JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 47 (2018) no. 2, pp. 89–113
ictp.acad.ro/jnaat

CAPUTO FRACTIONAL APPROXIMATION BY SUBLINEAR OPERATORS

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Abstract. Here we consider the approximation of functions by sublinear positive operators with applications to a big variety of Max-Product operators under Caputo fractional differentiability. Our study is based on our general fractional results about positive sublinear operators. We produce Jackson type inequalities under simple initial conditions. So our approach is quantitative by producing inequalities with their right hand sides involving the modulus of continuity of fractional derivative of the function under approximation.

MSC 2010. 26A33, 41A17, 41A25, 41A36.
Keywords. positive sublinear operators, Max-product operators, modulus of continuity, Caputo fractional derivative.

1. INTRODUCTION

The main motivation here is the monograph by B. Bede, L. Coroianu and S. Gal [5, 2016].

Let $N \in \mathbb{N}$, the well-known Bernstein polynomials [11] are positive linear operators, defined by the formula

$$B_N(f)(x) = \sum_{k=0}^{N} \binom{N}{k} x^k (1-x)^{N-k} f\left(\tfrac{k}{N}\right), \quad x \in [0,1], \ f \in C([0,1]). \tag{1.1}$$

T. Popoviciu in [12] (1935), proved for $f \in C([0,1])$ that

$$|B_N(f)(x) - f(x)| \leq \tfrac{5}{4}\omega_1\left(f, \tfrac{1}{\sqrt{N}}\right), \quad \forall\, x \in [0,1], \tag{1.2}$$

where

$$\omega_1(f,\delta) = \sup_{\substack{x,y\in[a,b]:\\ |x-y|\leq\delta}} |f(x) - f(y)|, \quad \delta > 0, \tag{1.3}$$

is the first modulus of continuity, here $[a,b] = [0,1]$.

G.G. Lorentz in [11, p. 21] (1986), proved for $f \in C^1([0,1])$ that

$$|B_N(f)(x) - f(x)| \leq \tfrac{3}{4\sqrt{N}}\omega_1\left(f', \tfrac{1}{\sqrt{N}}\right), \quad \forall\, x \in [0,1], \tag{1.4}$$

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In [5, p. 1], the authors introduced the basic Max-product Bernstein operators,

$$B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} p_{N,k}(x)}, \quad N \in \mathbb{N}, \tag{1.5}$$

where $\bigvee$ stands for maximum, and $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$ and $f : [0,1] \to \mathbb{R}_+ = [0,\infty)$.

These are nonlinear and piecewise rational operators.

The authors in [5] studied similar such nonlinear operators such as: the Max-product Favard-Szász-Mirakjan operators and their truncated version, the Max-product Baskakov operators and their truncated version, also many other similar specific operators. The study in [5] is based on presented there general theory of sublinear operators. These Max-product operators tend to converge faster to the on hand function.

So we mention from [5, p. 30], that for $f : [0,1] \to \mathbb{R}_+$ continuous, we have the estimate

$$\left| B_N^{(M)}(f)(x) - f(x) \right| \leq 12\, \omega_1\left(f, \frac{1}{\sqrt{N+1}}\right), \quad \text{for all } N \in \mathbb{N}, \ x \in [0,1], \tag{1.6}$$

Also from [5, p. 36], we mention that for $f : [0,1] \to \mathbb{R}_+$ being concave function we get that

$$\left| B_N^{(M)}(f)(x) - f(x) \right| \leq 2\, \omega_1\left(f, \frac{1}{N}\right), \quad \text{for all } x \in [0,1], \tag{1.7}$$

a much faster convergence.

In this article we expand the study in [5] by considering Caputo fractional smoothness of functions. So our inequalities are with respect to $\omega_1(D^\alpha f, \delta)$, $\delta > 0$, where $D^\alpha f$ with $\alpha > 0$ is the Caputo fractional derivative.

2. MAIN RESULTS

We need

Definition 2.1. *Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see* [7, p. 49], [10], [13]*) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \quad \forall\, x \in [a,b], \tag{2.1}$$

where Γ is the gamma function $\Gamma(v) = \int_0^\infty e^{-t} t^{v-1} dt$, $v > 0$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall\, x \in [a,b]$.

Lemma 2.2. [2] *Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}([a,b])$ and $f^{(n)} \in L_\infty([a,b])$. Then $D_{*a}^{\nu} f(a) = 0$.*

We need

Definition 2.3. *(see also* [1], [8], [10]*) Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \quad \forall\, x \in [a,b]. \tag{2.2}$$

We set $D_{b-}^{0} f(x) = f(x)$.

Lemma 2.4. [2] *Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then $D_{b-}^{\alpha} f(b) = 0$.*

Convention 2.5. *We assume that*

$$D_{*x_0}^{a} f(x) = 0, \text{ for } x < x_0, \tag{2.3}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{2.4}$$

for all $x, x_0 \in [a,b]$.

We mention

Proposition 2.6. [2] *Let $f \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a,b]$.*

Proposition 2.7. [2] *Let $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(x)$ is continuous in $x \in [a,b]$.*

The modulus of continuity $\omega_1(f,\delta)$ is defined the same way for bounded functions, see (1.3), and it is finite.

We make

Remark 2.8. [2] Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then

$$\omega_1\left(D_{*a}^{\nu} f, \delta\right) \leq \frac{2\left\|f^{(n)}\right\|_\infty}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{2.5}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, then

$$\omega_1\left(D_{b-}^{\alpha} f, \delta\right) \leq \frac{2\left\|f^{(m)}\right\|_\infty}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{2.6}$$

That is $\omega_1\left(D_{*a}^{\nu} f, \delta\right)$, $\omega_1\left(D_{b-}^{\alpha} f, \delta\right)$ are finite.

Clearly, above $D_{*a}^{\nu} f$ and $D_{b-}^{\alpha} f$ are bounded, from

$$\left|D_{*a}^{\nu} f(x)\right| \leq \frac{\|f^{(n)}\|_\infty}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}, \quad \forall\, x \in [a,b], \tag{2.7}$$

see [2]. □

We need

Definition 2.9. *Let* $D_{x_0}^{\alpha}f$ *denote any of* $D_{x_0-}^{\alpha}f$, $D_{*x_0}^{\alpha}f$, *and* $\delta>0$. *We set*

$$\omega_1\left(D_{x_0}^{\alpha}f,\delta\right):=\max\left\{\omega_1\left(D_{x_0-}^{\alpha}f,\delta\right)_{[a,x_0]},\omega_1\left(D_{*x_0}^{\alpha}f,\delta\right)_{[x_0,b]}\right\}, \tag{2.8}$$

where $x_0\in[a,b]$. *Here the moduli of continuity are considered over* $[a,x_0]$ *and* $[x_0,b]$, *respectively.*

We need

Theorem 2.10. *Let* $\alpha>0$, $\alpha\notin\mathbb{N}$, $m=\lceil\alpha\rceil$, $x_0\in[a,b]\subset\mathbb{R}$, $f\in AC^m([a,b],\mathbb{R}_+)$ *(i.e.* $f^{(m-1)}\in AC([a,b])$, *absolutely continuous functions on* $[a,b]$*), and* $f^{(m)}\in L_\infty([a,b])$. *Furthermore we assume that* $f^{(k)}(x_0)=0$, $k=1,...,m-1$. *Then*

$$|f(x)-f(x_0)|\leq\frac{\omega_1\left(D_{x_0}^{\alpha}f,\delta\right)}{\Gamma(\alpha+1)}\left[|x-x_0|^{\alpha}+\frac{|x-x_0|^{\alpha+1}}{(\alpha+1)\delta}\right],\quad \delta>0, \tag{2.9}$$

for all $a\leq x\leq b$.

If $0<\alpha<1$, *then we do not need initial conditions.*

Proof. From [7, p. 54], we get by left Caputo Taylor formula that

$$f(x)=\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-z)^{\alpha-1}D_{*x_0}^{\alpha}f(z)\,dz, \tag{2.10}$$

for all $x_0\leq x\leq b$.

Also from [1], using the right Caputo fractional Taylor formula we get

$$f(x)=\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{1}{\Gamma(\alpha)}\int_{x}^{x_0}(z-x)^{\alpha-1}D_{x_0-}^{\alpha}f(z)\,dz, \tag{2.11}$$

for all $a\leq x\leq x_0$.

By the assumption $f^{(k)}(x_0)=0$, $k=1,...,m-1$, we get

$$f(x)-f(x_0)=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-z)^{\alpha-1}D_{*x_0}^{\alpha}f(z)\,dz, \tag{2.12}$$

for all $x_0\leq x\leq b$.

And it holds

$$f(x)-f(x_0)=\frac{1}{\Gamma(\alpha)}\int_{x}^{x_0}(z-x)^{\alpha-1}D_{x_0-}^{\alpha}f(z)\,dz, \tag{2.13}$$

for all $a\leq x\leq x_0$.

Notice that when $0<\alpha<1$, then $m=1$, and (2.12) and (2.13) are valid without initial conditions.

Since $D_{x_0-}^{\alpha}f(x_0)=D_{*x_0}^{\alpha}f(x_0)=0$, we get

$$f(x)-f(x_0)=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-z)^{\alpha-1}\left(\left(D_{*x_0}^{\alpha}f\right)(z)-D_{*x_0}^{\alpha}f(x_0)\right)dz, \tag{2.14}$$

$x_0\leq x\leq b$, and

$$f(x)-f(x_0)=\frac{1}{\Gamma(\alpha)}\int_{x}^{x_0}(z-x)^{\alpha-1}\left(D_{x_0-}^{\alpha}f(z)-D_{x_0-}^{\alpha}f(x_0)\right)dz, \tag{2.15}$$

$a \le x \le x_0$.

We have that ($x_0 \le x \le b$)

$$|f(x) - f(x_0)| \le$$
$$\le \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-z)^{\alpha-1} \left|\left(D_{*x_0}^{\alpha} f\right)(z) - D_{*x_0}^{\alpha} f(x_0)\right| dz$$
$$\underset{(\delta_1>0)}{\le} \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-z)^{\alpha-1} \omega_1 \left(D_{*x_0}^{\alpha} f, \frac{\delta_1 |z-x_0|}{\delta_1}\right)_{[x_0,b]} dz \tag{2.16}$$
$$\le \frac{\omega_1\left(D_{*x_0}^{\alpha} f, \delta_1\right)_{[x_0,b]}}{\Gamma(\alpha)} \int_{x_0}^{x} (x-z)^{\alpha-1} \left(1 + \frac{(z-x_0)}{\delta_1}\right) dz$$
$$= \frac{\omega_1\left(D_{*x_0}^{\alpha} f, \delta_1\right)_{[x_0,b]}}{\Gamma(\alpha)} \left[\frac{(x-x_0)^{\alpha}}{\alpha} + \frac{1}{\delta_1} \int_{x_0}^{x} (x-z)^{\alpha-1} (z-x_0)^{2-1} dz\right]$$
$$= \frac{\omega_1\left(D_{*x_0}^{\alpha} f, \delta_1\right)_{[x_0,b]}}{\Gamma(\alpha)} \left[\frac{(x-x_0)^{\alpha}}{\alpha} + \frac{1}{\delta_1} \frac{\Gamma(\alpha)\Gamma(2)}{\Gamma(\alpha+2)} (x-x_0)^{\alpha+1}\right] \tag{2.17}$$
$$= \frac{\omega_1\left(D_{*x_0}^{\alpha} f, \delta_1\right)_{[x_0,b]}}{\Gamma(\alpha)} \left[\frac{(x-x_0)^{\alpha}}{\alpha} + \frac{1}{\delta_1} \frac{1}{(\alpha+1)\alpha} (x-x_0)^{\alpha+1}\right]$$
$$= \frac{\omega_1\left(D_{*x_0}^{\alpha} f, \delta_1\right)_{[x_0,b]}}{\Gamma(\alpha+1)} \left[(x-x_0)^{\alpha} + \frac{(x-x_0)^{\alpha+1}}{(\alpha+1)\delta_1}\right].$$

We have proved that

$$|f(x) - f(x_0)| \le \frac{\omega_1\left(D_{*x_0}^{\alpha} f, \delta_1\right)_{[x_0,b]}}{\Gamma(\alpha+1)} \left[(x-x_0)^{\alpha} + \frac{(x-x_0)^{\alpha+1}}{(\alpha+1)\delta_1}\right], \tag{2.18}$$

$\delta_1 > 0$, and $x_0 \le x \le b$.

Similarly acting, we get ($a \le x \le x_0$)

$$|f(x) - f(x_0)| \le$$
$$\le \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (z-x)^{\alpha-1} \left|D_{x_0-}^{\alpha} f(z) - D_{x_0-}^{\alpha} f(x_0)\right| dz$$
$$= \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (z-x)^{\alpha-1} \omega_1 \left(D_{x_0-}^{\alpha} f, \frac{\delta_2 (x_0-z)}{\delta_2}\right)_{[a,x_0]} dz \tag{2.19}$$
$$\le \frac{\omega_1\left(D_{x_0-}^{\alpha} f, \delta_2\right)_{[a,x_0]}}{\Gamma(\alpha)} \left[\int_{x}^{x_0} (z-x)^{\alpha-1} \left(1 + \frac{x_0-z}{\delta_2}\right) dz\right]$$
$$= \frac{\omega_1\left(D_{x_0-}^{\alpha} f, \delta_2\right)_{[a,x_0]}}{\Gamma(\alpha)} \left[\frac{(x_0-x)^{\alpha}}{\alpha} + \frac{1}{\delta_2} \int_{x}^{x_0} (x_0-z)^{2-1} (z-x)^{\alpha-1} dz\right]$$
$$= \frac{\omega_1\left(D_{x_0-}^{\alpha} f, \delta_2\right)_{[a,x_0]}}{\Gamma(\alpha)} \left[\frac{(x_0-x)^{\alpha}}{\alpha} + \frac{1}{\delta_2} \frac{\Gamma(\alpha)\Gamma(2)}{\Gamma(\alpha+2)} (x_0-x)^{\alpha+1}\right]$$
$$= \frac{\omega_1\left(D_{x_0-}^{\alpha} f, \delta_2\right)_{[a,x_0]}}{\Gamma(\alpha)} \left[\frac{(x_0-x)^{\alpha}}{\alpha} + \frac{1}{\delta_2} \frac{(x_0-x)^{\alpha+1}}{(\alpha+1)\alpha}\right] \tag{2.20}$$
$$= \frac{\omega_1\left(D_{x_0-}^{\alpha} f, \delta_2\right)_{[a,x_0]}}{\Gamma(\alpha+1)} \left[(x_0-x)^{\alpha} + \frac{(x_0-x)^{\alpha+1}}{(\alpha+1)\delta_2}\right].$$

We have proved that

$$|f(x)-f(x_0)| \le \frac{\omega_1\left(D^{\alpha}_{x_0-}f,\delta_2\right)_{[a,x_0]}}{\Gamma(\alpha+1)}\left[(x_0-x)^{\alpha}+\frac{(x_0-x)^{\alpha+1}}{(\alpha+1)\delta_2}\right], \tag{2.21}$$

$\delta_2 > 0$, and $(a \le x \le x_0)$. Choosing $\delta = \delta_1 = \delta_2 > 0$, by (2.18) and (2.21), we get (2.9). □

We need

DEFINITION 2.11. *Here* $C_+([a,b]) := \{f : [a,b] \to \mathbb{R}_+,\ f \text{ continuous}\}$. *Let* $L_N : C_+([a,b]) \to C_+([a,b])$, *operators,* $\forall\ N \in \mathbb{N}$, *such that*

(i)

$$L_N(\alpha f) = \alpha L_N(f),\ \forall \alpha \ge 0, \forall f \in C_+([a,b]), \tag{2.22}$$

(ii) if $f, g \in C_+([a,b]) : f \le g$, *then*

$$L_N(f) \le L_N(g),\ \ \forall N \in \mathbb{N}, \tag{2.23}$$

(iii)

$$L_N(f+g) \le L_N(f) + L_N(g),\ \ \forall\ f, g \in C_+([a,b]). \tag{2.24}$$

We call $\{L_N\}_{N\in\mathbb{N}}$ *positive sublinear operators.*

We need a Hölder's type inequality, see next:

THEOREM 2.12. *(see* [3]*) Let* $L : C_+([a,b]) \to C_+([a,b])$, *be a positive sublinear operator and* $f, g \in C_+([a,b])$, *furthermore let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Assume that* $L((f(\cdot))^p)(s_*),\ L((g(\cdot))^q)(s_*) > 0$ *for some* $s_* \in [a,b]$. *Then*

$$L(f(\cdot)\,g(\cdot))(s_*) \le \left(L((f(\cdot))^p)(s_*)\right)^{\frac{1}{p}} \left(L((g(\cdot))^q)(s_*)\right)^{\frac{1}{q}}. \tag{2.25}$$

We make

REMARK 2.13. By [5, p. 17],we get: let $f, g \in C_+([a,b])$, then

$$|L_N(f)(x) - L_N(g)(x)| \le L_N(|f-g|)(x),\ \ \forall\ x \in [a,b]. \tag{2.26}$$

Furthermore, we also have that

$$|L_N(f)(x) - f(x)| \le L_N(|f(\cdot) - f(x)|)(x) + |f(x)|\,|L_N(e_0)(x) - 1|, \tag{2.27}$$

$\forall\ x \in [a,b]$; $e_0(t) = 1$.

From now on we assume that $L_N(1) = 1$. Hence it holds

$$|L_N(f)(x) - f(x)| \le L_N(|f(\cdot) - f(x)|)(x),\ \ \forall\ x \in [a,b]. \tag{2.28}$$

Using Theorem 2.10 and (2.9) with (2.28) we get:

$$\begin{aligned} &|L_N(f)(x_0) - f(x_0)| \le \\ &\le \frac{\omega_1\left(D^{\alpha}_{x_0}f,\delta\right)}{\Gamma(\alpha+1)}\left[L_N\left(|\cdot - x_0|^{\alpha}\right)(x_0) + \frac{L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)}{(\alpha+1)\delta}\right],\ \ \delta > 0. \end{aligned} \tag{2.29}$$

□

We have proved

THEOREM 2.14. *Let* $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $x_0 \in [a,b] \subset \mathbb{R}$, $f \in AC^m([a,b], \mathbb{R}_+)$, *and* $f^{(m)} \in L_\infty([a,b])$. *Furthermore we assume that* $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. *Let* $L_N : C_+([a,b]) \to C_+([a,b])$, $\forall N \in \mathbb{N}$, *be positive sublinear operators, such that* $L_N(1) = 1$, $\forall N \in \mathbb{N}$. *Then*

(2.30)
$$|L_N(f)(x_0) - f(x_0)| \leq \frac{\omega_1\left(D_{x_0}^{\alpha} f, \delta\right)}{\Gamma(\alpha+1)} \cdot \left[L_N\left(|\cdot - x_0|^{\alpha}\right)(x_0) + \frac{L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)}{(\alpha+1)\delta}\right]$$

$\delta > 0$, $\forall N \in \mathbb{N}$.

In particular (2.30) *is true for* $\alpha > 1$, $\alpha \notin \mathbb{N}$.

COROLLARY 2.15. *Let* $0 < \alpha < 1$, $x_0 \in [a,b] \subset \mathbb{R}$, $f \in AC([a,b], \mathbb{R}_+)$, *and* $f' \in L_\infty([a,b])$. *Let* $L_N : C_+([a,b]) \to C_+([a,b])$, $\forall N \in \mathbb{N}$, *be positive sublinear operators, such that* $L_N(1) = 1$, $\forall N \in \mathbb{N}$. *Then* (2.30) *is valid.*

We give

THEOREM 2.16. *Let* $0 < \alpha < 1$, $x_0 \in [a,b] \subset \mathbb{R}$, $f \in AC([a,b], \mathbb{R}_+)$, *and* $f' \in L_\infty([a,b])$. *Let* L_N *from* $C_+([a,b])$ *into itself be positive sublinear operators, such that* $L_N(1) = 1$, $\forall N \in \mathbb{N}$. *Assume that* $L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0) > 0$, $\forall N \in \mathbb{N}$. *Then*

$$|L_N(f)(x_0) - f(x_0)| \leq$$
$$\leq \frac{(\alpha+2)\omega_1\left(D_{x_0}^{\alpha} f, \left(L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+2)} \left(L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{\alpha}{\alpha+1}}. \tag{2.31}$$

Proof. By Theorem 2.12, see (2.25), we get

$$L_N\left(|\cdot - x_0|^{\alpha}\right)(x_0) \leq \left(L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{\alpha}{\alpha+1}}. \tag{2.32}$$

Choose

$$\delta := \left(L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{1}{\alpha+1}} > 0, \tag{2.33}$$

i.e. $\delta^{\alpha+1} = L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)$.

By (2.30) we obtain

$$|L_N(f)(x_0) - f(x_0)| \leq$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\omega_1\left(D_{x_0}^{\alpha} f, \left(L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{1}{\alpha+1}}\right) \cdot$$
$$\cdot \left[\left(L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{(\alpha+1)}\left(L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{\alpha}{\alpha+1}}\right] =$$

$$= \frac{\omega_1\left(D^{\alpha}_{x_0}f, \left(L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)} \cdot$$

$$\cdot \left(L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{\alpha}{\alpha+1}} \left[1 + \frac{1}{\alpha+1}\right] \tag{2.34}$$

$$= \frac{\omega_1\left(D^{\alpha}_{x_0}f, \left(L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)} \left(L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{\alpha}{\alpha+1}} \left(\frac{\alpha+2}{\alpha+1}\right)$$

$$= \frac{(\alpha+2)\omega_1\left(D^{\alpha}_{x_0}f, \left(L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+2)} \left(L_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{\alpha}{\alpha+1}},$$

proving (2.31). ☐

3. APPLICATIONS

I) Case $0 < \alpha < 1$.

Here we apply Theorem 2.16 to well known Max-product operators.

We make

REMARK 3.1. The Max-product Bernstein operators $B_N^{(M)}(f)(x)$ are defined by (1.5), see also [5, p. 10]; here $f : [0,1] \to \mathbb{R}_+$ is a continuous function.

We have $B_N^{(M)}(1) = 1$, and

$$B_N^{(M)}(|\cdot - x|)(x) \leq \frac{6}{\sqrt{N+1}}, \ \forall\, x \in [0,1], \forall\, N \in \mathbb{N}, \tag{3.1}$$

see [5, p. 31].

$B_N^{(M)}$ are positive sublinear operators and thus they possess the monotonicity property, also since $|\cdot - x| \leq 1$, then $|\cdot - x|^{\beta} \leq 1$, $\forall\, x \in [0,1]$, $\forall\, \beta > 0$.

Therefore it holds

$$B_N^{(M)}\left(|\cdot - x|^{1+\beta}\right)(x) \leq \frac{6}{\sqrt{N+1}}, \ \forall\, x \in [0,1], \forall\, N \in \mathbb{N}, \forall\, \beta > 0. \tag{3.2}$$

Furthermore, clearly it holds that

$$B_N^{(M)}\left(|\cdot - x|^{1+\beta}\right)(x) > 0, \forall\, N \in \mathbb{N}, \forall\, \beta \geq 0 \text{ and any } x \in (0,1). \tag{3.3}$$

The operator $B_N^{(M)}$ maps $C_+([0,1])$ into itself. ☐

We present

THEOREM 3.2. *Let $0 < \alpha < 1$, any $x \in (0,1)$, $f \in AC([0,1], \mathbb{R}_+)$, and $f' \in L_\infty([0,1])$. Then*

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \frac{(\alpha+2)\omega_1\left(D^{\alpha}_{x}f, \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+2)} \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{\alpha}{\alpha+1}}, \tag{3.4}$$

$\forall\, N \in \mathbb{N}$.

As $N \to +\infty$, we get $B_N^{(M)}(f)(x) \to f(x)$, any $x \in (0,1)$.

Proof. By Theorem 2.16 ☐

We continue with

REMARK 3.3. The truncated Favard-Szász-Mirakjan operators are given by

$$T_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} s_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} s_{N,k}(x)}, \quad x \in [0,1], \ N \in \mathbb{N}, \ f \in C_+([0,1]), \tag{3.5}$$

$s_{N,k}(x) = \frac{(Nx)^k}{k!}$, see also [5, p. 11].

By [5, p. 178-179],we get that

$$T_N^{(M)}(|\cdot - x|)(x) \leq \frac{3}{\sqrt{N}}, \quad \forall \ x \in [0,1], \ \forall \ N \in \mathbb{N}. \tag{3.6}$$

Clearly it holds

$$T_N^{(M)}\left(|\cdot - x|^{1+\beta}\right)(x) \leq \frac{3}{\sqrt{N}}, \quad \forall \ x \in [0,1], \ \forall \ N \in \mathbb{N}, \forall \ \beta > 0. \tag{3.7}$$

The operators $T_N^{(M)}$ are positive sublinear operators mapping $C_+([0,1])$ into itself, with $T_N^{(M)}(1) = 1$.

Furthermore it holds

(3.8)

$$T_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) = \frac{\bigvee_{k=0}^{N} \frac{(Nx)^k}{k!} \left|\frac{k}{N} - x\right|^{\lambda}}{\bigvee_{k=0}^{N} \frac{(Nx)^k}{k!}} > 0, \quad \forall \ x \in (0,1], \ \forall \ \lambda \geq 1, \forall \ N \in \mathbb{N}.$$

□

We give

THEOREM 3.4. *Let $0 < \alpha < 1$, any $x \in (0,1]$, $f \in AC([0,1], \mathbb{R}_+)$, and $f' \in L_\infty([0,1])$. Then*

$$\left|T_N^{(M)}(f)(x) - f(x)\right| \leq \frac{(\alpha+2)\omega_1\left(D_x^\alpha f, \left(\frac{3}{\sqrt{N}}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+2)} \left(\frac{3}{\sqrt{N}}\right)^{\frac{\alpha}{\alpha+1}}, \quad \forall \ N \in \mathbb{N}. \tag{3.9}$$

As $N \to +\infty$, we get $T_N^{(M)}(f)(x) \to f(x)$, for any $x \in (0,1]$.

Proof. By Theorem 2.16. □

We make

REMARK 3.5. Next we study the truncated Max-product Baskakov operators (see [5, p. 11])

$$U_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} b_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} b_{N,k}(x)}, \quad x \in [0,1], \ f \in C_+([0,1]), \ N \in \mathbb{N}, \tag{3.10}$$

where

$$b_{N,k}(x) = \binom{N+k-1}{k} \frac{x^k}{(1+x)^{N+k}}. \tag{3.11}$$

From [5, pp. 217-218], we get ($x \in [0,1]$)

$$\left(U_N^{(M)}\left(|\cdot - x|\right)\right)(x) \leq \frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}, \quad N \geq 2,\ N \in \mathbb{N}. \tag{3.12}$$

Let $\lambda \geq 1$, clearly then it holds

$$\left(U_N^{(M)}\left(|\cdot - x|^{\lambda}\right)\right)(x) \leq \frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}, \quad \forall\ N \geq 2,\ N \in \mathbb{N}. \tag{3.13}$$

Also it holds $U_N^{(M)}(1) = 1$, and $U_N^{(M)}$ are positive sublinear operators from $C_+([0,1])$ into itself. Furthermore it holds

$$U_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) > 0, \quad \forall\ x \in (0,1],\ \forall\ \lambda \geq 1,\ \forall\ N \in \mathbb{N}. \tag{3.14}$$

□

We give

Theorem 3.6. *Let* $0 < \alpha < 1$, *any* $x \in (0,1]$, $f \in AC([0,1], \mathbb{R}_+)$, *and* $f' \in L_\infty([0,1])$. *Then*

(3.15)

$$\left|U_N^{(M)}(f)(x) - f(x)\right| \leq \frac{(\alpha+2)\omega_1\left(D_x^{\alpha} f, \left(\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+2)} \cdot \left(\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}\right)^{\frac{\alpha}{\alpha+1}},$$

$\forall\ N \geq 2$, $N \in \mathbb{N}$.

As $N \to +\infty$, *we get* $U_N^{(M)}(f)(x) \to f(x)$, *for any* $x \in (0,1]$.

Proof. By Theorem 2.16. □

We continue with

Remark 3.7. Here we study the Max-product Meyer-Köning and Zeller operators (see [5, p. 11]) defined by

$$Z_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} s_{N,k}(x)\, f\left(\frac{k}{N+k}\right)}{\bigvee_{k=0}^{\infty} s_{N,k}(x)}, \quad \forall N \in \mathbb{N},\ f \in C_+([0,1]), \tag{3.16}$$

$s_{N,k}(x) = \binom{N+k}{k} x^k$, $x \in [0,1]$.

By [5, p. 253], we get that

$$Z_N^{(M)}(|\cdot - x|)(x) \leq \frac{8\left(1+\sqrt{5}\right)}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}}, \quad \forall\ x \in [0,1],\ \forall\ N \geq 4,\ N \in \mathbb{N}. \tag{3.17}$$

As before we get that (for $\lambda \geq 1$)

$$Z_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) \leq \frac{8\left(1+\sqrt{5}\right)}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}} := \rho(x), \tag{3.18}$$

$\forall\ x \in [0,1]$, $N \geq 4$, $N \in \mathbb{N}$.

Also it holds $Z_N^{(M)}(1) = 1$, and $Z_N^{(M)}$ are positive sublinear operators from $C_+([0,1])$ into itself. Also it holds

$$Z_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) > 0, \quad \forall\, x \in (0,1),\ \forall\, \lambda \geq 1, \forall\, N \in \mathbb{N}. \tag{3.19}$$

We give

THEOREM 3.8. *Let $0 < \alpha < 1$, any $x \in (0,1)$, $f \in AC([0,1], \mathbb{R}_+)$, and $f' \in L_\infty([0,1])$. Then*

$$\left|Z_N^{(M)}(f)(x) - f(x)\right| \leq \frac{(\alpha+2)\omega_1\left(D_x^{\alpha} f, (\rho(x))^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+2)}(\rho(x))^{\frac{\alpha}{\alpha+1}} \tag{3.20}$$

$\forall\, N \geq 4$, $N \in \mathbb{N}$.

As $N \to +\infty$, we get $Z_N^{(M)}(f)(x) \to f(x)$, for any $x \in (0,1)$.

Proof. By Theorem 2.16. □

We continue with

REMARK 3.9. Here we deal with the Max-product truncated sampling operators (see [5, p. 13]) defined by

$$W_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi}}, \tag{3.21}$$

and

$$K_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}}, \tag{3.22}$$

$\forall\, x \in [0,\pi]$, $f : [0,\pi] \to \mathbb{R}_+$ a continuous function.

Following [5, p. 343], and making the convention $\frac{\sin(0)}{0} = 1$ and denoting $s_{N,k}(x) = \frac{\sin(Nx-k\pi)}{Nx-k\pi}$, we get that $s_{N,k}\left(\frac{k\pi}{N}\right) = 1$, and $s_{N,k}\left(\frac{j\pi}{N}\right) = 0$, if $k \neq j$, furthermore $W_N^{(M)}(f)(\frac{j\pi}{N}) = f(\frac{j\pi}{N})$, for all $j \in \{0, ..., N\}$.

Clearly $W_N^{(M)}(f)$ is a well-defined function for all $x \in [0,\pi]$, and it is continuous on $[0,\pi]$, also $W_N^{(M)}(1) = 1$.

By [5, p. 344], $W_N^{(M)}$ are positive sublinear operators.

Call $I_N^+(x) = \{k \in \{0,1,...,N\}; s_{N,k}(x) > 0\}$, and set $x_{N,k} := \frac{k\pi}{N}$, $k \in \{0,1,...,N\}$.

We see that

$$W_N^{(M)}(f)(x) = \frac{\bigvee_{k\in I_N^+(x)} s_{N,k}(x) f(x_{N,k})}{\bigvee_{k\in I_N^+(x)} s_{N,k}(x)}. \tag{3.23}$$

By [5, p. 346],we have

$$W_N^{(M)}\left(\left|\cdot-x\right|\right)(x)\leq\frac{\pi}{2N},\quad\forall\ N\in\mathbb{N},\ \forall\ x\in\left[0,\pi\right].\tag{3.24}$$

Notice also $\left|x_{N,k}-x\right|\leq\pi$, $\forall\ x\in\left[0,\pi\right]$.

Therefore $(\lambda\geq 1)$ it holds

$$W_N^{(M)}\left(\left|\cdot-x\right|^{\lambda}\right)(x)\leq\frac{\pi^{\lambda-1}\pi}{2N}=\frac{\pi^{\lambda}}{2N},\quad\forall\ x\in\left[0,\pi\right],\ \forall\ N\in\mathbb{N}.\tag{3.25}$$

If $x\in\left(\frac{j\pi}{N},\frac{(j+1)\pi}{N}\right)$, with $j\in\{0,1,...,N\}$, we obtain $nx-j\pi\in(0,\pi)$ and thus $s_{N,j}(x)=\frac{\sin(Nx-j\pi)}{Nx-j\pi}>0$, see [5, pp. 343-344].

Consequently it holds $(\lambda\geq 1)$

$$W_N^{(M)}\left(\left|\cdot-x\right|^{\lambda}\right)(x)=\frac{\bigvee_{k\in I_N^{+}(x)}s_{N,k}(x)\left|x_{N,k}-x\right|^{\lambda}}{\bigvee_{k\in I_N^{+}(x)}s_{N,k}(x)}>0,\quad\forall\ x\in\left[0,\pi\right],\tag{3.26}$$

such that $x\neq x_{N,k}$, for any $k\in\{0,1,...,N\}$. □

We give

Theorem 3.10. *Let $0<\alpha<1$, any $x\in[0,\pi]$ be such that $x\neq\frac{k\pi}{N}$, $k\in\{0,1,...,N\}$, $\forall\ N\in\mathbb{N}$; $f\in AC\left([0,\pi],\mathbb{R}_{+}\right)$, and $f'\in L_{\infty}\left([0,\pi]\right)$. Then*

$$\left|W_N^{(M)}(f)(x)-f(x)\right|\leq\frac{(\alpha+2)\omega_1\left(D_x^{\alpha}f,\left(\frac{\pi^{\alpha+1}}{2N}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+2)}\left(\frac{\pi^{\alpha+1}}{2N}\right)^{\frac{\alpha}{\alpha+1}},\quad\forall\ N\in\mathbb{N}.\tag{3.27}$$

As $N\to+\infty$, we get $W_N^{(M)}(f)(x)\to f(x)$.

Proof. By Theorem 2.16. □

We make

Remark 3.11. Here we continue with the Max-product truncated sampling operators (see [5, p. 13]) defined by

$$K_N^{(M)}(f)(x)=\frac{\bigvee_{k=0}^{N}\frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N}\frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}},\tag{3.28}$$

$\forall\ x\in[0,\pi]$, $f:[0,\pi]\to\mathbb{R}_{+}$ a continuous function.

Following [5, p. 350], and making the convention $\frac{\sin(0)}{0}=1$ and denoting $s_{N,k}(x)=\frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}$, we get that $s_{N,k}\left(\frac{k\pi}{N}\right)=1$, and $s_{N,k}\left(\frac{j\pi}{N}\right)=0$, if $k\neq j$, furthermore $K_N^{(M)}(f)\left(\frac{j\pi}{N}\right)=f\left(\frac{j\pi}{N}\right)$, for all $j\in\{0,...,N\}$.

Since $s_{N,j}\left(\frac{j\pi}{N}\right)=1$ it follows that $\bigvee_{k=0}^{N}s_{N,k}\left(\frac{j\pi}{N}\right)\geq 1>0$, for all $j\in\{0,1,...,N\}$. Hence $K_N^{(M)}(f)$ is well-defined function for all $x\in[0,\pi]$, and it is continuous on $[0, \pi]$, also $K_N^{(M)}(1) = 1$. By [5, p. 350], $K_N^{(M)}$ are positive sublinear operators.

Denote $x_{N,k} := \frac{k\pi}{N}$, $k \in \{0, 1, ..., N\}$.

By [5, p. 352], we have

$$K_N^{(M)}(|\cdot - x|)(x) \leq \frac{\pi}{2N}, \quad \forall\ N \in \mathbb{N},\ \forall\ x \in [0, \pi]. \tag{3.29}$$

Notice also $|x_{N,k} - x| \leq \pi$, $\forall\ x \in [0, \pi]$.

Therefore ($\lambda \geq 1$) it holds

$$K_N^{(M)}\left(|\cdot - x|^\lambda\right)(x) \leq \frac{\pi^{\lambda-1}\pi}{2N} = \frac{\pi^\lambda}{2N}, \quad \forall\ x \in [0, \pi], \forall\ N \in \mathbb{N}. \tag{3.30}$$

If $x \in \left(\frac{j\pi}{N}, \frac{(j+1)\pi}{N}\right)$, with $j \in \{0, 1, ..., N\}$, we obtain $nx - j\pi \in (0, \pi)$ and thus $s_{N,j}(x) = \frac{\sin^2(Nx - j\pi)}{(Nx - j\pi)^2} > 0$, see [5, pp. 350].

Consequently it holds ($\lambda \geq 1$)

$$K_N^{(M)}\left(|\cdot - x|^\lambda\right)(x) = \frac{\bigvee_{k=0}^{N} s_{N,k}(x)\left|x_{N,k} - x\right|^\lambda}{\bigvee_{k=0}^{N} s_{N,k}(x)} > 0, \quad \forall\ x \in [0, \pi], \tag{3.31}$$

such that $x \neq x_{N,k}$, for any $k \in \{0, 1, ..., N\}$. □

We give

THEOREM 3.12. *Let $0 < \alpha < 1$, $x \in [0, \pi]$ be such that $x \neq \frac{k\pi}{N}$, $k \in \{0, 1, ..., N\}$, $\forall\ N \in \mathbb{N}$; $f \in AC([0, \pi], \mathbb{R}_+)$, and $f' \in L_\infty([0, \pi])$. Then*
(3.32)

$$\left|K_N^{(M)}(f)(x) - f(x)\right| \leq \frac{(\alpha+2)\omega_1\left(D_x^\alpha f, \left(\frac{\pi^{\alpha+1}}{2N}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+2)} \left(\frac{\pi^{\alpha+1}}{2N}\right)^{\frac{\alpha}{\alpha+1}}, \quad \forall\ N \in \mathbb{N}.$$

As $N \to +\infty$, we get $K_N^{(M)}(f)(x) \to f(x)$.

Proof. By Theorem 2.16. □

When $\alpha = \frac{1}{2}$ we get:

COROLLARY 3.13. *Let $f \in AC([0, 1], \mathbb{R}_+)$, $f' \in L_\infty([0, 1])$. Then*

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \frac{10\sqrt[3]{6}\omega_1\left(D_x^{\frac{1}{2}} f, \frac{\sqrt[3]{36}}{\sqrt[3]{N+1}}\right)}{3\sqrt{\pi}\sqrt[6]{N+1}}, \quad \forall\ x \in (0, 1), \forall\ N \in \mathbb{N}. \tag{3.33}$$

Proof. By Theorem 3.2. □

Due to lack of space we avoid to give other applications when $\alpha = \frac{1}{2}$ from the other Max-product operators.

II) Case $\alpha > 1$, $\alpha \notin \mathbb{N}$.

Here we apply Theorem 2.14 to well known Max-product operators.

We present

Theorem 3.14. *Let* $\alpha > 1$, $\alpha \notin \mathbb{N}$ $m = \lceil \alpha \rceil$, $x \in [0,1]$, $f \in AC^m([0,1], \mathbb{R}_+)$, *and* $f^{(m)} \in L_\infty([0,1])$. *Furthermore we assume that* $f^{(k)}(x) = 0$, $k = 1, ..., m-1$. *Then*

(3.34)

$$\left| B_N^{(M)}(f)(x) - f(x) \right| \le \frac{\omega_1\left(D_x^\alpha f, \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)} \cdot \left[\frac{6}{\sqrt{N+1}} + \frac{1}{(\alpha+1)}\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{\alpha}{\alpha+1}}\right],$$

$\forall\ N \in \mathbb{N}$.

We get $\lim_{N \to +\infty} B_N^{(M)}(f)(x) = f(x)$.

Proof. Applying (2.30) for $B_N^{(M)}$ and using (3.2), we get

$$\left| B_N^{(M)}(f)(x) - f(x) \right| \le \frac{\omega_1(D_x^\alpha f, \delta)}{\Gamma(\alpha+1)}\left[\frac{6}{\sqrt{N+1}} + \frac{\frac{6}{\sqrt{N+1}}}{(\alpha+1)\delta}\right]. \qquad (3.35)$$

Choose $\delta = \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{1}{\alpha+1}}$, then $\delta^{\alpha+1} = \frac{6}{\sqrt{N+1}}$, and apply it to (3.35). Clearly we derive (3.34). □

We continue with

Theorem 3.15. *Same assumptions as in Theorem 3.14. Then*

$$\left| T_N^{(M)}(f)(x) - f(x) \right| \le \frac{\omega_1\left(D_x^\alpha f, \left(\frac{3}{\sqrt{N}}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)}\left[\frac{3}{\sqrt{N}} + \frac{1}{(\alpha+1)}\left(\frac{3}{\sqrt{N}}\right)^{\frac{\alpha}{\alpha+1}}\right], \qquad (3.36)$$

$\forall\ N \in \mathbb{N}$.

We get $\lim_{N \to +\infty} T_N^{(M)}(f)(x) = f(x)$.

Proof. Use of Theorem 2.14, similar to the proof of Theorem 3.14. □

We give

Theorem 3.16. *Same assumptions as in Theorem 3.14. Then*

$$\left| U_N^{(M)}(f)(x) - f(x) \right| \le$$

$$\le \frac{\omega_1\left(D_x^\alpha f, \left(\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)} \cdot \left[\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}} + \frac{1}{(\alpha+1)}\left(\frac{2\sqrt{3}(\sqrt{2}+2)}{\sqrt{N+1}}\right)^{\frac{\alpha}{\alpha+1}}\right], \qquad (3.37)$$

$\forall\ N \in \mathbb{N},\ N \ge 2$.

We get $\lim_{N \to +\infty} U_N^{(M)}(f)(x) = f(x)$.

Proof. Use of Theorem 2.14, similar to the proof of Theorem 3.14. □

We give

THEOREM 3.17. *Same assumptions as in Theorem 3.14. Then*

(3.38)

$$\left|Z_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^\alpha f, (\rho(x))^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)} \cdot \left[\rho(x) + \frac{1}{(\alpha+1)}(\rho(x))^{\frac{\alpha}{\alpha+1}}\right],$$

$\forall\ N \in \mathbb{N},\ N \geq 4.$

We get $\lim_{N\to+\infty} Z_N^{(M)}(f)(x) = f(x)$, *where* $\rho(x)$ *is as in* (3.18).

Proof. Use of Theorem 2.14, similar to the proof of Theorem 3.14. □

We continue with

THEOREM 3.18. *Let* $\alpha > 1$, $\alpha \notin \mathbb{N}$, $m = \lceil\alpha\rceil$, $x \in [0,\pi] \subset \mathbb{R}$, $f \in AC^m([0,\pi], \mathbb{R}_+)$, *and* $f^{(m)} \in L_\infty([0,\pi])$. *Furthermore we assume that* $f^{(k)}(x) = 0$, $k = 1, ..., m-1$. *Then*

(3.39)

$$\left|W_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^\alpha f, \left(\frac{\pi^{\alpha+1}}{2N}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)} \cdot \left[\frac{\pi^\alpha}{2N} + \frac{1}{(\alpha+1)}\left(\frac{\pi^{\alpha+1}}{2N}\right)^{\frac{\alpha}{\alpha+1}}\right],$$

$\forall\ N \in \mathbb{N}.$

We have that $\lim_{N\to+\infty} W_N^{(M)}(f)(x) = f(x)$.

Proof. Applying (2.30) for $W_N^{(M)}$ and using (3.25), we get

$$\left|W_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1(D_x^\alpha f, \delta)}{\Gamma(\alpha+1)}\left[\frac{\pi^\alpha}{2N} + \frac{\frac{\pi^{\alpha+1}}{2N}}{(\alpha+1)\delta}\right]. \tag{3.40}$$

Choose $\delta = \left(\frac{\pi^{\alpha+1}}{2N}\right)^{\frac{1}{\alpha+1}}$, i.e. $\delta^{\alpha+1} = \frac{\pi^{\alpha+1}}{2N}$, and $\delta^\alpha = \left(\frac{\pi^{\alpha+1}}{2N}\right)^{\frac{\alpha}{\alpha+1}}$. We use the last into (3.40) and we obtain (3.39). □

We also have

THEOREM 3.19. *Let* $\alpha > 1$, $\alpha \notin \mathbb{N}$, $m = \lceil\alpha\rceil$, $x \in [0,\pi] \subset \mathbb{R}$, $f \in AC^m([0,\pi], \mathbb{R}_+)$, *and* $f^{(m)} \in L_\infty([0,\pi])$. *Furthermore we assume that* $f^{(k)}(x) = 0$, $k = 1, ..., m-1$. *Then*

(3.41)

$$\left|K_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^\alpha f, \left(\frac{\pi^{\alpha+1}}{2N}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)} \cdot \left[\frac{\pi^\alpha}{2N} + \frac{1}{(\alpha+1)}\left(\frac{\pi^{\alpha+1}}{2N}\right)^{\frac{\alpha}{\alpha+1}}\right],$$

$\forall\ N \in \mathbb{N}.$

We have that $\lim_{N\to+\infty} K_N^{(M)}(f)(x) = f(x)$.

Proof. As in Theorem 3.18. □

We make

Remark 3.20. We mention the interpolation Hermite-Fejer polynomials on Chebyshev knots of the first kind (see [5, p. 4]): Let $f : [-1,1] \to \mathbb{R}$ and based on the knots $x_{N,k} = \cos\left(\frac{(2(N-k)+1)}{2(N+1)}\pi\right) \in (-1,1)$, $k \in \{0, ..., N\}$, $-1 < x_{N,0} < x_{N,1} < ... < x_{N,N} < 1$, which are the roots of the first kind Chebyshev polynomial $T_{N+1}(x) = \cos((N+1)\arccos x)$, we define (see Fejér [9])

$$H_{2N+1}(f)(x) = \sum_{k=0}^{N} h_{N,k}(x) f(x_{N,k}), \tag{3.42}$$

where

$$h_{N,k}(x) = (1 - x \cdot x_{N,k}) \left(\frac{T_{N+1}(x)}{(N+1)(x - x_{N,k})}\right)^2, \tag{3.43}$$

the fundamental interpolation polynomials.

The Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind (see [5, p. 12]) are defined by

$$H_{2N+1}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x) f(x_{N,k})}{\bigvee_{k=0}^{N} h_{N,k}(x)}, \quad \forall\, N \in \mathbb{N}, \tag{3.44}$$

where $f : [-1,1] \to \mathbb{R}_+$ is continuous.

Call

$$E_N(x) := H_{2N+1}^{(M)}(|\cdot - x|)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x) |x_{N,k} - x|}{\bigvee_{k=0}^{N} h_{N,k}(x)}, \quad x \in [-1,1]. \tag{3.45}$$

Then by [5, p. 287], we obtain that

$$E_N(x) \le \frac{2\pi}{N+1}, \ \forall\, x \in [-1,1], \ N \in \mathbb{N}. \tag{3.46}$$

For $m > 1$, we get

$$H_{2N+1}^{(M)}(|\cdot - x|^m)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x) |x_{N,k} - x|^m}{\bigvee_{k=0}^{N} h_{N,k}(x)} =$$

$$= \frac{\bigvee_{k=0}^{N} h_{N,k}(x) |x_{N,k} - x| |x_{N,k} - x|^{m-1}}{\bigvee_{k=0}^{N} h_{N,k}(x)} \le 2^{m-1} \frac{\bigvee_{k=0}^{N} h_{N,k}(x) |x_{N,k} - x|}{\bigvee_{k=0}^{N} h_{N,k}(x)}$$

$$\le \frac{2^m \pi}{N+1}, \quad \forall x \in [-1,1], \ N \in \mathbb{N}. \tag{3.47}$$

Hence it holds

$$H_{2N+1}^{(M)}(|\cdot - x|^m)(x) \le \frac{2^m \pi}{N+1}, \quad \forall x \in [-1,1], \ m > 1, \forall\, N \in \mathbb{N}. \tag{3.48}$$

Furthermore we have

$$H_{2N+1}^{(M)}(1)(x)=1,\ \forall\ x\in[-1,1], \tag{3.49}$$

and $H_{2N+1}^{(M)}$ maps continuous functions to continuous functions over $[-1,1]$ and for any $x\in\mathbb{R}$ we have $\bigvee_{k=0}^{N}h_{N,k}(x)>0$.

We also have $h_{N,k}(x_{N,k})=1$, and $h_{N,k}(x_{N,j})=0$, if $k\neq j$, furthermore it holds $H_{2N+1}^{(M)}(f)(x_{N,j})=f(x_{N,j})$, for all $j\in\{0,...,N\}$, see [5, p. 282].

$H_{2N+1}^{(M)}$ are positive sublinear operators, [5, p. 282]. □

We give

Theorem 3.21. *Let* $\alpha>1$, $\alpha\notin\mathbb{N}$, $m=\lceil\alpha\rceil$, $x\in[-1,1]$, $f\in AC^{m}([-1,1],\mathbb{R}_{+})$, *and* $f^{(m)}\in L_{\infty}([-1,1])$. *Furthermore we assume that* $f^{(k)}(x)=0$, $k=1,...,m-1$. *Then*

(3.50)

$$\left|H_{2N+1}^{(M)}(f)(x)-f(x)\right|\leq\frac{\omega_1\left(D_x^{\alpha}f,\left(\frac{2^{\alpha+1}\pi}{N+1}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)}\cdot\left[\frac{2^{\alpha}\pi}{N+1}+\frac{1}{(\alpha+1)}\left(\frac{2^{\alpha+1}\pi}{N+1}\right)^{\frac{\alpha}{\alpha+1}}\right],$$

$\forall\ N\in\mathbb{N}$.

Furthermore it holds $\lim_{N\to+\infty}H_{2N+1}^{(M)}(f)(x)=f(x)$.

Proof. By Theorem 2.14, choose $\delta:=\left(\frac{2^{\alpha+1}\pi}{N+1}\right)^{\frac{1}{\alpha+1}}$, use (2.30), (3.48). □

We continue with

Remark 3.22. Here we deal with Lagrange interpolation polynomials on Chebyshev knots of second kind plus the endpoints ±1 (see [5, p. 5]). These polynomials are linear operators attached to $f:[-1,1]\to\mathbb{R}$ and to the knots $x_{N,k}=\cos\left(\left(\frac{N-k}{N-1}\right)\pi\right)\in[-1,1]$, $k=1,...,N$, $N\in\mathbb{N}$, which are the roots of $\omega_N(x)=\sin(N-1)t\sin t$, $x=\cos t$. Notice that $x_{N,1}=-1$ and $x_{N,N}=1$. Their formula is given by [5, p. 377]

$$L_N(f)(x)=\sum_{k=1}^{N}l_{N,k}(x)f(x_{N,k}), \tag{3.51}$$

where

$$l_{N,k}(x)=\frac{(-1)^{k-1}\omega_N(x)}{\left(1+\delta_{k,1}+\delta_{k,N}\right)(N-1)\left(x-x_{N,k}\right)}, \tag{3.52}$$

$N\geq2$, $k=1,...,N$, and $\omega_N(x)=\prod_{k=1}^{N}(x-x_{N,k})$ and $\delta_{i,j}$ denotes the Kronecher's symbol, that is $\delta_{i,j}=1$, if $i=j$, and $\delta_{i,j}=0$, if $i\neq j$.

The Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints ±1, are defined by [5, p. 12]

$$L_N^{(M)}(f)(x)=\frac{\bigvee_{k=1}^{N} l_{N,k}(x)f(x_{N,k})}{\bigvee_{k=1}^{N} l_{N,k}(x)},\quad x\in[-1,1], \tag{3.53}$$

where $f:[-1,1]\to\mathbb{R}_+$ continuous.

First we see that $L_N^{(M)}(f)(x)$ is well defined and continuous for any $x\in[-1,1]$. Following [5, p. 289], because $\sum_{k=1}^{N} l_{N,k}(x)=1$, $\forall\ x\in\mathbb{R}$, for any x there exists $k\in\{1,...,N\}:l_{N,k}(x)>0$, hence $\bigvee_{k=1}^{N} l_{N,k}(x)>0$. We have that $l_{N,k}(x_{N,k})=1$, and $l_{N,k}(x_{N,j})=0$, if $k\neq j$. Furthermore it holds $L_N^{(M)}(f)(x_{N,j})=f(x_{N,j})$, all $j\in\{1,...,N\}$, and $L_N^{(M)}(1)=1$.

Call $I_N^{+}(x)=\{k\in\{1,...,N\};l_{N,k}(x)>0\}$, then $I_N^{+}(x)\neq\emptyset$.

So for $f\in C_+([-1,1])$ we get

$$L_N^{(M)}(f)(x)=\frac{\bigvee_{k\in I_N^{+}(x)} l_{N,k}(x)f(x_{N,k})}{\bigvee_{k\in I_N^{+}(x)} l_{N,k}(x)}\geq 0. \tag{3.54}$$

Notice here that $|x_{N,k}-x|\leq 2$, $\forall\ x\in[-1,1]$.

By [5, p. 297], we get that

(3.55)

$$L_N^{(M)}(|\cdot-x|)(x)=\frac{\bigvee_{k=1}^{N} l_{N,k}(x)|x_{N,k}-x|}{\bigvee_{k=1}^{N} l_{N,k}(x)}=\frac{\bigvee_{k\in I_N^{+}(x)} l_{N,k}(x)|x_{N,k}-x|}{\bigvee_{k\in I_N^{+}(x)} l_{N,k}(x)}\leq\frac{\pi^2}{6(N-1)},$$

$N\geq 3$, $\forall\ x\in(-1,1)$, N is odd.

We get that $(m>1)$

$$L_N^{(M)}(|\cdot-x|^m)(x)=\frac{\bigvee_{k\in I_N^{+}(x)} l_{N,k}(x)|x_{N,k}-x|^m}{\bigvee_{k\in I_N^{+}(x)} l_{N,k}(x)}\leq\frac{2^{m-1}\pi^2}{6(N-1)}, \tag{3.56}$$

$N\geq 3$ odd, $\forall\ x\in(-1,1)$.

$L_N^{(M)}$ are positive sublinear operators, [5, p. 290]. □

We give

Theorem 3.23. *Same assumptions as in Theorem* 3.21. *Then*

(3.57)

$$\left|L_N^{(M)}(f)(x)-f(x)\right|\leq\frac{\omega_1\left(D_x^{\alpha}f,\left(\frac{2^{\alpha}\pi^2}{6(N-1)}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)}\left[\frac{2^{\alpha-1}\pi^2}{6(N-1)}+\frac{1}{(\alpha+1)}\left(\frac{2^{\alpha}\pi^2}{6(N-1)}\right)^{\frac{\alpha}{\alpha+1}}\right],$$

$\forall\ N \in \mathbb{N} : N \geq 3$, odd.

It holds $\lim\limits_{N \to +\infty} L_N^{(M)}(f)(x) = f(x)$.

Proof. By Theorem 2.14, choose $\delta := \left(\frac{2^{\alpha}\pi^2}{6(N-1)}\right)^{\frac{1}{\alpha+1}}$, use of (2.30) and (3.56). At ± 1 the left hand side of (3.57) is zero, thus (3.57) is trivially true. □

We make

REMARK 3.24. Let $f \in C_+([-1,1])$, $N \geq 4$, $N \in \mathbb{N}$, N even.

By [5, p. 298], we get

$$L_N^{(M)}(|\cdot - x|)(x) \leq \frac{4\pi^2}{3(N-1)} = \frac{2^2\pi^2}{3(N-1)}, \quad \forall x \in (-1,1). \tag{3.58}$$

Hence $(m > 1)$

$$L_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{2^{m+1}\pi^2}{3(N-1)}, \quad \forall\ x \in (-1,1). \tag{3.59}$$

□

We present

THEOREM 3.25. *Same assumptions as in Theorem* 3.21. *Then*

$$\left|L_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^{\alpha}f, \left(\frac{2^{\alpha+2}\pi^2}{3(N-1)}\right)^{\frac{1}{\alpha+1}}\right)}{\Gamma(\alpha+1)} \left[\frac{2^{\alpha+1}\pi^2}{3(N-1)} + \frac{1}{(\alpha+1)}\left(\frac{2^{\alpha+2}\pi^2}{3(N-1)}\right)^{\frac{\alpha}{\alpha+1}}\right], \tag{3.60}$$

$\forall\ N \in \mathbb{N},\ N \geq 4$, N is even.

It holds $\lim\limits_{N \to +\infty} L_N^{(M)}(f)(x) = f(x)$.

Proof. By Theorem 2.14, choose $\delta := \left(\frac{2^{\alpha+2}\pi^2}{3(N-1)}\right)^{\frac{1}{\alpha+1}}$, use of (2.30) and (3.59). At ± 1, (3.60) is trivially true. □

We need

DEFINITION 3.26. *(*[6, p. 41]*). Let $I \subset \mathbb{R}$ be an interval of finite or infinite length, and $f : I \to \mathbb{R}$ a bounded or uniformly continuous function. We define the first modulus of continuity*

$$\omega_1(f,\delta)_I = \sup_{\substack{x,y \in I \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{3.61}$$

Clearly, it holds $\omega_1(f,\delta)_I < +\infty$.

We also have

$$\omega_1(f, r\delta)_I \leq (r+1)\,\omega_1(f,\delta)_I, \quad \textit{any } r \geq 0. \tag{3.62}$$

Convention 3.27. *Let a real number* $m > 1$*, from now on we assume that* $D^m_{x_0-} f$ *is either bounded or uniformly continuous function on* $(-\infty, x_0]$*, similarly from now on we assume that* $D^m_{*x_0} f$ *is either bounded or uniformly continuous function on* $[x_0, +\infty)$*.*

We need

Definition 3.28. *Let* $D^m_{x_0} f$ *(real number* $m > 1$*) denote any of* $D^m_{x_0-} f$*,* $D^m_{*x_0} f$ *and* $\delta > 0$*. We set*

$$\omega_1 \left(D^m_{x_0} f, \delta\right)_{\mathbb{R}} := \max \left\{ \omega_1 \left(D^m_{x_0-} f, \delta\right)_{(-\infty, x_0]}, \omega_1 \left(D^m_{*x_0} f, \delta\right)_{[x_0, +\infty)} \right\}, \tag{3.63}$$

where $x_0 \in \mathbb{R}$*. Notice that* $\omega_1 \left(D^m_{x_0} f, \delta\right)_{\mathbb{R}} < +\infty$*.*

We will use

Theorem 3.29. *Let the real number* $m > 0$*,* $m \notin \mathbb{N}$*,* $\lambda = \lceil m \rceil$*,* $x_0 \in \mathbb{R}$*,* $f \in AC^{\lambda} ([a, b], \mathbb{R}_+)$ *(i.e.* $f^{(\lambda-1)} \in AC[a, b]$*, absolutely continuous functions on* $[a, b]$*),* $\forall\ [a, b] \subset \mathbb{R}$*, and* $f^{(\lambda)} \in L_\infty (\mathbb{R})$*. Furthermore we assume that* $f^{(k)} (x_0) = 0$*,* $k = 1, ..., \lambda - 1$*. The Convention* 3.27 *is imposed here. Then*

$$|f(x) - f(x_0)| \leq \frac{\omega_1 \left(D^m_{x_0} f, \delta\right)_{\mathbb{R}}}{\Gamma(m+1)} \left[|x - x_0|^m + \frac{|x - x_0|^{m+1}}{(m+1)\delta} \right], \quad \delta > 0, \tag{3.64}$$

for all $x \in \mathbb{R}$*.*

If $0 < m < 1$*, then we do not need initial conditions.*

Proof. Similar to Theorem 2.10. □

We continue with

Remark 3.30. Let $b : \mathbb{R} \to \mathbb{R}_+$ be a centered (it takes a global maximum at 0) bell-shaped function, with compact support $[-T, T]$, $T > 0$ (that is $b(x) > 0$ for all $x \in (-T, T)$) and $I = \int_{-T}^{T} b(x)\, dx > 0$.

The Cardaliaguet-Euvrard neural network operators are defined by (see [4])

$$C_{N,\alpha} (f)(x) = \sum_{k=-N^2}^{N^2} \frac{f\left(\frac{k}{n}\right)}{IN^{1-\alpha}} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right), \tag{3.65}$$

$0 < \alpha < 1$, $N \in \mathbb{N}$ and typically here $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded or uniformly continuous on $\mathbb{R}$.

$CB(\mathbb{R})$ denotes the continuous and bounded function on $\mathbb{R}$, and

$$CB_+ (\mathbb{R}) = \{f : \mathbb{R} \to [0, \infty);\ f \in CB(\mathbb{R})\}.$$

The corresponding max-product Cardaliaguet-Euvrard neural network operators will be given by

$$C^{(M)}_{N,\alpha} (f)(x) = \frac{\bigvee_{k=-N^2}^{N^2} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) f\left(\frac{k}{N}\right)}{\bigvee_{k=-N^2}^{N^2} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)}, \tag{3.66}$$

$x \in \mathbb{R}$, typically here $f \in CB_+(\mathbb{R})$, see also [4].

Next we follow [4].

For any $x \in \mathbb{R}$, denoting

$$J_{T,N}(x) = \left\{ k \in \mathbb{Z};\ -N^2 \le k \le N^2,\ N^{1-\alpha}\left(x - \frac{k}{N}\right) \in (-T,T) \right\},$$

we can write

$$C_{N,\alpha}^{(M)}(f)(x) = \frac{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) f\left(\frac{k}{N}\right)}{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)}, \tag{3.67}$$

$x \in \mathbb{R}$, $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$, where $J_{T,N}(x) \neq \emptyset$. Indeed, we have $\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) > 0$, $\forall\, x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$.

We have that $C_{N,\alpha}^{(M)}(1)(x) = 1$, $\forall\, x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$. □

See in [4, Lemma 2.1, Corollary 2.2 and Remarks].

We need

Theorem 3.31. [4]. *Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T,T]$, $T > 0$, $0 < \alpha < 1$ and $C_{N,\alpha}^{(M)}$ be defined as in* (3.66).

(i) If $|f(x)| \le c$ for all $x \in \mathbb{R}$ then $\left|C_{N,\alpha}^{(M)}(f)(x)\right| \le c$, for all $x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$ and $C_{N,\alpha}^{(M)}(f)(x)$ is continuous at any point $x \in \mathbb{R}$, for all $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$;

(ii) If $f, g \in CB_+(\mathbb{R})$ satisfy $f(x) \le g(x)$ for all $x \in \mathbb{R}$, then $C_{N,\alpha}^{(M)}(f)(x) \le C_{N,\alpha}^{(M)}(g)(x)$ for all $x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$;

(iii) $C_{N,\alpha}^{(M)}(f+g)(x) \le C_{N,\alpha}^{(M)}(f)(x) + C_{N,\alpha}^{(M)}(g)(x)$ for all $f, g \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$;

(iv) For all $f, g \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$, we have

$$\left|C_{N,\alpha}^{(M)}(f)(x) - C_{N,\alpha}^{(M)}(g)(x)\right| \le C_{N,\alpha}^{(M)}(|f - g|)(x);$$

(v) $C_{N,\alpha}^{(M)}$ is positive homogeneous, that is $C_{N,\alpha}^{(M)}(\lambda f)(x) = \lambda C_{N,\alpha}^{(M)}(f)(x)$ for all $\lambda \ge 0$, $x \in \mathbb{R}$, $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$ and $f \in CB_+(\mathbb{R})$.

We make

Remark 3.32. We have that

$$E_{N,\alpha}(x) := C_{N,\alpha}^{(M)}(|\cdot - x|)(x) = \frac{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)\left|x - \frac{k}{N}\right|}{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)}, \tag{3.68}$$

$\forall\ x \in \mathbb{R}$, and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$. □

We mention from [4] the following:

THEOREM 3.33. [4]. *Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T, T]$, $T > 0$ and $0 < \alpha < 1$. In addition, suppose that the following requirements are fulfilled:*

(i) There exist $0 < m_1 \le M_1 < \infty$ such that $m_1 (T - x) \le b(x) \le M_1 (T - x)$, $\forall\ x \in [0, T]$;

(ii) There exist $0 < m_2 \le M_2 < \infty$ such that $m_2 (x + T) \le b(x) \le M_2 (x + T)$, $\forall\ x \in [-T, 0]$.

Then for all $f \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ and for all $N \in \mathbb{N}$ satisying $N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$, we have the estimate

$$\left|C_{N,\alpha}^{(M)}(f)(x) - f(x)\right| \le c\,\omega_1\left(f, N^{\alpha-1}\right)_{\mathbb{R}}, \tag{3.69}$$

where

$$c := 2\left(\max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} + 1\right),$$

and

$$\omega_1(f, \delta)_{\mathbb{R}} := \sup_{\substack{x,y\in\mathbb{R}:\\ |x-y|\le\delta}} |f(x) - f(y)|.$$

We make

REMARK 3.34. In [4], was proved that

$$E_{N,\alpha}(x) \le \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} N^{\alpha-1}, \quad \forall\ N > \max\left\{T + |x|, \left(\tfrac{2}{T}\right)^{\frac{1}{\alpha}}\right\}. \tag{3.70}$$

That is

$$C_{N,\alpha}^{(M)}(|\cdot - x|)(x) \le \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} N^{\alpha-1}, \quad \forall\ N > \max\left\{T + |x|, \left(\tfrac{2}{T}\right)^{\frac{1}{\alpha}}\right\}. \tag{3.71}$$

From (3.68) we have that $\left|x - \frac{k}{N}\right| \le \frac{T}{N^{1-\alpha}}$.

Hence $(m > 1)$ ($\forall\ x \in \mathbb{R}$ and $N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$)

$$C_{N,\alpha}^{(M)}(|\cdot - x|^m)(x) = \frac{\bigvee_{k\in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)\left|x - \frac{k}{N}\right|^m}{\bigvee_{k\in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)}$$

$$\le \left(\frac{T}{N^{1-\alpha}}\right)^{m-1} \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} N^{\alpha-1}, \tag{3.72}$$

$$\forall\ N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}.$$

Then $(m>1)$ it holds

$$C_{N,\alpha}^{(M)}\left(\left|\cdot-x\right|^{m}\right)(x)\leq T^{m-1}\max\left\{\frac{TM_2}{2m_2},\frac{TM_1}{2m_1}\right\}\frac{1}{N^{m(1-\alpha)}},$$

$$\forall\ N>\max\left\{T+\left|x\right|,\left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}. \tag{3.73}$$

Call

$$\theta:=\max\left\{\frac{TM_2}{2m_2},\frac{TM_1}{2m_1}\right\}>0. \tag{3.74}$$

Consequently $(m>1)$ we derive

$$C_{N,\alpha}^{(M)}\left(\left|\cdot-x\right|^{m}\right)(x)\leq\frac{\theta T^{m-1}}{N^{m(1-\alpha)}},\quad\forall\ N>\max\left\{T+\left|x\right|,\left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}. \tag{3.75}$$

□

We need

THEOREM 3.35. *All here as in Theorem 3.29, where $x=x_0\in\mathbb{R}$ is fixed. Let b be a centered bell-shaped function, continuous and with compact support $[-T,T]$, $T>0$, $0<\alpha<1$ and $C_{N,\alpha}^{(M)}$ be defined as in* (3.66). *Then*

$$\left|C_{N,\alpha}^{(M)}(f)(x)-f(x)\right|\leq\frac{\omega_1\left(D_x^m f,\delta\right)_{\mathbb{R}}}{\Gamma(m+1)}\left[C_{N,\alpha}^{(M)}\left(\left|\cdot-x\right|^{m}\right)(x)+\frac{C_{N,\alpha}^{(M)}\left(\left|\cdot-x\right|^{m+1}\right)(x)}{(m+1)\delta}\right],$$

$\forall\ N\in\mathbb{N}:N>\max\left\{T+\left|x\right|,T^{-\frac{1}{\alpha}}\right\}$.

Proof. By Theorem 3.29 and (3.64) we get

$$\left|f(\cdot)-f(x)\right|\leq\frac{\omega_1\left(D_x^m f,\delta\right)_{\mathbb{R}}}{\Gamma(m+1)}\left[\left|\cdot-x\right|^{m}+\frac{\left|\cdot-x\right|^{m+1}}{(m+1)\delta}\right],\quad\delta>0, \tag{3.76}$$

true over $\mathbb{R}$.

As in Theorem 3.31 and using similar reasoning and $C_{N,\alpha}^{(M)}(1)=1$, we get

$$\left|C_{N,\alpha}^{(M)}(f)(x)-f(x)\right|\leq C_{N,\alpha}^{(M)}\left(\left|f(\cdot)-f(x)\right|\right)(x)\overset{(3.76)}{\leq}$$

$$\overset{(3.76)}{\leq}\frac{\omega_1\left(D_x^m f,\delta\right)_{\mathbb{R}}}{\Gamma(m+1)}\left[C_{N,\alpha}^{(M)}\left(\left|\cdot-x\right|^{m}\right)(x)+\frac{C_{N,\alpha}^{(M)}\left(\left|\cdot-x\right|^{m+1}\right)(x)}{(m+1)\delta}\right], \tag{3.77}$$

$\forall\ N\in\mathbb{N}:N>\max\left\{T+\left|x\right|,T^{-\frac{1}{\alpha}}\right\}$. □

We continue with

THEOREM 3.36. *Here all as in Theorem 3.29, where $x=x_0\in\mathbb{R}$ is fixed and $m>1$. Also the same assumptions as in Theorem 3.33. Then*

$$\left|C_{N,\alpha}^{(M)}(f)(x)-f(x)\right|\leq\frac{1}{\Gamma(m+1)}\omega_1\left(D_x^m f,\left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{1}{m+1}}\right)_{\mathbb{R}}\cdot$$

$$\cdot\left[\frac{\theta T^{m-1}}{N^{m(1-\alpha)}}+\frac{1}{(m+1)}\left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{m}{m+1}}\right], \tag{3.78}$$

$\forall\ N \in \mathbb{N} : N > \max\left\{T + |x|, (\frac{2}{T})^{\frac{1}{\alpha}}\right\}$.

We have that $\lim_{N\to+\infty} C_{N,\alpha}^{(M)}(f)(x) = f(x)$.

Proof. We apply Theorem 3.35. In (3.35) we choose

$$\delta := \left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{1}{m+1}},$$

thus $\delta^{m+1} = \frac{\theta T^m}{N^{(m+1)(1-\alpha)}}$, and

$$\delta^m = \left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{m}{m+1}}. \tag{3.79}$$

Therefore we have

$$\left|C_{N,\alpha}^{(M)}(f)(x) - f(x)\right| \overset{(3.75)}{\leq}$$

$$\overset{(3.75)}{\leq} \frac{1}{\Gamma(m+1)}\omega_1\left(D_x^m f, \left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{1}{m+1}}\right)_{\mathbb{R}} \cdot \left[\frac{\theta T^{m-1}}{N^{m(1-\alpha)}} + \frac{1}{(m+1)\delta}\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right]$$

$$= \frac{1}{\Gamma(m+1)}\omega_1\left(D_x^m f, \left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{1}{m+1}}\right)_{\mathbb{R}}\left[\frac{\theta T^{m-1}}{N^{m(1-\alpha)}} + \frac{1}{(m+1)\delta}\delta^{m+1}\right]$$

(3.80)

$$\overset{(3.79)}{=} \frac{1}{\Gamma(m+1)}\omega_1\left(D_x^m f, (\frac{\theta T^m}{N^{(m+1)(1-\alpha)}})^{\frac{1}{m+1}}\right)_{\mathbb{R}}\left[\frac{\theta T^{m-1}}{N^{m(1-\alpha)}} + \frac{1}{(m+1)}(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}})^{\frac{m}{m+1}}\right]$$

$\forall\ N \in \mathbb{N} : N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$, proving the inequality (3.78). □

We finish with (case of $\alpha = 1.5$)

Corollary 3.37. *Let* $x \in [0,1]$, $f \in AC^2([0,1], \mathbb{R}_+)$ *and* $f^{(2)} \in L_\infty([0,1])$. *Assume that* $f'(x) = 0$. *Then*

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \frac{4\omega_1\left(D_x^{1.5}f, (\frac{6}{\sqrt{N+1}})^{\frac{2}{5}}\right)}{3\sqrt{\pi}}\left[\frac{6}{\sqrt{N+1}} + \frac{2}{5}\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{3}{5}}\right], \tag{3.81}$$

$\forall\ N \in \mathbb{N}$.

Proof. By Theorem 3.14, apply (3.34). □

Due to lack of space we do not give other example applications.

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Received by the editors: September 25, 2017. Accepted: October 12, 2018. Published online: February 17, 2019.