) \right| \leq C_{N,\alpha}^{(M)}(|f(\cdot) - f(x)|)(x) \leq \frac{\omega_1(D_{m}^{\alpha} f, \delta)_{\mathbb{R}}}{1(m+1)} \left[ C_{N,\alpha}^{(M)}(\cdot - x|^{m})(x) + \frac{C_{N,\alpha}^{(M)}(|-x|^{m+1})(x)}{(m+1)\delta} \right], \]
\(\forall \; N \in \mathbb{N} : N > \max \left\{ T + |x|, T^{-\frac{1}{m}} \right\} \). \(\Box\)

We continue with

**Theorem 3.36.** Here all as in Theorem 3.29 where \(x = x_0 \in \mathbb{R}\) is fixed and \(m > 1\). Also the same assumptions as in Theorem 3.33. Then
(3.78) \[ \left| C_{N,\alpha}^{(M)}(f)(x) - f(x) \right| \leq \frac{1}{1(m+1)} \omega_1 \left( D_{x}^{m} f, \left( \frac{\theta T^{m}}{N^{m(1-\alpha)} (1-\alpha)} \right) \frac{1}{m+1} \right)_{\mathbb{R}}, 
\]
\[ \cdot \left[ \frac{\theta T^{m-1}}{N^{m(1-\alpha)}} + \frac{1}{(m+1)} \left( \frac{\theta T^{m}}{N^{m(1-\alpha)} (1-\alpha)} \frac{1}{m+1} \right) \right], \]
\(\forall \; N \in \mathbb{N} : N > \max \left\{ T + |x|, T^{-\frac{1}{m}} \right\}. \)
\[ \forall N \in \mathbb{N} : N > \max \left\{ T + |x|, \left( \frac{2}{T} \right)^{\frac{1}{\alpha}} \right\}. \]

We have that \( \lim_{N \to +\infty} C_{N,\alpha}^{(M)} (f) (x) = f (x). \)

**Proof.** We apply Theorem 3.35. In (3.35) we choose
\[ \delta := \left( \frac{\theta T^m}{N^{(m+1)(1-\alpha)}} \right)^{\frac{1}{m+1}}, \]
thus \( \delta^{m+1} = \frac{\theta T^m}{N^{(m+1)(1-\alpha)}}, \) and
\[ \delta^m = \left( \frac{\theta T^m}{N^{(m+1)(1-\alpha)}} \right)^{\frac{m}{m+1}}. \]

Therefore we have
\[ \left| C_{N,\alpha}^{(M)} (f) (x) - f (x) \right| \leq \frac{1}{\Gamma (m+1)} \omega_1 \left( D^m_x f, \left( \frac{\theta T^m}{N^{(m+1)(1-\alpha)}} \right)^{\frac{1}{m+1}} \right) \left[ \frac{\theta T^{m-1}}{N^{(m+1)(1-\alpha)}} + \frac{1}{(m+1)^{\alpha}} N^{(m+1)(1-\alpha)} \right] \]
\[ = \frac{1}{\Gamma (m+1)} \omega_1 \left( D^m_x f, \left( \frac{\theta T^m}{N^{(m+1)(1-\alpha)}} \right)^{\frac{1}{m+1}} \right) \left[ \frac{\theta T^{m-1}}{N^{(m+1)(1-\alpha)}} + \frac{1}{(m+1)^{\alpha}} \delta^{m+1} \right], \]
(3.79)
\[ \forall N \in \mathbb{N} : N > \max \left\{ T + |x|, \left( \frac{2}{T} \right)^{\frac{1}{\alpha}} \right\}, \] proving the inequality (3.78). \( \square \)

We finish with (case of \( \alpha = 1.5 \))

**Corollary 3.37.** Let \( x \in [0, 1], f \in AC^2 ([0, 1], \mathbb{R}_+) \) and \( f^{(2)} \in L_\infty ([0, 1]). \)
Assume that \( f'(x) = 0. \) Then
\[ \left| D_{N}^{(M)} (f) (x) - f (x) \right| \leq 4 \omega_1 \left( D_{x}^{1.5} f, \left( \frac{6}{\sqrt{N+1}} \right)^{\frac{3}{2}} \right) \left[ \frac{6}{\sqrt{N+1}} + \frac{2}{5} \left( \frac{6}{\sqrt{N+1}} \right)^{\frac{3}{2}} \right], \]
(3.81)
\[ \forall N \in \mathbb{N}. \]

**Proof.** By Theorem 3.14 apply (3.34). \( \square \)

Due to lack of space we do not give other example applications.

**REFERENCES**


Received by the editors: September 25, 2017. Accepted: October 12, 2018. Published online: February 17, 2019.