JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 48 (2019) no. 1, pp. 62–88
ictp.acad.ro/jnaat

SECOND DERIVATIVE GENERAL LINEAR METHOD IN NORDSIECK FORM

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Abstract. This paper considers the construction of *second derivative general linear methods* (SD-GLM) from hybrid LMM and their transformation to Nordsieck GLM. We show how the Runge-Kutta starters for the methods can be derived. The representation of the methods in Nordsieck form has the advantage of easy implementation in variable stepsize.

MSC 2010. 65L05, 65L06.

Keywords. *General linear methods* (GLM), Nordsieck vectors/methods, second derivative GLM, RK-starters, variable stepsize.

1. INTRODUCTION

The representation of second derivative hybrid linear multistep methods (SD-HLMM) as SD-GLM and transformation to its Nordsieck form for easy implementation will be the main focus herein. Nordsieck method was introduced by Arnold Nordsieck in 1962 as an easy means of changing step size in LMM [1]. Variable step-size implementation is readily achieved for formulas of Nordsieck kinds and in fact, LMM can be represented by Nordsieck formulas, details can be found in [1], [3], [10], [12], [18] and [19]. Hybrid LMM which can be with one or more off-step points are aimed at bypassing the Dahlquist [7] order barrier for the conventional LMM in computing the numerical solution of the autonomous initial value problems (IVPs),

$$(1) \qquad \begin{cases} y' = f(y(x)), & x \in [x_0, X] \\ y(x_0) = y_0, \quad y \in \mathbb{R}^m, & f : \mathbb{R} \times \mathbb{R}^m, m \geq 1, \end{cases}$$

Examples of hybrid LMM are in [3], [4] [13] and [15]. Indeed, it is somewhat difficult to implement the hybrid LMM in variable step-size because of the problems associated with attempt to put the input and output data in their standard representations at each step of approximations of the solution of (1). The problems associated with the direct implementation of the hybrid LMM make it less attractive compared to the Nordsieck GLM [3],[4]. Indeed, over

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the years there is now a greater sophistication in the understanding of the GLM,

$$(2a)\qquad \begin{aligned} Y &= hAF(Y) + Uy^{[n-1]}, \\ y^{[n]} &= hBF(Y) + Vy^{[n-1]}, \end{aligned}$$

where $n = 1, 2, \ldots;$ $h = (x_{n+1} - x_n)$ is the stepsize. The GLM (2a) is readily represented also as,

$$(2b)\qquad \left(\begin{array}{c} Y \\ \hline y^{[n]} \end{array}\right) = \left(\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right) \left(\begin{array}{c} hF(Y) \\ \hline y^{[n-1]} \end{array}\right)$$

where, k is the number of input quantities, s is the number of stages, $Y = (Y_1, Y_2, ..., Y_s)^T$ denote the stage values computed in step n, then $y^{[n-1]} = (y_1^{[n-1]}, y_2^{[n-1]} \cdots, y_k^{[n-1]})^T$ represent the vector of input quantities at the start of step n, while $y^{[n]} = (y_1^{[n]}, y_2^{[n]} \cdots, y_k^{[n]})^T$ is the output solution and denotes the vector quantities exported to the next step, $n+1$. The $F(Y) = (f(Y_1), f(Y_2), ..., f(Y_s))^T$ is the stage derivatives computed in step n and the matrices defining the structure of the method are $A = \{a_{ij}\} \in \mathbb{R}^{(s\times s)}$, $B = \{b_{ij}\} \in \mathbb{R}^{(k\times s)}$, $U = \{u_{ij}\} \in \mathbb{R}^{(s\times k)}$, and $V = \{v_{ij}\} \in \mathbb{R}^{(k\times k)}$. The $c = [c_1, \ldots, c_s]^T$ is a vector of the stage abscissas. The Nordsieck form of (2b) is

$$(3\text{a})\qquad \left(\begin{array}{c} Y \\ \hline \overline{y}^{[n]} \end{array}\right) = \left(\begin{array}{c|c} A & UT \\ \hline T^{-1}B & T^{-1}VT \end{array}\right) \left(\begin{array}{c} hF(Y) \\ \hline \overline{y}^{[n-1]} \end{array}\right),$$

where,

$$(3a)\qquad y^{[n-1]} = T\overline{y}^{[n-1]}, \quad y^{[n]} = T\overline{y}^{[n]}, \quad T = [t_{ij}]_{ij=0}^{p,p},$$

with the Nordsieck vectors in the past and present being respectively,

$$(3b)\qquad \overline{y}^{[n-1]} = \begin{bmatrix} y(x_{n-1}) \\ hy'(x_{n-1}) \\ h^2y''(x_{n-1}) \\ h^3y^{(3)}(x_{n-1}) \\ \vdots \\ h^py^{(p)}(x_{n-1}) \end{bmatrix}, \quad \overline{y}^{[n]} = \begin{bmatrix} y(x_n) \\ hy'(x_n) \\ h^2y''(x_n) \\ h^3y^{(3)}(x_n) \\ \vdots \\ h^py^{(p)}(x_n) \end{bmatrix}.$$

Hybrid LMM can in general be easily represented in the GLM formalism as in (2b) and when it is possible to define Nordsieck vector in (3b), then it is readily implementable as a variable stepsize Nordsieck GLM in (3a). The [4] has given a practical example of the transformation of hybrid LMM into the Nordsieck form (3a) of a GLM in (2b).We are therefore motivated following [4] to consider the transformation of a certain class of hybrid second derivative LMM into its Nordsieck SD-GLM for the purpose of easy implementation in variable step-size. In this regard, the multi-derivative GLM is

$$\left(\begin{array}{c} Y \\ \hline y^{[n]} \end{array}\right) = \left(\begin{array}{cccc} A^{(1)} & \cdots & A^{(\ell-1)} \; A^{(\ell)} & U \\ \hline B^{(1)} & \cdots & B^{(\ell-1)} \; B^{(\ell)} & V \end{array}\right) \left(\begin{array}{c} hF(Y) \\ h^2F'(Y) \\ \vdots \\ h^{(\ell)}F^{(\ell-1)}(Y) \\ \hline y^{[n-1]} \end{array}\right), \quad \ell \geq 1, \tag{4}$$

in [11], [16]. This method engages directly the higher order derivatives of the solution of (1) in the implementation of (2b). In particular, the SD-GLM in the Butcher sense is

$$\left(\begin{array}{c} Y \\ \hline y^{[n]} \end{array}\right) = \left(\begin{array}{c|c|c} A^{(1)} & A^{(2)} & U \\ \hline B^{(1)} & B^{(2)} & V \end{array}\right) \left(\begin{array}{c} hF(Y) \\ h^2F'(Y) \\ \hline y^{[n-1]} \end{array}\right). \tag{5a}$$

This class of methods introduced in [2] which incorporates second derivatives of the solution of (1), enhances (2) in order and stability wise for a comparable number of stages with respect to (2b). The terms in (5a) which extend (2) are the matrices $A^{(1)} = \{a_{ij}^{(1)}\} \in \mathbb{R}^{(s\times s)}$, $A^{(2)} = \{a_{ij}^{(2)}\} \in \mathbb{R}^{(s\times s)}$, $B^{(1)} = \{b_{ij}^{(1)}\} \in \mathbb{R}^{(k\times s)}$, $B^{(2)} = \{b_{ij}^{(2)}\} \in \mathbb{R}^{(k\times s)}$, $U = \{u_{ij}^{(j)}\} \in \mathbb{R}^{(s\times k)}$, $V = \{v_{ij}\} \in \mathbb{R}^{(k\times k)}$ and the second derivatives stages $F'(Y) = (f'(Y_1), f'(Y_2), ..., f'(Y_s))^T$ respectively. The picture of (5a) in Nordsieck form is now

$$\left(\begin{array}{c} Y \\ \hline \overline{y}^{[n]} \end{array}\right) = \left(\begin{array}{c|c|c} A^{(1)} & A^{(2)} & UT \\ \hline T^{-1}B^{(1)} & T^{-1}B^{(2)} & T^{-1}VT \end{array}\right) \left(\begin{array}{c} hF(Y) \\ h^2F'(Y) \\ \hline \overline{y}^{[n-1]} \end{array}\right). \tag{5b}$$

The transformation matrix T is as in equation (3a). The stability matrix of (5a) is

$$M(z) = V + z(B^{(1)} + zB^{(2)})(I - zA^{(1)} - z^2A^{(2)})^{-1}U, \quad z = \lambda h,$$

and the stability polynomial is given by

$$\Pi(w, z) = \det(wI - M(z)).$$

It is readily obvious that (5a) and (5b) exhibit the same stability characteristics. The same for (2b) and (3a). An example of the explicit SD-GLM (5a) is from [14],

$$\left(\begin{array}{c|c|c} A^{(1)} & A^{(2)} & U \\ \hline B^{(1)} & B^{(2)} & V \end{array}\right) = \left(\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & \frac{1}{27} & 0 & 0 & 1 \\ \frac{7}{16} & \frac{9}{16} & 0 & \frac{1}{16} & \frac{1}{16} & 0 & 1 \\ \hline \frac{7}{16} & \frac{9}{16} & 0 & \frac{1}{16} & \frac{1}{16} & 0 & 1 \end{array}\right), \quad c = \begin{pmatrix} 0 \\ \frac{2}{3} \\ 1 \end{pmatrix}.$$

The interval of absolute stability of this SD-GLM [14] is $(-12, 0)$. Again, see [2] and [14] for some further examples of the SD-GLM in (5a). Sequential SD-GLM have been derived in [17] and inherent Runge-Kutta stable methods are proposed in [18].

A direct generalisation of GLM of Butcher and O'Sullivan [4] incorporating multiple hybrid points is

(6a)

$$y(x_{n-1}+c_jh)=\sum_{i=1}^{k}u_{ji}^{(j)}y_{n-i}+h\sum_{i=1}^{k}u_{ji}^{(j+1)}f_{n-i}+h\sum_{i=\ell}^{j+\ell-1}a_{ji}^{(1)}f_{n+c_i-1},$$
$$p=2k+2(j+\ell-1),\quad 0<c_j\le 1,\quad j=\ell(1)s-1;\quad \ell=0,1,$$

with the last stage and also serving as an output method is given as

(6b)

$$y(x_{n-1}+th)=\sum_{i=1}^{k}v_{1i}(t)y_{n-i}+h\sum_{i=1}^{k}v_{2i}(t)f_{n-i}+h\sum_{i=\ell}^{s+\ell-1}b_{ji}^{(1)}(t)f_{n+c_i-1},$$
$$i=2(1)k,\quad p=2k+2(s+\ell-1)-(k-1)-1,\quad v_{11}(t)=1,$$
$$\{v_{1i}(t)=0\}_{i=2}^{k},\quad t=c_s=1,$$

where $f_{n+c_i-1}=f(y(x_{n-1}+c_ih))$ and $x_{n-1}=x_n-h$. A new SD-GLM extension of this class of methods becomes

$$\begin{aligned}y(x_{n-1}+c_jh)=&\sum_{i=1}^{k}u_{ji}^{(j)}y_{n-i}+h\sum_{i=1}^{k}u_{ji}^{(j+1)}f_{n-i}+h^2\sum_{i=1}^{k}u_{ji}^{(j+2)}f'_{n-i}\\&+h\sum_{i=\ell}^{j+\ell-1}a_{ji}^{(1)}f_{n+c_i-1}+h^2\sum_{i=\ell}^{j+\ell-1}a_{ji}^{(2)}f'_{n+c_i-1},\\&p=3k+2(j+\ell-1)-1,\quad 0<c_j\le 1,\quad f'=f_yf,\end{aligned}\tag{7a}$$

with last stage as the method,

(7b)

$$\begin{aligned}y(x_{n-1}+th)=&\sum_{i=1}^{k}v_{1i}(t)y_{n-i}+h\sum_{i=1}^{k}v_{2i}(t)f_{n-i}+h^2\sum_{i=1}^{k}v_{3i}(t)f'_{n-i}\\&+h\sum_{i=\ell}^{s+\ell-1}b_{ji}^{(1)}(t)f_{n+c_i-1}+h^2\sum_{i=\ell}^{s+\ell-1}b_{ji}^{(2)}(t)f'_{n+c_i-1},\quad i=2(1)k,\\&p=3k+2(s+\ell-1)-(k-1)-1,\quad v_{11}(t)=1,\\&\{v_{1i}(t)=0\}_{i=2}^{k},\quad t=c_s=1.\end{aligned}$$

This class of methods is of interest in the next section. This will be implemented as *first same as last* (FSAL), see [3]. To construct an explicit or diagonally implicit SD-GLM from (6a), (6b), and (7a), (7b) set the flag ℓ accordingly as,

$$\ell=\begin{cases}0; & \text{gives an explicit SD-GLM}\\ 1; & \text{gives an implicit SD-GLM}\end{cases}$$

By setting $\ell=0$, $k=2$, $s=3$ and the coefficients $\{a_{ji}^{(1)}\}_{j=0}$, $\{u_{ji}^{(j+2)}\}_{j=0}^{k}$, $\{a_{ji}^{(2)}\}_{j=0}^{k}$, $\{v_{3i}(t)\}_{j=0}^{k}$ and $\{a_{ji}^{(2)}(t)\}_{j=0}^{k}$ in (6a) and (6b) respectively to zero,

[4] using the Cauchy integral formula we obtain the hybrid LMM
(8)

$$
\begin{aligned}
y_{n-\frac{7}{15}} &= -\tfrac{529}{3375} y_{n-1} + \tfrac{3904}{3375} y_{n-2} + h\left(\tfrac{4232}{3375} f_{n-1} + \tfrac{1472}{3375} f_{n-2}\right), \quad p=3,\\
\widehat{y}_n &= \tfrac{152}{25} y_{n-1} - \tfrac{127}{25} y_{n-2} + h\left(\tfrac{189}{92} f_{n-\frac{7}{15}} - \tfrac{419}{100} f_{n-1} - \tfrac{1118}{575} f_{n-2}\right), \quad p=3,\\
y_n &= y_{n-1} + h\left(\tfrac{3375}{5152} f_{n-\frac{7}{15}} + \tfrac{25}{168} \widehat{f}_n + \tfrac{19}{96} f_{n-1} - \tfrac{1}{552} f_{n-2}\right), \quad p=5.
\end{aligned}
$$

By (3a) and (3b) the Nordsieck form of this becomes,
(9)

$$
\left(\begin{array}{c|c} A^{(1)} & UT \\ \hline T^{-1}B^{(1)} & T^{-1}VT \end{array}\right) = \left(\begin{array}{ccccccccc}
0 & 0 & 0 & 1 & \frac{8}{15} & \frac{64}{225} & \frac{512}{3375} & \frac{-1984}{3375} & \frac{128}{125}\\
\frac{189}{92} & 0 & 0 & 1 & \frac{-97}{92} & \frac{-137}{115} & \frac{-433}{575} & \frac{1551}{575} & \frac{-2669}{575}\\
\frac{3375}{5152} & \frac{25}{168} & 0 & 1 & \frac{433}{2208} & \frac{1}{276} & \frac{-1}{184} & \frac{1}{138} & \frac{-5}{552}\\
\hline
\frac{3375}{5152} & \frac{25}{168} & 0 & 1 & \frac{433}{2208} & \frac{1}{276} & \frac{-1}{184} & \frac{1}{138} & \frac{-5}{552}\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{-3375}{896} & \frac{-575}{672} & 3 & 0 & \frac{623}{384} & \frac{35}{48} & \frac{-7}{32} & \frac{-7}{24} & \frac{77}{96}\\
\frac{-111375}{20608} & \frac{-275}{224} & \frac{13}{4} & 0 & \frac{9957}{2944} & \frac{633}{368} & \frac{-335}{736} & \frac{-149}{184} & \frac{1527}{736}\\
\frac{-57375}{20608} & \frac{-425}{672} & \frac{3}{2} & 0 & \frac{16927}{8832} & \frac{1363}{1104} & \frac{-167}{736} & \frac{-431}{552} & \frac{3949}{2208}\\
\frac{-10125}{20608} & \frac{-25}{224} & \frac{1}{4} & 0 & \frac{1039}{2944} & \frac{91}{368} & \frac{3}{736} & \frac{-47}{184} & \frac{373}{736}
\end{array}\right),
$$

The abscissa vector for the GLM in (9) is $c = \left(\frac{8}{15}, 1, 1\right)^T$. The T transformation in this representation is

$$
T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
1 & -1 & 1 & -1 & 1 & -1\\
1 & -2 & 4 & -8 & 16 & -32\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & -2 & 3 & -4 & 5\\
0 & 1 & -4 & 12 & -32 & 80
\end{pmatrix}. \tag{10}
$$

The interval of absolute stability of (9) is $(-1.231, 0)$, see Fig. 1 for its stability plot. The region of absolute stability is the closed region in the *left half plane* (LHP) of the complex plane in Fig. 1.

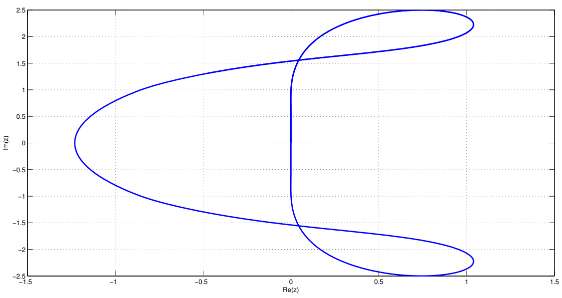

Fig. 1. Region of absolute stability (closed region in the LHP) of the method in (9).

An implicit counterpart of (9) is

$$\left(\begin{array}{c|c} A^{(1)} & UT \\ \hline T^{-1}B^{(1)} & T^{-1}VT \end{array}\right) = \left(\begin{array}{ccccccccc}
\frac{92}{465} & 0 & 0 & 1 & \frac{52}{155} & \frac{512}{6975} & \frac{-1792}{104625} & \frac{-4096}{104625} & \frac{3328}{34875} \\
\frac{-189}{2300} & \frac{22}{75} & 0 & 1 & \frac{1303}{6900} & \frac{112}{8625} & \frac{-96}{14375} & \frac{16}{43125} & \frac{256}{43125} \\
\frac{3375}{5152} & 0 & \frac{25}{168} & 1 & \frac{433}{2208} & \frac{1}{276} & \frac{-1}{184} & \frac{1}{138} & \frac{-5}{552} \\
\hline
\frac{3375}{5152} & 0 & \frac{25}{168} & 1 & \frac{433}{2208} & \frac{1}{276} & \frac{-1}{184} & \frac{1}{138} & \frac{-5}{552} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-3375}{896} & 0 & \frac{1441}{672} & 0 & \frac{623}{384} & \frac{35}{48} & \frac{-7}{32} & \frac{-7}{24} & \frac{77}{96} \\
\frac{-111375}{20608} & 0 & \frac{453}{224} & 0 & \frac{9957}{2944} & \frac{633}{368} & \frac{-335}{736} & \frac{-149}{184} & \frac{1527}{736} \\
\frac{-57375}{20608} & 0 & \frac{583}{672} & 0 & \frac{16927}{8832} & \frac{1363}{1104} & \frac{-167}{736} & \frac{-431}{552} & \frac{3949}{2208} \\
\frac{-10125}{20608} & 0 & \frac{31}{224} & 0 & \frac{1039}{2944} & \frac{91}{368} & \frac{3}{736} & \frac{-47}{184} & \frac{373}{736}
\end{array}\right).$$

The $c = \left(\frac{8}{15}, \frac{2}{5}, 1\right)^T$ is the abscissa vector for the implicit counterpart of the method in (9) with the angle of absolute stability, $\alpha = 77°$ and having (10) as its T.

This paper is organized as follows. Section 2 is on the transformation of SD-HLMM(7) into its Nordsieck form which is illustrated for step numbers $k = 1, 2$. In section 3 we derive the SD-HLMM (7) of step number $k \leq 3$, while section 4 presents the variable step-size implementation. The section 5 presents how the RK-starters for the methods can be obtained and with examples. Section 6, presents results of numerical experiments.

2. THE SD-HLMM (7a,b)

Consider an extension of the methods of [5] provided in (7a) and (7b) to SD-GLM. The explicit stage predictors are,

$$y(x_{n-1} + c_1 h) = \sum_{j=1}^{k} u_{1,j}^{(1)} y_{n-j} + h \sum_{j=1}^{k} u_{1,j}^{(2)} f_{n-j} + h^2 \sum_{j=1}^{k} u_{1,j}^{(3)} f'_{n-j}, \tag{11}$$

and

$$\begin{aligned} \widehat{y}(x_{n-1} + c_2 h) = &\sum_{j=1}^{k} u_{2,j}^{(1)} y_{n-j} + h \sum_{j=1}^{k} u_{2,j}^{(2)} f_{n-j} + h a_{2,1}^{(1)} f_{n+c_1-1} \\ &+ h^2 \sum_{j=1}^{k} u_{2,j}^{(3)} f'_{n-j} + h^2 a_{2,1}^{(2)} f'_{n+c_1-1}, \end{aligned} \tag{12}$$

for computing the off-step points at x_{n+c_1-1} and x_{n+c_2-1} respectively. The last stage that will also serve as the output method is,

$$y(x_{n-1} + th) = \sum_{j=1}^{k} v_{1,j}(t) y_{n-j} + h \sum_{j=1}^{k} v_{2,j}(t) f_{n-j} + h b_{11}^{(1)}(t) f_{n+c_1-1} + \tag{13}$$

$$+ hb_{12}^{(1)}(t)f_{n+c_2-1} + h^2\sum_{j=1}^{k} v_{3,j}(t)f'_{n-j} + h^2 b_{11}^{(2)}(t)f'_{n+c_1-1}$$
$$+ h^2 b_{12}^{(2)}(t)f'_{n+c_2-1}, \quad t = 1.$$

The $c = (c_1, c_2) \in [0,1]$, is the stage abscissas vector, the continuous coefficients $v_{1,1}(t)$ and $v_{1,2}(t)$ are assumed to be one and zero respectively. The first and the second stage derivatives are $F(Y) = [f(y_{n+c_1-1}) \quad f(y_{n+c_2-1})]^T$ and $F'(Y) = [f'(y_{n+c_1-1}) \quad f'(y_{n+c_2-1})]^T$ respectively. Here $f_{n+c_i-1} = f(y(x_{n-1} + c_i h)), i = 1(1)s$, $f'_{n+c_i-1} = f'(y(x_{n-1} + c_i h)), i = 1(1)s$ and $x_{n-1} = x_n - h$. The order of the algorithms in (11), (12) and (13) are $3k-1$, $3k+1$ and $3k+2$ respectively.

Similarly, the equivalent methods of the implicit case are

$$y(x_{n-1} + c_1 h) = \sum_{j=1}^{k} u_{1,j}^{(1)} y_{n-j} + h\sum_{j=1}^{k} u_{1,j}^{(2)} f_{n-j} + h^2 \sum_{j=1}^{k} u_{1,j}^{(3)} f'_{n-j} + ha_{1,1}^{(1)} f_{n+c_1-1} + h^2 \widehat{a}_{1,1}^{(2)} f'_{n+c_1-1}, \tag{14}$$

and

(15)

$$\widehat{y}(x_{n-1} + c_2 h) = \sum_{j=1}^{k} u_{2,j}^{(1)} y_{n-j} + \sum_{j=1}^{k} u_{2,j}^{(2)} f_{n-j} + ha_{2,1}^{(1)} f_{n+c_1-1} + ha_{2,2}^{(1)} f_{n+c_2-1}$$
$$+ h^2 \sum_{j=1}^{k} u_{2,j}^{(3)} f'_{n-j} + h^2 a_{2,1}^{(2)} f'_{n+c_1-1} + h^2 a_{2,2}^{(2)} f'_{n+c_2-1},$$

with the last stage as an output method,

$$y(x_{n-1} + th) = \sum_{j=1}^{k} v_{1,j}(t) y_{n-j} + h\sum_{j=1}^{k} v_{2,j}(t) f_{n-j} + hb_{11}^{(1)}(t) f_{n+c_1-1}$$
$$+ hb_{12}^{(1)}(t) f_{n+c_2-1} + hb_{13}^{(1)}(t) f_{n+t-1} + h^2 \sum_{j=1}^{k} v_{3,j}(t) f'_{n-j}$$
$$+ h^2 b_{11}^{(2)}(t) f'_{n+c_1-1} + h^2 b_{12}^{(2)}(t) f'_{n+c_2-1} + h^2 b_{13}^{(2)}(t) f'_{n+t-1}, \quad t = 1. \tag{16}$$

The prospect of these classes of SD-HLMM is the fact that the explicit (13) and implicit (16) methods have a common T transformation (3a) for a fixed k in their Nordsieck GLM. Moreover, the explicit method (11)-(13) can serve as a starter for the implicit one in (14)-(16). The SD-HLMM can be transformed into Nordsieck methods. To get started, the SD-GLM representation of the

SD-HLMM (11), (12), (13) when $k=2$ is

$$(17a)\qquad \left(\frac{A^{(1)}\; A^{(2)}\; U}{B^{(1)}\; B^{(2)}\; V}\right) = \left(\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & u_{11}^{(1)} & u_{12}^{(1)} & u_{11}^{(2)} & u_{12}^{(2)} & u_{11}^{(3)} & u_{12}^{(3)} \\ a_{21}^{(1)} & 0 & 0 & a_{21}^{(2)} & 0 & 0 & u_{21}^{(1)} & u_{22}^{(1)} & u_{21}^{(2)} & u_{22}^{(2)} & u_{21}^{(3)} & u_{22}^{(3)} \\ b_{11}^{(1)} & b_{12}^{(1)} & 0 & b_{11}^{(2)} & b_{12}^{(2)} & 0 & 1 & 0 & v_{11}^{(2)} & v_{12}^{(2)} & v_{11}^{(3)} & v_{12}^{(3)} \\ \hline b_{11}^{(1)} & b_{12}^{(1)} & 0 & b_{11}^{(2)} & b_{12}^{(2)} & 0 & 1 & 0 & v_{11}^{(2)} & v_{12}^{(2)} & v_{11}^{(3)} & v_{12}^{(3)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right),$$

while the implicit SD-GLM partitioned matrix of the SD-HLMM (14), (15),(16) when $k=2$ is

$$(17b)\qquad \left(\begin{array}{c|c|c} A^{(1)} & A^{(2)} & U \\ \hline B^{(1)} & B^{(2)} & V \end{array}\right) = \left(\begin{array}{ccccccc} a_{11}^{(1)} & 0 & 0 & a_{11}^{(2)} & 0 & 0 & \\ a_{21}^{(1)} & a_{22}^{(1)} & 0 & a_{21}^{(2)} & a_{22}^{(2)} & 0 & U \\ b_{11}^{(1)} & b_{12}^{(1)} & b_{13}^{(1)} & b_{11}^{(2)} & b_{12}^{(2)} & b_{13}^{(2)} & \\ \hline & B^{(1)} & & & B^{(2)} & & V \end{array}\right),$$

where $B^{(1)}$, $B^{(2)}$, U, V are of the same structure as in (17a). The stages, stage derivatives, the inputs and the output approximations for the implicit method in (17a, b) are:

$$\begin{aligned} Y =& [y(x_{n-1}+c_1h), y(x_{n-1}+c_2h), y(x_{n-1}+th)]^T, \\ F(Y) =& [f(y(x_{n-1}+c_1h)), f(y(x_{n-1}+c_2h)), f(y(x_{n-1}+th))]^T, \\ F'(Y) =& [f'(y(x_{n-1}+c_1h)), f'(y(x_{n-1}+c_2h)), f'(y(x_{n-1}+th))]^T, \text{ and} \\ y^{[n]} =& [y_n, y_{n-1}, hf_n, hf_{n-1}, h^2f'_n, h^2f'_{n-1}]^T \end{aligned}$$

respectively. To achieve our aim we seek a transformation of (17a) into the form

$$(18a)\qquad \left(\frac{A^{(1)}\quad A^{(2)}\quad UT}{T^{-1}B^{(1)}\; T^{-1}B^{(2)}\; T^{-1}VT}\right) = \left(\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & u_{11} & \cdots & u_{19} \\ a_{21}^{(1)} & 0 & 0 & a_{21}^{(2)} & 0 & 0 & 1 & u_{21} & \cdots & u_{29} \\ b_{11}^{(1)} & b_{12}^{(1)} & 0 & b_{11}^{(2)} & b_{12}^{(2)} & 0 & 1 & v_{11} & \cdots & v_{19} \\ \hline b_{11}^{(1)} & b_{12}^{(1)} & 0 & b_{11}^{(2)} & b_{12}^{(2)} & 0 & 1 & v_{11} & \cdots & v_{19} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ b_{41} & b_{42} & b_{43} & b_{41} & b_{42} & b_{43} & 0 & v_{41} & \cdots & v_{49} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ b_{91} & b_{92} & b_{93} & b_{91} & b_{92} & b_{93} & 0 & v_{91} & \cdots & v_{99} \end{array}\right),$$

using the Nordsieck vectors of (3b). Similarly, the implicit case (17b) is

$$(18b)\qquad \left(\frac{A^{(1)}\quad A^{(2)}\quad UT}{T^{-1}B^{(1)}\; T^{-1}B^{(2)}\; T^{-1}VT}\right) = \left(\begin{array}{ccccccc} a_{11}^{(1)} & 0 & 0 & a_{11}^{(2)} & 0 & 0 & \\ a_{21}^{(1)} & a_{22}^{(1)} & 0 & a_{21}^{(2)} & a_{22}^{(2)} & 0 & UT \\ b_{11}^{(1)} & b_{12}^{(1)} & b_{13}^{(1)} & b_{11}^{(2)} & b_{12}^{(2)} & b_{13}^{(2)} & \\ \hline & T^{-1}B^{(1)} & & & T^{-1}B^{(2)} & & T^{-1}VT \end{array}\right).$$

The UT, $T^{-1}B^{(1)}$, $T^{-1}B^{(2)}$ and $T^{-1}VT$ are of similar form to that in (18a). To arrive at this the stage values and the stage derivatives at step number n are

$$Y = \begin{pmatrix} y_{n+c_1-1} \\ y_{n+c_2-1} \\ y_{n+t-1} \end{pmatrix}, \quad F(Y) = \begin{pmatrix} f_{n+c_1-1} \\ f_{n+c_2-1} \\ f_{n+t-1} \end{pmatrix}, \quad F'(Y) = \begin{pmatrix} f'_{n+c_1-1} \\ f'_{n+c_2-1} \\ f'_{n+t-1} \end{pmatrix}.$$

Thus, the $(3+9)\times(3+3+9)$ partitioned matrix of the SD-GLM (17a) is

$$(19a) \qquad \left(\begin{array}{ccc} A^{(1)} & A^{(2)} & U \\ \hline B^{(1)} & B^{(2)} & V \end{array}\right) = \left(\begin{array}{ccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & u_{11}^{(1)} & u_{12}^{(1)} & 0 & u_{11}^{(2)} & u_{12}^{(2)} & 0 & u_{11}^{(3)} & u_{12}^{(3)} & 0 \\
a_{21}^{(1)} & 0 & 0 & a_{21}^{(2)} & 0 & 0 & u_{21}^{(1)} & u_{22}^{(1)} & 0 & u_{21}^{(2)} & u_{22}^{(2)} & 0 & u_{21}^{(3)} & u_{22}^{(3)} & 0 \\
b_{11}^{(1)} & b_{12}^{(1)} & 0 & b_{11}^{(2)} & b_{12}^{(2)} & 0 & 1 & 0 & 0 & v_{11}^{(2)} & v_{12}^{(2)} & 0 & v_{11}^{(3)} & v_{12}^{(3)} & 0 \\
\hline
b_{11}^{(1)} & b_{12}^{(1)} & 0 & b_{11}^{(2)} & b_{12}^{(2)} & 0 & 1 & 0 & 0 & v_{11}^{(2)} & v_{12}^{(2)} & 0 & v_{11}^{(3)} & v_{12}^{(3)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right).$$

In the same manner,

$$(19b) \qquad \left(\begin{array}{c|c|c} A^{(1)} & A^{(2)} & U \\ \hline B^{(1)} & B^{(2)} & V \end{array}\right) = \left(\begin{array}{ccccccc}
a_{11}^{(1)} & 0 & 0 & a_{11}^{(2)} & 0 & 0 & \\
a_{21}^{(1)} & a_{22}^{(1)} & 0 & a_{21}^{(2)} & a_{22}^{(2)} & 0 & U \\
b_{11}^{(1)} & b_{12}^{(1)} & b_{13}^{(1)} & b_{11}^{(2)} & b_{12}^{(2)} & b_{13}^{(2)} & \\
\hline
 & B^{(1)} & & & B^{(2)} & & V
\end{array}\right),$$

for the implicit method in (18b), where y_{n-3}, hf_{n-3} and $h^2 f'_{n-3}$ are the additional input data with coefficients zeros, which will be included to write the method (19a,b) in this $(3+9)\times(3+3+9)$ partitioned matrix, following the ideas of [4]. The incoming and the corresponding outgoing vectors are now,

$$(20) \qquad y^{[n-1]} = \begin{pmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ y_3^{[n-1]} \\ y_4^{[n-1]} \\ y_5^{[n-1]} \\ y_6^{[n-1]} \\ y_7^{[n-1]} \\ y_8^{[n-1]} \\ y_9^{[n-1]} \end{pmatrix} = \begin{pmatrix} y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ hf_{n-1} \\ hf_{n-2} \\ hf_{n-3} \\ h^2 f'_{n-1} \\ h^2 f'_{n-2} \\ h^2 f'_{n-3} \end{pmatrix}, \quad y^{[n]} = \begin{pmatrix} y_1^{[n]} \\ y_2^{[n]} \\ y_3^{[n]} \\ y_4^{[n]} \\ y_5^{[n]} \\ y_6^{[n]} \\ y_7^{[n]} \\ y_8^{[n]} \\ y_9^{[n]} \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n-1} \\ y_{n-2} \\ hf_n \\ hf_{n-1} \\ hf_{n-2} \\ h^2 f'_n \\ h^2 f'_{n-1} \\ h^2 f'_{n-2} \end{pmatrix}.$$

The T obtained through Taylor's expansion of $y^{[n]}$ in (20) about x_n using (3a) is

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -2 & 4 & -8 & 16 & -32 & 64 & -128 & 256 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 & -4 & 5 & -6 & 7 & -8 \\ 0 & 1 & -4 & 12 & -32 & 80 & -192 & 448 & -1024 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -6 & 12 & -20 & 30 & -42 & 56 \\ 0 & 0 & 2 & -12 & 48 & -160 & 480 & -1344 & 3584 \end{pmatrix}. \tag{21}$$

Carrying out the operations UT, $T^{-1}B^{(1)}$, $T^{-1}B^{(2)}$, and $T^{-1}VT$ on (19a,b) yields the Nordsieck algorithms in (18a,b). The transformations above are illustrated by practical examples in the next section.

3. THE TRANSFORMATION OF SD-HLMM TO SD-NORDSIECK METHODS

First, consider the derivation of the SD-HLMM in (11) for $k = 1$ and its transformation to the explicit Nordsieck methods (18a). For the derivation let the polynomial solution of (1) be

$$y(x) = \sum_{j=0}^{N} a_j x^j, \quad N = 3k + 1, \tag{22}$$

to set up the following interpolatory system of linear equations,

$$XA = B, \tag{23}$$

where

$$X = \begin{pmatrix} 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^N \\ 0 & 1 & 2x_{n-1} & \cdots & Nx_{n-1}^{N-1} \\ 0 & 1 & 2x_{n+c_1-1} & \cdots & Nx_{n+c_1-1}^{N-1} \\ 0 & 0 & 2 & \cdots & N(N-1)x_{n-1}^{N-2} \\ 0 & 0 & 2 & \cdots & N(N-1)x_{n+c_1-1}^{N-2} \end{pmatrix}, \quad A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \quad B = \begin{pmatrix} y_{n-1} \\ f_{n-1} \\ f_{n+c_1-1} \\ f'_{n-1} \\ f'_{n+c_1-1} \end{pmatrix},$$

for the algorithm in (13). For $k = 1$ in (11) the (23) reduces to

$$\begin{pmatrix} 1 & x_{n-1} & x_{n-1}^2 \\ 0 & 1 & 2x_{n-1} \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_{n-1} \\ f_{n-1} \\ f'_{n-1} \end{pmatrix}, \quad N = 2. \tag{24}$$

Solving this resulting system of linear equations for $a_j, j = 0(1)2$ and substituting the resulting values into (11) with $c_1 = (x - x_{n-1})/h$ yields the following coefficients:

$$u_{11}^{(1)} = 1,\ u_{11}^{(2)} = c_1,\ u_{11}^{(3)} = \tfrac{c_1^2}{2}. \tag{25}$$

A predictor exists for any choice of c_1. Consider for example, fixing $c_1 = \frac{1}{2}$ into (25) which gives thus

$$y_{n-\frac{1}{2}} = y_{n-1} + \tfrac{h}{2} f_{n-1} + \tfrac{h^2}{8} f'_{n-1}, \quad p = 2. \tag{26}$$

Similarly, setting $N = 4$ in (22) yields (23). Solving (23) for $a_j, j = 0(1)4$ and substituting the result into (22) lead to

$$\begin{aligned} &u_{11}^{(1)} = 1, && u_{11}^{(2)} = c_2 - \tfrac{c_2^3}{c_1^2} + \tfrac{c_2^4}{2c_1^3}, && u_{11}^{(3)} = \tfrac{c_2^2}{2} - \tfrac{2c_2^3}{3c_1} + \tfrac{c_2^4}{4c_1^2}, \\ &t = c_2, && b_{11}^{(1)} = \tfrac{c_2^3}{c_1^2} - \tfrac{c_2^4}{2c_1^3}, && b_{11}^{(2)} = -\tfrac{c_2^3}{3c_1} + \tfrac{c_2^4}{4c_1^2}. \end{aligned} \tag{27}$$

These coefficients in (27) show that an infinite number of output methods (13) exists for step number $k = 1$. Inserting $c = \left[\frac{1}{2}, 1\right]$ into (27) yields

$$y_n = y_{n-1} + h f_{n-1} + h^2 \left(\tfrac{1}{6} f'_{n-1} + \tfrac{1}{3} f'_{n-\frac{1}{2}}\right), \quad p = 4. \tag{28}$$

Following the approach in (17)–(20) the Nordsieck form of (26),(28) is

$$\left(\begin{array}{c|c|c} A^{(1)} & A^{(2)} & UT \\ \hline T^{-1}B^{(1)} & T^{-1}B^{(2)} & T^{-1}VT \end{array}\right) = \left(\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 1 & 1 & \frac{1}{6} & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{3} & 0 & 1 & 1 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & \frac{10}{3} & 3 & 0 & 6 & \frac{7}{6} & 0 & 0 & 0 \\ 0 & -8 & 5 & 3 & 0 & 8 & \frac{3}{2} & 0 & 0 & 0 \\ 0 & -3 & 2 & 1 & 0 & 3 & \frac{1}{2} & 0 & 0 & 0 \end{array}\right). \tag{29}$$

The transformation matrix T in (29) is

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -6 & 12 & -20 \end{pmatrix}. \tag{30}$$

The eigenvalues of the stability matrix of (29) are

$$\{1 + z + \tfrac{z^2}{2} + \tfrac{z^3}{6} + \tfrac{z^4}{24}, 0, 0, 0, 0, 0, 0\}$$

and the fact that only one of them is not identically zero means that the method will behave like a Runge-Kutta method with stability function $R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}$, see [6]. The interval of absolute stability of the algorithm in (29) is thus $[-2.785, 0]$. For the stability plot, see Fig. 2.

Similarly, the abscissa vector $c = [\frac{2}{3}, 1]^T$ in (25) and (27) gives another fourth order SD-HLMM for step number $k = 1$ as

$$\begin{aligned} y_{n-\frac{1}{3}} &= y_{n-1} + \tfrac{2h}{3} f_{n-1} + \tfrac{2h^2}{9} f_{n-1}, \quad p = 2, \\ y_n &= y_{n-1} + \tfrac{h}{16}\left(9 f_{n-\frac{1}{3}} + 7 f_{n-1}\right) + \tfrac{h^2}{16}\left(f'_{n-\frac{1}{3}} + f'_{n-1}\right), \quad p = 4. \end{aligned} \tag{31}$$

The Nordsieck form of this is

$$
(32) \qquad \left(\begin{array}{c|c|c} A^{(1)} & A^{(2)} & UT \\ \hline T^{-1}B^{(1)} & T^{-1}B^{(2)} & T^{-1}VT \end{array}\right) = \left(\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{2}{9} & 0 & 0 & 0 \\ \frac{9}{16} & 0 & \frac{1}{16} & 0 & 1 & \frac{7}{16} & \frac{1}{16} & 0 & 0 & 0 \\ \hline \frac{9}{16} & 0 & \frac{1}{16} & 0 & 1 & \frac{7}{16} & \frac{1}{16} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{45}{8} & -6 & \frac{5}{8} & 3 & 0 & \frac{3}{8} & \frac{1}{8} & 0 & 0 & 0 \\ \frac{135}{16} & -8 & \frac{15}{16} & 3 & 0 & -\frac{7}{16} & -\frac{1}{16} & 0 & 0 & 0 \\ \frac{27}{8} & -3 & \frac{3}{8} & 1 & 0 & -\frac{3}{8} & -\frac{1}{8} & 0 & 0 & 0 \end{array}\right).
$$

The transformation matrix T in (32) is as in (30). The eigenvalues of stability matrix of (32) are $\{1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{72}, 0, 0, 0, 0, 0, 0\}$, see [19]. There is only one non-zero eigenvalue and is thus RK is nearly stable because of the term $\frac{z^4}{72}$. The interval of absolute stability of the method in (32) is $[-3.117, 0]$, see Fig. 2.

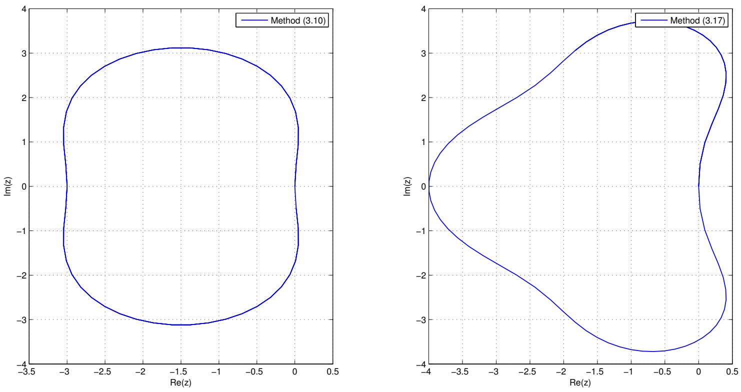

Fig. 2. Region of absolute stability of the Nordsieck methods in (29) and (32).

The coefficients of the first predictor of the SD-HLMM(11) for step number $k = 2$ are:

$$
(33) \qquad \begin{aligned} u_{11}^{(1)} &= (1 + c_1)^3 \left(1 - 3c_1 + 6c_1^2\right), & u_{12}^{(1)} &= -c_1^3 \left(10 + 3c_1 \left(5 + 2c_1\right)\right), \\ u_{11}^{(2)} &= -c_1 \left(1 + c_1\right)^3 \left(-1 + 3c_1\right), & u_{12}^{(2)} &= -c_1^3 \left(1 + c_1\right)\left(4 + 3c_1\right), \\ u_{11}^{(3)} &= \tfrac{1}{2} c_1^2 \left(1 + c_1\right)^3, & u_{12}^{(3)} &= -\tfrac{1}{2} c_1^3 \left(1 + c_1\right)^2. \end{aligned}
$$

If $c_1 \neq -1$, a family of predictors exist and $0 \leq c_1 \leq 1$. Inserting $c_1 = \frac{2}{3}$ into the coefficients and substituting the resulting values into (11) gives the first predictor scheme

$$
\begin{aligned}
y_{n-\frac{1}{3}} =& \tfrac{625}{81} y_{n-1} - \tfrac{544}{81} y_{n-2} + h\left(-\tfrac{250}{81} f_{n-1} - \tfrac{80}{27} f_{n-2}\right) \\
&+ h^2\left(\tfrac{250}{243} f'_{n-1} - \tfrac{100}{243} f'_{n-2}\right), \quad p=5.
\end{aligned} \tag{34}
$$

The relevant equations for the second predictor (12) using (22) are thus obtained and following the above process leads to the coefficients in (A) of the appendix. Fixing $c_1 = \frac{2}{3}$ and $c_2 = 1$ into (A) gives (12) as

$$
\begin{aligned}
\widehat{y}_n =& 2y_{n-1} - y_{n-2} + h\left(\tfrac{243}{1000} f_{n-\frac{1}{3}} + \tfrac{1}{8} f_{n-1} - \tfrac{46}{125} f_{n-2}\right) \\
&+ h^2\left(\tfrac{27}{200} f'_{n-\frac{1}{3}} + \tfrac{3}{8} f'_{n-1} - \tfrac{1}{25} f'_{n-2}\right), \quad p=7.
\end{aligned} \tag{35}
$$

The coefficients in terms of (c,t) are very large expressions hence we fix the abscissa value of the output method as $t=1$ as defined in section 2 to give the coefficients of the output method in (13) in terms of the other abscissa values $c=[c_1,c_2]$. The expressions are in (B) of the appendix. Now inserting $c=[\frac{2}{3},1]$ into (B) and substituting the resulting values into (13) gives

$$
\begin{aligned}
y_n =& y_{n-1} + h\left(\tfrac{60993}{140000} f_{n-\frac{1}{3}} + \tfrac{883}{3360} \widehat{f}_n + \tfrac{1009}{3360} f_{n-1} + \tfrac{521}{420000} f_{n-2}\right) \\
&+ h^2\left(\tfrac{-1863}{28000} f'_{n-\frac{1}{3}} - \tfrac{59}{3360} \widehat{f'}_n + \tfrac{107}{3360} f'_{n-1} + \tfrac{1}{4000} f'_{n-2}\right), \quad p=8.
\end{aligned} \tag{36}
$$

Composing the methods in (34), (35) and (36) and reorganizing their data in SD-GLM format as in (19a) yields

$$
\left(\begin{array}{c|c|c} A^{(1)} & A^{(2)} & U \\ \hline B^{(1)} & B^{(2)} & V \end{array}\right) = \left(\begin{array}{cccccc c}
0 & 0 & 0 & 0 & 0 & 0 & \\
\frac{243}{1000} & 0 & 0 & \frac{27}{200} & 0 & 0 & U \\
1 & 0 & \frac{1009}{3360} & \frac{521}{420000} & \frac{107}{3360} & \frac{1}{4000} & \\
\hline
\frac{60993}{140000} & \frac{883}{3360} & 0 & \frac{-1863}{28000} & \frac{-59}{3360} & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 1 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & V \\
0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right) \tag{37}
$$

where

$$
U = \begin{pmatrix}
\frac{625}{81} & -\frac{544}{81} & -\frac{250}{81} & -\frac{80}{27} & \frac{250}{243} & -\frac{100}{243} \\
2 & -1 & \frac{1}{8} & -\frac{46}{125} & \frac{3}{8} & -\frac{1}{25} \\
1 & 0 & \frac{1009}{3360} & \frac{521}{420000} & \frac{107}{3360} & \frac{1}{4000}
\end{pmatrix}, \quad V = \begin{pmatrix}
1 & 0 & \frac{1009}{3360} & \frac{521}{420000} & \frac{107}{3360} & \frac{1}{4000} \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
$$

Putting all these details together as in (18) with T in (21) we find the coefficient matrices of the Nordsieck method to be

$$
\left(\begin{array}{cccccc c}
0 & 0 & 0 & 0 & 0 & 0 & \\
\frac{243}{1000} & 0 & 0 & \frac{27}{200} & 0 & 0 & UT \\
\frac{60993}{140000} & \frac{883}{3360} & 0 & \frac{-1863}{28000} & \frac{-59}{3360} & 0 & \\
\hline
\multicolumn{3}{c}{T^{-1}B^{(1)}} & \multicolumn{3}{c}{T^{-1}B^{(2)}} & T^{-1}VT
\end{array}\right), \tag{38}
$$

with $k = 2$, where UT, $T^{-1}B^{(1)}, T^{-1}B^{(2)}, T^{-1}VT$ are in appendix C. By verification $\mathrm{Det}(wI - \overline{V}) = w^{k-1}(w-1)$ thus guaranteeing zero stability of the Nordsieck method in (19a). The stability polynomial of (38) is

$$w(w-1) + \tfrac{51z}{16} - \tfrac{67wz}{16} + \tfrac{757z^2}{560} + \tfrac{187wz^2}{140} + \tfrac{3881z^3}{10080} - \tfrac{50wz^3}{63} + \tfrac{1507z^4}{20160} + \tfrac{119wz^4}{960} + \tfrac{11z^5}{1890} - \tfrac{4769wz^5}{120960} - \tfrac{59z^6}{60480} + \tfrac{59wz^6}{24192}.$$

The interval of absolute stability is $(-0.2342, 0)$, see Fig. 4. The stability interval of explicit Nordsieck methods as with explicit LMM generally are usually very severely limited compared to their implicit counterpart; this is compensated by the flexibility in their automatic stepsize implementation allowed by such methods.

From (33), (35) and (B) with $c = [\frac{1}{2}, 1, 1]^T$, the SD-HLMM for $k = 2$ is

(39)

$$\begin{aligned}
y_{n-\frac{1}{2}} &= \tfrac{27}{8} y_{n-1} - \tfrac{19}{8} y_{n-2} + h\Big(-\tfrac{27}{32} f_{n-1} - \tfrac{33}{32} f_{n-2}\Big) + h^2\Big(\tfrac{27}{64} f'_{n-1} - \tfrac{9}{64} f'_{n-2}\Big), \quad p=5,\\
y_n &= \tfrac{128}{29} y_{n-1} - \tfrac{99}{29} y_{n-2} + h\Big(-\tfrac{1792}{783} f_{n-\frac{1}{2}} + \tfrac{32}{29} f_{n-1} - \tfrac{962}{783} f_{n-2}\Big)\\
&\quad + h^2\left(\tfrac{64}{87} f'_{n-\frac{1}{2}} + \tfrac{44}{29} f'_{n-1} - \tfrac{34}{261} f'_{n-2}\right), \quad p = 7,\\
y_n &= y_{n-1} + h\Big(\tfrac{32}{63} f_{n-\frac{1}{2}} + \tfrac{23}{112} f_n + \tfrac{2}{7} f_{n-1} + \tfrac{1}{1008} f_{n-2}\Big)\\
&\quad + h^2\Big(\tfrac{8}{315} f'_{n-\frac{1}{2}} - \tfrac{1}{80} f'_n + \tfrac{1}{35} f'_{n-1} + \tfrac{1}{5040} f'_{n-2}\Big), \quad p = 8.
\end{aligned}$$

Its Nordsieck GLM form is

$$(40) \qquad \left(\begin{array}{ccc|ccc|c}
0 & 0 & 0 & 0 & 0 & 0 & \\
-\frac{1792}{783} & 0 & 0 & \frac{64}{87} & 0 & 0 & UT \\
\frac{32}{63} & \frac{23}{112} & 0 & \frac{8}{315} & -\frac{1}{80} & 0 & \\
\hline
 & T^{-1}B^{(1)} & & & T^{-1}B^{(2)} & & T^{-1}VT
\end{array}\right),$$

where UT, $T^{-1}B^{(1)}, T^{-1}B^{(2)}, T^{-1}VT$ are in appendix D. To distinguish between the wide range of special subcases within (38) and (40), with a view to making a choice for practical use, we consider the stability region for the given abscissae vector $c = [\frac{1}{2}, 1, 1]^T$. The stability polynomial $\det(wI - M(z))$ of (40) is

$$w(w-1) + \tfrac{79z}{56} - \tfrac{135wz}{56} - \tfrac{475z^2}{1624} + \tfrac{2759wz^2}{1624} + \tfrac{311z^3}{6090} - \tfrac{1516wz^3}{1015} + \tfrac{2419z^4}{24360} + \tfrac{79wz^4}{203} + \tfrac{769z^5}{48720} - \tfrac{1357wz^5}{16240} - \tfrac{3z^6}{2320} + \tfrac{9wz^6}{2320}.$$

The interval of absolute stability is $(-0.4, 0)$, see Fig. 3.

4. VARIABLE STEPSIZE IMPLEMENTATION OF THE NORDSIECK GLM IN (5B)

Take note that Section 3 has shown how (11)-(13) can practically be represented as the Nordsieck method in (5b) and found in (18a,b). An effective

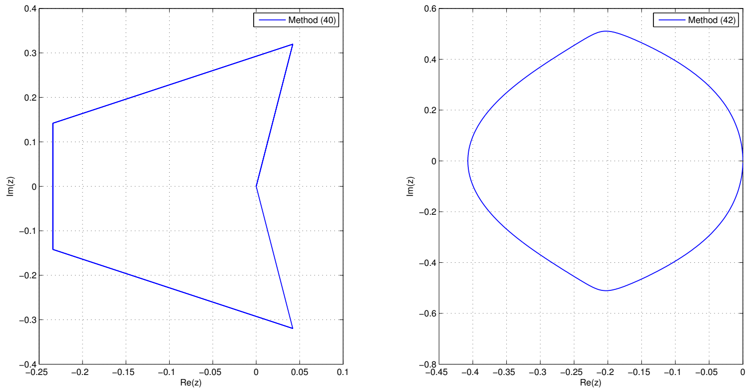

Fig. 3. Region of absolute stability of the Nordsieck methods in (38) and (40).

implementation of (18a,b) requires step-size changing and rescaling of the vector of the external approximations as it becomes necessary, see [17]. The variable stepsize form of (5b) is

$$\begin{aligned} Y &= h_n A^{(1)} F(Y) + h_n^2 A^{(2)} F'(Y) + UTy^{[n-1]}, \\ y^{[n]} &= h_n D(r) T^{-1} B^{(1)} F(Y) + h_n^2 D(r) T^{-1} B^{(2)} F'(Y) + D(r) T^{-1} V T y^{[n-1]} \end{aligned} \tag{41}$$

where the rescaling matrix is given by

$$D(r) = \text{diag}\left(1, r, r^2, \cdots, r^p\right), \quad r = h_n/h_{n-1}, \tag{42}$$

and

$$h_n = rh_{n-1}; \quad r = \min\left\{\max\left\{0.5, 0.9 * \left(\tfrac{TOL}{\|\Delta_n\|}\right)^{1/(p+1)}\right\}, 2.0\right\}, \tag{43}$$

where $n = 1, 2, \ldots, \quad TOL = 10^{-m}; \quad m \gg 1$. (43) is the step-size required to advance from step $n-1$ to n, if the last step was successful, while h_{n-1} represents the step-size in the last attempt either a successful or a failed step, p denotes the order of the corrector formula and TOL is the allowable error estimate specified by the user. The Δ_n in (43) is the local error estimate with the condition $\|\Delta_n\| < TOL$. If it passes this condition, the step is successful and the next step-size will be obtained using the step-size increment formula in (43). We compute the local error estimate using a linear combination of the stage derivatives of the current step only. That is,

$$\Omega_{p+1} h_n^{p+1} y^{(p+1)}_{(x_{n-1})} = h_n \sum_{i=1}^{s} \Theta_i F(Y) + h_n^2 \sum_{i=1}^{s} \psi_i F'(Y) \tag{44}$$

where Ω_{p+1} is the error constant of the output point of the Nordsieck method. Expanding (44) using the Taylor series, it follows that the values of Θ_i and ψ_i

satisfy the resulting system of equations

$$\sum_{i=1}^{p}\Theta_i=0,\quad \sum_{i=1}^{p}\psi_i+\sum_{i=1}^{p}\Theta_i c_i=0,\quad \sum_{i=1}^{p}\psi_i c_i+\sum_{i=1}^{p}\frac{\Theta_i c_i^2}{2!}=0,$$
$$\sum_{i=1}^{p}\frac{\psi_i c_i^2}{2!}+\sum_{i=1}^{p}\frac{\Theta_i c_i^3}{3!}=0,\cdots,\sum_{i=1}^{p}\frac{\psi_i c_i^{q-2}}{(q-2)!}+\sum_{i=1}^{p}\frac{\Theta_i c_i^{q-1}}{(q-1)!}=0, \tag{45}$$
$$\sum_{i=1}^{p}\frac{\psi_i c_i^{q-1}}{(q-1)!}+\sum_{i=1}^{p}\frac{\Theta_i c_i^{q}}{(q)!}=1,\quad q=5,6,\ldots$$

see [12], page 86-87. Consider for example, the method (29) of order $p=4$, $\Omega_5=\frac{1}{720}$ with abscissa vector $c=[\frac{1}{2},1]^T$ in (43). Solving the resulting system of linear equations in (45) yields no value for the variables Θ_1, Θ_2, ψ_1 and ψ_2. No local error estimate exist for the methods in (32), (38) and (40) from (44), therefore, an alternative approach to estimate the error is adopted. To control the error in (29) and (32), $N=3$ in (22) to obtain a method of order 3, one order less than the corrector. The resulting error estimators for (29)($j=1$) and (32) ($j=2$) will therefore be based on $\|y_n^{e_j}-y_1^{[n]}\|$, $j=1,2$ with

$$\begin{aligned}
y_n^{e_1}=&y_{n-1}+\tfrac{h}{3}\left(4f_{n-\frac{1}{2}}-f_{n-1}\right)-\tfrac{h^2}{6}f'_{n-1},\quad p=3,\\
y_n^{e_2}=&y_{n-1}+\tfrac{h}{4}\left(f_{n-1}+3f_{n-\frac{1}{3}}\right),\quad f_n=f(x_n,y_1^{[n]}),\quad p=2.
\end{aligned}$$

Similarly, for the Nordsieck methods in (38) ($j=3$) and (40) ($j=4$) the methods adopted to control the error can be:

$$\begin{aligned}
y_n^{e_3}=&y_{n-1}+h\left(\tfrac{1}{5250}f_{n-2}+\tfrac{473}{1680}f_{n-1}+\tfrac{5589}{14000}f_{n-\frac{1}{3}}+\tfrac{67}{210}f_n\right)\\
&+h^2\left(\tfrac{43}{1680}f'_{n-1}-\tfrac{243}{2800}f'_{n-\frac{1}{3}}-\tfrac{1}{42}f'_n\right),\quad f_n=f(x_n,y_1^{[n]}),\quad p=7,\\
y_n^{e_4}=&y_{n-1}+h\left(\tfrac{1}{7560}f_{n-2}+\tfrac{9}{35}f_{n-1}+\tfrac{61}{280}f_n+\tfrac{496}{945}f_{n-\frac{1}{2}}\right)\\
&+h^2\left(\tfrac{3}{140}f'_{n-1}+\tfrac{4}{315}f'_{n-\frac{1}{2}}-\tfrac{1}{70}f_n\right),\quad p=7.
\end{aligned}$$

The Δ_n in (43) is now $\|y_n^{e_j}-y_1^{[n]}\|$, $j=3,4$ and is the error estimate for the proposed Nordsieck methods in (38), and (40). This is defined as the difference between the output methods of proposed Nordsieck methods in (38), and (40), and the alternative scheme used to control the error. As a result of the asymptotic behaviour of $\|y_n^{e_j}-y_1^{[n]}\|, j=3,4$ we use r in (43) to increase or decrease the error predicted in the next step by multiplying it by h_{n-1}, (see [3], pp. 130 - 131, Algorithm 271α). The availability of initial data $y_i^{[n-1]}$ facilitates easy adjustment of stepsize. The Nordsieck vectors (3b) in (41) are the outgoing and incoming approximation at step number n respectively. The application of (41) to the scalar test problem $y'(x)=0$, yields

$$y^{[n]}=D(r)T^{-1}VTy^{[n-1]}. \tag{46}$$

The zero stability property of the method (41) is determined by the eigenvalues of the matrix $D(r)T^{-1}VT$, see [12] and [17]. Consider for example the matrix $T^{-1}VT$ in (29a) given by

$$T^{-1}VT = \begin{pmatrix} 1 & 1 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & \frac{7}{6} & 0 & 0 & 0 \\ 0 & 8 & \frac{3}{2} & 0 & 0 & 0 \\ 0 & 3 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $D(r)T^{-1}VT$ from (29a) are found to be $\{1, 0, 0, 0, 0, 0\}$. We obtain similarly the eigenvalues of the methods in (38) and (40) as $\{1, \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0\}$ and $\{1, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0\}$ respectively. This means that the Nordsieck methods are zero-stable.

5. THE RK AND SD-RK STARTERS

To start up the methods (5b), we find an approximation to the Nordsieck vector

$$y^{[0]} = \begin{pmatrix} y(x_0) & h_0 y'(x_0) & h_0^2 y''(x_0) & \dots & h_0^p y^{(p)}(x_0) \end{pmatrix}^T, \tag{47}$$

by use of a RK starter

$$\begin{aligned} Y &= h\overline{A}F(Y) + \overline{U}y_0, \\ y^{[0]} &= h\overline{B}F(Y) + \overline{V}y_0. \end{aligned} \tag{48a}$$

In the search of a high order starter for a given stage number, the SD-RK starter

$$\begin{aligned} Y &= h^2 AF'(Y) + h\overline{A}F(Y) + \overline{U}y_0, \\ y^{[0]} &= h^2 BF'(Y) + h\overline{B}F(Y) + \overline{V}y_0, \end{aligned} \tag{48b}$$

comes in handy to advance the solution at x_0 through one single step to $x_0 + h$. The vectors of $\overline{U}$ and $\overline{V}$ in (48a) and (48b) are $(1, 1, \dots, 1)^T$, and $(1, 0, \dots, 0)^T$ respectively. In the Nordsieck vector, the first and second components of (47) are known from the initial values defining the ODE (1), but we can determine the other components using a Runge-Kutta method with multiple outputs. To obtain (48a) we use the internal stages of the method given by

$$Y_i = h \sum_{j=1}^{i-1} \overline{a}_{ij} F(Y_j) + y_0, \quad \widehat{c}_i \in [0, 1], \tag{49a}$$

and for (48b) we use

$$Y_i = h^2 \sum_{j=1}^{i-1} a_{ij} F'(Y_j) + h \sum_{j=1}^{i-1} \overline{a}_{ij} F(Y_j) + y_0, \quad \widehat{c}_i \in [0, 1], \tag{49b}$$

where $Y_i = y(x_0 + \widehat{c}_i h)$. Expanding (49a) by Taylor series and equating the powers of h yields the first stage order condition as

$$\overline{u}_{11} = 1, \overline{a}_{11} = 0, \tag{50}$$

for the second stage we have

$$\overline{u}_{21} = 1, \sum_{j=1}^{1} \overline{a}_{2j} = \widehat{c}_2, \tag{51}$$

while the third stage gives

$$\overline{u}_{31} = 1, \sum_{j=1}^{2} \overline{a}_{3j} = \widehat{c}_3, \sum_{j=1}^{2} \overline{a}_{3j}\widehat{c}_j = \frac{\widehat{c}_3^2}{2!}, \tag{52}$$

and the fourth stage order conditions are

$$\overline{u}_{41} = 1,\ \sum_{j=1}^{3} \overline{a}_{4j} = \widehat{c}_4,\ \sum_{j=1}^{3} \overline{a}_{4j}\widehat{c}_j = \frac{\widehat{c}_4^2}{2!},\ \frac{1}{2!}\sum_{j=1}^{3} \overline{a}_{4j}\widehat{c}_j^2 = \frac{\widehat{c}_4^3}{3!}. \tag{53}$$

Also, the fifth stage order conditions are

$$\begin{gathered} \overline{u}_{51} = 1,\ \sum_{j=1}^{4} \overline{a}_{5j} = \widehat{c}_5, \\ \sum_{j=1}^{4} \overline{a}_{5j}\widehat{c}_j = \frac{\widehat{c}_5^2}{2!},\ \frac{1}{2!}\sum_{j=1}^{4} \overline{a}_{5j}\widehat{c}_j^2 = \frac{\widehat{c}_5^3}{3!},\ \frac{1}{3!}\sum_{j=1}^{4} \overline{a}_{5j}\widehat{c}_j^3 = \frac{\widehat{c}_5^4}{4!}. \end{gathered} \tag{54}$$

Let $\widehat{c} = [0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1]^T$ in (50)-(54) and solving the resulting system of linear equations we obtain the value $\overline{a}'_{ij}s$, of matrix $\overline{A}$.

Similarly, the components of the output method in (48a) are obtained using

$$y^{[0]} = y(x_0 + th) = \sum_{j=1}^{s} \overline{b}_{ij} F(Y_j) + y_0. \tag{55}$$

The $t = 0$, gives the output value at $x = x_0$. If we set $t = 1$ the output values will be for the first point $x_1 = x_0 + h$. Expanding (55) by Taylor series and equating both sides in powers of h, the first component $y_1^{[0]} = y(x_0 + th)$, yields

$$\begin{gathered} \sum_{j=1}^{5} \overline{b}_{1j} = t, \qquad \sum_{j=1}^{5} \overline{b}_{1j}\widehat{c}_j = \frac{1}{2!}t^2, \\ \frac{1}{2!}\sum_{j=1}^{5} \overline{b}_{1j}\widehat{c}_j^2 = \frac{1}{3!}t^3, \qquad \frac{1}{3!}\sum_{j=1}^{5} \overline{b}_{1j}\widehat{c}_j^3 = \frac{1}{4!}t^4, \qquad \frac{1}{4!}\sum_{j=1}^{5} \overline{b}_{1j}\widehat{c}_j^4 = \frac{1}{5!}t^5. \end{gathered} \tag{56}$$

The system of linear equations for the second component $y_2^{[0]} = hy'(x_0 + th)$ is

$$\sum_{j=1}^{5} \bar{b}_{2j} = 1, \qquad \sum_{j=1}^{5} \bar{b}_{2j}\widehat{c}_j = t, \tag{57}$$
$$\frac{1}{2!}\sum_{j=1}^{5} \bar{b}_{2j}\widehat{c}_j^2 = \frac{t^2}{2!}, \qquad \frac{1}{3!}\sum_{j=1}^{5} \bar{b}_{2j}\widehat{c}_j^3 = \frac{t^3}{3!}, \qquad \frac{1}{4!}\sum_{j=1}^{5} \bar{b}_{2j}\widehat{c}_j^4 = \frac{t^4}{4!},$$

The third component $y_3^{[0]} = h^2y''(x_0 + th)$ gives

$$\sum_{j=1}^{5} \bar{b}_{3j} = 0, \sum_{j=1}^{5} \bar{b}_{3j}\widehat{c}_j = 1, \tag{58}$$
$$\frac{1}{2!}\sum_{j=1}^{5} \bar{b}_{3j}\widehat{c}_j^2 = t, \frac{1}{3!}\sum_{j=1}^{5} \bar{b}_{3j}\widehat{c}_j^3 = \frac{t^2}{2!}, \frac{1}{4!}\sum_{j=1}^{5} \bar{b}_{3j}\widehat{c}_j^4 = \frac{t^3}{3!},$$

In the same manner, the fourth component $y_4^{[0]} = h^3y'''(x_0 + th)$ of the output method (48a) gives

$$\sum_{j=1}^{5} \bar{b}_{4j} = 0, \qquad \sum_{j=1}^{5} \bar{b}_{4j}\widehat{c}_j = 0, \tag{59}$$
$$\frac{1}{2!}\sum_{j=1}^{5} \bar{b}_{4j}\widehat{c}_j^2 = 1, \qquad \frac{1}{3!}\sum_{j=1}^{5} \bar{b}_{4j}\widehat{c}_j^3 = t, \qquad \frac{1}{4!}\sum_{j=1}^{5} \bar{b}_{4j}\widehat{c}_j^4 = \frac{t^2}{2!}.$$

Similarly, the fifth component $y_5^{[0]} = h^4y^{(4)}(x_0 + th)$ yields

$$\sum_{j=1}^{5} \bar{b}_{5j} = 0, \qquad \sum_{j=1}^{5} \bar{b}_{5j}\widehat{c}_j = 0, \tag{60}$$
$$\frac{1}{2!}\sum_{j=1}^{5} \bar{b}_{5j}\widehat{c}_j^2 = 0, \qquad \frac{1}{3!}\sum_{j=1}^{5} \bar{b}_{5j}\widehat{c}_j^3 = 1, \qquad \frac{1}{4!}\sum_{j=1}^{5} \bar{b}_{5j}\widehat{c}_j^4 = t,$$

while the sixth component $y_6^{[0]} = h^5y^{(5)}(x_0 + th)$ gives

$$\sum_{j=1}^{5} \bar{b}_{6j} = 0, \qquad \sum_{j=1}^{5} \bar{b}_{6j}\widehat{c}_j = 0, \tag{61}$$
$$\frac{1}{2!}\sum_{j=1}^{5} \bar{b}_{6j}\widehat{c}_j^2 = 0, \qquad \frac{1}{3!}\sum_{j=1}^{5} \bar{b}_{6j}\widehat{c}_j^3 = 0, \qquad \frac{1}{4!}\sum_{j=1}^{5} \bar{b}_{6j}\widehat{c}_j^4 = 1.$$

The GLM representation of (48b) is defined by

$$\left(\begin{array}{c|c} \overline{A} & \overline{U} \\ \hline \overline{B_t} & \overline{V} \end{array}\right); \quad \left(\begin{array}{c|c|c} \overline{A} & A & \overline{U} \\ \hline \overline{B_t} & B_t & \overline{V} \end{array}\right). \tag{62}$$

respectively. Again, let $\widehat{c} = [0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1]^T$ and $t = 0$ in (56)-(61) and solving the resulting system of linear equations respectively gives $\overline{b}'_{ij}s$ and their values are in matrix $\overline{B}$ in (62), given here as,

$$\left(\begin{array}{c|c} \overline{A} & \overline{U} \\ \hline \overline{B}_{t=0} & \overline{V} \end{array}\right) = \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ \frac{3}{16} & 0 & \frac{9}{16} & 0 & 0 & 1 \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 1 \\ \hline & & \overline{B}_{t=0} & & & \overline{V} \end{array}\right),$$

(63a)
$$\left(\overline{B}_{t=0} \mid \overline{V} \right) = \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{25}{3} & 16 & -12 & \frac{16}{3} & -1 & 0 \\ \frac{140}{3} & -\frac{416}{3} & 152 & -\frac{224}{3} & \frac{44}{3} & 0 \\ -160 & 576 & -768 & 448 & -96 & 0 \\ 256 & -1024 & 1536 & -1024 & 256 & 0 \end{array}\right).$$

At $t = 1$ we have

$$\left(\begin{array}{c|c} \overline{A} & \overline{U} \\ \hline \overline{B}_{t=1} & \overline{V} \end{array}\right) = \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ \frac{3}{16} & 0 & \frac{9}{16} & 0 & 0 & 1 \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 1 \\ \hline & & \overline{B}_{t=1} & & & \overline{V} \end{array}\right),$$

(63b)
$$\left(\overline{B}_{t=1} \mid \overline{V} \right) = \left(\begin{array}{cccccc} \frac{7}{90} & \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -\frac{16}{3} & 12 & -16 & \frac{25}{3} & 0 \\ \frac{44}{3} & -\frac{224}{3} & 152 & -\frac{416}{3} & \frac{140}{3} & 0 \\ 96 & -448 & 768 & -576 & 160 & 0 \\ 256 & -1024 & 1536 & -1024 & 256 & 0 \end{array}\right).$$

Following this, a starting method for the Nordsieck method in (38) and (40) is derived. The outputs of the scheme (62) are the starter for the explicit Nordsieck method in (29) and (32) respectively. As was noted in [12], this idea of using a RK method with multiple outputs as starter for a Nordsieck method was introduced by Gear[8]. This can readily and analogously extended to SD-RK starters through (48b).

6. NUMERICAL EXPERIMENTS AND CONCLUSION

To illustrate the application of the schemes, and for the purpose of comparison consider implementation of the methods

(i). order four Adams-Bashforth methods (ABF4) in [3], page 116, Table 244(I),

(ii). order four Runge-Kutta method (RK4) in [3], page 102, (235i),

(iii). order four second derivative Nordsieck method (SDNM4) in (32), (iv). order four second derivative GLM with nearly Almost Runge-Kutta methods stability (SDNARK4) in [20] on the following DETEST problems:

Problem 1:

$$y'(x) = -\frac{y^3}{2}, \quad y(0) = 1, \quad y(x) = 1/\sqrt{1+x}, \quad x \in [0, 5].$$

Problem 2:

$$\begin{cases} y_1'(x) = y_2^2 - 2y_1, & y_1(0) = 0, \quad y_1(x) = xe^{-2x}, \\ y_2'(x) = y_1 - y_2 - xy_2^2, & y_2(0) = 1, \quad y_2(x) = e^{-x}, \\ x \in [0, 1]. \end{cases}$$

Problem 3: A nonlinear chemical reaction.

$$\begin{cases} y_1'(x) = -y_1, & y_1(0) = 1, \\ y_2'(x) = y_1 - y_2^2, & y_2(0) = 0, \\ y_3'(x) = y_2^2, & y_3(0) = 0, \\ x \in [0, 5]. \end{cases}$$

In Tables 1–3, TOL is the tolerance, ns represent the number of total step, nfs is the number of failed step, nfe denotes the number of function evaluations, $Error$ is the $|y_{exact} - y_n|$, while $\|\Delta_n\|_\infty$ represents the maximum error of the error estimate. The y_{exact} and y_n are the exact and the numerical solutions respectively. The results obtained by each of the fourth order methods for Problems 1–3 with $Tol = 10^{-j}, j = 2, 4, 6$ are given in Tables 1–3.

Method	Tol	ns	nrs	nfe
ABF4	10^{-2}	11	0	40
RK4	10^{-2}	9	0	32
SDNM4	10^{-2}	8	0	21
SDNARK4	10^{-2}	64	0	252
ABF4	10^{-4}	39	2	160
RK4	10^{-4}	25	1	100
SDNM4	10^{-4}	16	0	45
SDNARK4	10^{-4}	6351	9	25436
ABF4	10^{-6}	316	4	1276
RK4	10^{-6}	99	2	400
SDNM4	10^{-6}	38	1	114
SDNARK4	10^{-6}	635065	11	2540300

Table 1. Results for solving Problem 1. Output $x = 5$, starting $h = 0.1$.

Table 1 shows that the SDNM4 (32) (*i.e.*, the order 4 second derivative Nordsieck method) performs better than ABF4, RK4 and SDNARK4 in terms of accuracy and computational cost. The main advantage over other RK methods is that its stage order is two. In addition, the transformation of the SDHLMM (11)–(13) to Nordsieck method (18a) also enhanced the performance of the SDNM4 (32) in variable step-size implementation.

The numerical results given in Table 2 are the results of evidences for the very good performance of the proposed method compared with the methods

Method	Tol	$Error$	$\|\|\Delta_n\|\|_\infty$	$Time$
ABF4	10^{-2}	7.4843×10^{-2}	3.2649×10^{-3}	0.010317
RK4	10^{-2}	3.3662×10^{-6}	1.8535×10^{-3}	0.006815
SDNM4	10^{-2}	5.8506×10^{-4}	1.4734×10^{-3}	0.005878
SDNARK4	10^{-2}	5.7540×10^{-6}	6.5610×10^{-3}	0.006422
ABF4	10^{-4}	7.5249×10^{-3}	5.4075×10^{-5}	0.010725
RK4	10^{-4}	3.1057×10^{-3}	4.4688×10^{-5}	0.009433
SDNM4	10^{-4}	1.2355×10^{-5}	3.4720×10^{-5}	0.006053
SDNARK4	10^{-4}	5.2965×10^{-3}	6.5610×10^{-5}	0.204725
ABF4	10^{-6}	2.7454×10^{-3}	6.3987×10^{-7}	0.019703
RK4	10^{-6}	4.1923×10^{-3}	5.9825×10^{-7}	0.010058
SDNM4	10^{-6}	3.3229×10^{-6}	5.2121×10^{-7}	0.006593
SDNARK4	10^{-6}	3.7533×10^{-3}	6.5610×10^{-7}	2525.875703

Table 2. Continuation of Table 1.

Method	Tol	ns	nrs	nfe
ABF4	10^{-2}	28	0	108
RK4	10^{-2}	9	0	32
SDNM4	10^{-2}	6	0	15
SDNARK4	10^{-2}	27	0	104
ABF4	10^{-4}	214	0	852
RK4	10^{-4}	23	0	88
SDNM4	10^{-4}	12	0	33
SDNARK4	10^{-4}	1796	2	7188
ABF4	10^{-6}	2078	3	8320
RK4	10^{-6}	89	0	352
SDNM4	10^{-6}	30	0	87
SDNARK4	10^{-6}	179623	9	718524

Table 3. Results for solving problem 2. Output $x = 1$, starting $h = 0.001$.

Method	Tol	Max Error	Min Error
ABF4	10^{-2}	2.1963×10^{-2}	5.0052×10^{-3}
RK4	10^{-2}	1.5545×10^{-4}	1.0623×10^{-4}
SDNM4	10^{-2}	7.3033×10^{-4}	2.8690×10^{-4}
SDNARK4	10^{-2}	4.4695×10^{-3}	1.7568×10^{-3}
ABF4	10^{-4}	2.2598×10^{-3}	4.4308×10^{-4}
RK4	10^{-4}	5.6210×10^{-7}	4.4258×10^{-7}
SDNM4	10^{-4}	9.7776×10^{-7}	3.5894×10^{-7}
SDNARK4	10^{-4}	2.5733×10^{-4}	5.6534×10^{-5}
ABF4	10^{-6}	1.2749×10^{-4}	9.2037×10^{-5}
RK4	10^{-6}	1.5644×10^{-9}	1.2320×10^{-9}
SDNM4	10^{-6}	3.9004×10^{-9}	1.1259×10^{-9}
SDNARK4	10^{-6}	2.5277×10^{-4}	6.6029×10^{-5}

Table 4. Continuation of Table 2.

considered for comparison. In terms of computational cost, the SDNM4 in (32) outperform the ABF4, RK4, and SDNARK4, for details, see Table 2.

Method	Tol	$\|\Delta_n\|_\infty$	$Time$
ABF4	10^{-2}	5.4844×10^{-3}	0.0210
RK4	10^{-2}	9.3170×10^{-4}	0.0124
SDNM4	10^{-2}	2.5921×10^{-4}	0.0134
SDNARK4	10^{-2}	6.54×10^{-3}	0.0149
ABF4	10^{-4}	6.4298×10^{-5}	0.0343
RK4	10^{-4}	3.7849×10^{-5}	0.0172
SDNM4	10^{-4}	4.3744×10^{-5}	0.0147
SDNARK4	10^{-4}	6.5610×10^{-5}	0.0683
ABF4	10^{-6}	6.5476×10^{-7}	0.1079
RK4	10^{-6}	5.5693×10^{-7}	0.0187
SDNM4	10^{-6}	5.6670×10^{-7}	0.0174
SDNARK4	10^{-6}	6.5610×10^{-7}	287.43

Table 5. Continuation of Table 2.

Method	Tol	ns	nrs	nfe	$\|\Delta_n\|_\infty$	$Time$
ABF4	10^{-2}	45	2	184	5.0653×10^{-3}	0.026
RK4	10^{-2}	13	0	48	2.4439×10^{-3}	0.021
SDNM4	10^{-2}	9	0	24	2.2116×10^{-3}	0.015
SDNARK4	10^{-2}	110	3	448	6.5609×10^{-3}	0.021
ABF4	10^{-4}	406	4	1636	6.3484×10^{-5}	0.040
RK4	10^{-4}	44	1	176	4.9075×10^{-5}	0.018
SDNM4	10^{-4}	21	0	60	4.0231×10^{-5}	0.020
SDNARK4	10^{-4}	10999	9	44028	6.5610×10^{-5}	1.254
ABF4	10^{-6}	4067	5	16284	6.5387×10^{-7}	0.342
RK4	10^{-6}	187	2	752	6.1335×10^{-7}	0.028
SDNM4	10^{-6}	57	1	171	5.4834×10^{-7}	0.017
SDNARK4	10^{-6}	541521	12	2166128	6.5610×10^{-7}	19329.266

Table 6. Results for solving problem 3. Output $x = 5$, starting $h = 0.1$.

The numerical results given in Table 3 confirm the improved performance of the SDNM4 over ABF4, RK4, and SDNARK4 in terms of accuracy and computational cost. In fact, in all the problems solved the SDNM4 has the smallest number of function evaluations when compared with other methods considered herein. However, the numerical results presented in Tables 1, 2 and 3 show the accuracy of the new scheme. Observe from Tables 1–3, as the tolerance becomes smaller, the better is the accuracy of the SDNM4 in (32); however, SDNARK4 at high tolerance gives poor response time.

Conclusively, we have described in this paper the construction of second derivative Nordsieck methods (5b) for nonstiff differential systems (1). This approach can be extended to a higher step numbers and off-step points in (11)–(13). We have shown thus that it is possible to extend the methods of [4] to multiple hybrid points with Nordsieck implementation.

Acknowledgements. The authors wish to express their most sincere thanks to Prof. J.C. Butcher (Univ. of Auckland, Auckland, New Zealand) for the help to implement these methods.

APPENDIX A

$$
\begin{aligned}
u_{21}^{(1)} =& \frac{2+7c_1^2(1+c_2)^3\left(1-3c_2+6c_2^2\right)+2c_2^5(21+5c_2(7+3c_2))}{2+7c_1(1+c_1)}\frac{-7c_1\left(-1+c_2^4(15+2c_2(12+5c_2))\right)}{2+7c_1(1+c_1)},\\
u_{22}^{(1)} =& \frac{c_2^3\left(-7c_1^2(10+3c_2(5+2c_2))-2c_2^2(21+5c_2(7+3\,c_2)\right)}{2+7c_1(1+c_1)} + \frac{7c_1c_2(15+2c_2(12+5c_2)))}{2+7c_1(1+c_1)},\\
u_{21}^{(2)} =& \frac{-\left(c_2(1+c_2)^3\left(-c_1^3(2+7c_1(1+c_1))+3c_1^3(2+7c_1(1+c_1))c_2\right.\right.}{\left(c_1^3(2+7c_1(1+c_1))\right)},\\
&+ \frac{c_1(2+c_1(2-c_1(17+35c_1)))c_2^2+(-1+c_1(-2+c_1(4+15c_1)))c_2^3)\big)}{\left(c_1^3(2+7c_1(1+c_1))\right)},\\
u_{22}^{(2)} =& \frac{(c_2^3(c_1^2(20+c_1(68+7c_1(11+4c_1)))}{((1+c_1)^3(2+7c_1(1+c_1)))} + \frac{c_1(-30+c_1(-69+c_1(-1+7c_1(13+7c_1))))c_2+12c_2^2)}{((1+c_1)^3(2+7c_1(1+c_1)))}\\
&- \frac{c_2^5(3c_1(-4+c_1(-41+7c_1(-7+(-1+c_1)c_1)))}{((1+c_1)^3(2+7c_1(1+c_1)))} + \frac{22c_2+c_1(53-7c_1(-1+c_1(9+5c_1)))c_2}{((1+c_1)^3(2+7c_1(1+c_1)))}\\
&+ \frac{(10+c_1t(35+c_1(41+15c_1)))c_2^2))}{((1+c_1)^3(2+7c_1(1+c_1)))},\\
u_{21}^{(3)} =& \frac{c_2^2(1+c_2)^3\left(3c_1^2(2+7c_1(1+c_1))\right.}{6c_1^2(2+7c_1(1+c_1))}\frac{c_1(8+c_1(31+35c_1))c_2+3(1+c_1(4+5c_1))c_2^2\big)}{6c_1^2(2+7c_1(1+c_1))},\\
u_{22}^{(3)} =& -\frac{c_2^3(1+c_2)^2(c_1^2(10+7c_1(4+3c_1))}{c_1^3(1+c_1)^3(2+7c_1(1+c_1))} - \frac{c_1(15+c_1(44+35c_1))c_2+3(2+c_1(6+5c_1))c_2^2)}{6(1+c_1)^2(2+7c_1(1+c_1))},\\
a_{21}^{(1)} =& -\frac{c_2^3(1+c_2)^3(c_2+c_1(-2+5c_2+c_1(-8-7c_1+5c_2)))}{c_1^3(1+c_1)^3(2+7c_1(1+c_1))},\\
a_{21}^{(2)} =& \frac{c_2^3(1+c_2)^3(3c_2+c_1(-4-7c_1+6c_2))}{6c_1^2(1+c_1)^2(2+7c_1(1+c_1))}.
\end{aligned}
$$

APPENDIX B

$$
\begin{aligned}
b_{11}^{(1)} =& \frac{(14c_1^3(142+7c_2(-49+31c_2))}{(420(c_1-c_2)^3c_2^3(1+c_2)^3)}\frac{-7c_1^2(5+2c_2(103+c_2(-253+155c_2))))}{(420c_1^3(1+c_1)^3(c_1-c_2)^3)},\\
b_{12}^{(1)} =& \frac{(c_2(970+7(5-284c_2)c_2)+14c_1^3(-49+5c_2(-1+31c_2))}{(420(c_1-c_2)^3c_2^3(1+c_2)^3)} + \frac{2c_1^2(568+c_2(793-7c_2(253+217c_2)))}{(420(c_1-c_2)^3c_2^3(1+c_2)^3)}\\
&+ \frac{c_1(-485+2c_2(-1195+7c_2(103+343c_2))))}{(420(c_1-c_2)^3c_2^3(1+c_2)^3)},\\
b_{11}^{(2)} =& \frac{(485+2c_2(-568+343c_2)-4c_1(142+7c_2(-49+31c_2)))}{(840c_1^2(1+c_1)^2(c_1-c_2)^2)},\\
v_{11}^{(1)} =& 1, \qquad v_{12}^{(1)} = 0,\\
b_{12}^{(2)} =& \frac{(485-568c_2+2c_1(-568+7c_1(49-62c_2)+686c_2))}{(840(c_1-c_2)^2c_2^2(1+c_2)^2)},\\
v_{11}^{(2)} =& \frac{(8c_1^2(-1+c_2)(142+7c_2(-49+31c_2))}{(420c_1^3c_2^3)} + \frac{c_2(485+2c_2(-568+343c_2))}{(420c_1^3c_2^3)}\\
&+ \frac{c_1(485+c_2(-2473+4(970-497c_2)c_2))}{(420c_1^3c_2^3)} + \frac{14c_1^3(49+c_2(-142+(124-15c_2)c_2)))}{(420c_1^3c_2^3)},\\
v_{12}^{(2)} =& \frac{(735+2c_2(-595-86c_2+350c_2^2)}{(420(1+c_1)^3(1+c_2)^3)} + \frac{c_1(-1190+c_2(1957+2(181-665c_2)c_2))}{(420(1+c_1)^3(1+c_2)^3)} +
\end{aligned}
$$

$$+\frac{2c_1^2(-86+c_2(181+7c_2(-19+11c_2)))}{(420(1+c_1)^3(1+c_2)^3)}+\frac{14c_1^3(50+c_2(-95+c_2(11+45c_2))))}{(420(1+c_1)^3(1+c_2)^3)},$$

$$v_{11}^{(3)}=\frac{(485+2c_2(-568+343c_2)-8c_1(142+7c_2(-49+31c_2))}{(840c_1^2c_2^2)}+\frac{14c_1^2(49+c_2(-124+85c_2)))}{(840c_1^2c_2^2)},$$

$$v_{12}^{(3)}=\frac{(225+4c_2(-130+77c_2)-4c_1(130+7c_2(-44+27c_2))}{(840(1+c_1)^2(1+c_2)^2)}+\frac{14c_1^2(22+c_2(-54+35c_2)))}{(840(1+c_1)^2(1+c_2)^2)}.$$

APPENDIX C

$$T^{-1}B^{(1)}=\begin{pmatrix} \frac{60993}{140000} & \frac{883}{3360} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{6038307}{560000} & \frac{29139}{4480} & -12 \\ \frac{66055419}{2240000} & \frac{318763}{17920} & -\frac{117}{4} \\ \frac{75204369}{2240000} & \frac{362913}{17920} & -\frac{501}{16} \\ \frac{43488009}{2240000} & \frac{629579}{53760} & -\frac{279}{16} \\ \frac{12625551}{2240000} & \frac{60927}{17920} & -\frac{79}{16} \\ \frac{182979}{280000} & \frac{883}{2240} & -\frac{9}{16} \end{pmatrix}, \qquad T^{-1}B^{(2)}=\begin{pmatrix} -\frac{1863}{28000} & -\frac{59}{3360} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{184437}{112000} & -\frac{1947}{4480} & \frac{9}{2} \\ -\frac{2017629}{448000} & -\frac{21299}{17920} & \frac{33}{4} \\ -\frac{2297079}{448000} & -\frac{24249}{17920} & \frac{63}{8} \\ -\frac{1328319}{448000} & -\frac{42067}{53760} & \frac{33}{8} \\ -\frac{385641}{448000} & -\frac{4071}{17920} & \frac{9}{8} \\ -\frac{5589}{56000} & -\frac{59}{2240} & \frac{1}{8} \end{pmatrix},$$

$$UT=\begin{pmatrix} 1 & \frac{2}{3} & -\frac{142}{243} & \frac{8}{27} & \frac{16}{81} & \frac{32}{243} & -\frac{104}{81} & \frac{88}{27} & -\frac{1472}{243} \\ 1 & \frac{757}{1000} & \frac{31}{1000} & \frac{17}{125} & -\frac{1}{125} & -\frac{1}{25} & \frac{1}{125} & \frac{13}{125} & -\frac{37}{125} \\ 1 & \frac{63323}{210000} & \frac{4181}{140000} & \frac{311}{140000} & -\frac{103}{52500} & \frac{101}{84000} & \frac{1}{17500} & -\frac{109}{60000} & \frac{107}{26250} \end{pmatrix},$$

$$T^{-1}VT=\begin{pmatrix} 1 & \frac{63323}{210000} & \frac{4181}{140000} & \frac{311}{140000} & -\frac{103}{52500} & \frac{101}{84000} & \frac{1}{17500} & -\frac{109}{60000} & \frac{107}{26250} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1480341}{280000} & -\frac{286081}{560000} & -\frac{669211}{560000} & \frac{14101}{70000} & \frac{31333}{112000} & -\frac{17401}{70000} & -\frac{23597}{80000} & \frac{47281}{35000} \\ 0 & -\frac{20190397}{1120000} & -\frac{3171977}{2240000} & -\frac{10163187}{2240000} & \frac{190317}{280000} & \frac{512461}{448000} & -\frac{261417}{280000} & -\frac{419349}{320000} & \frac{782377}{140000} \\ 0 & -\frac{25214247}{1120000} & -\frac{2264827}{2240000} & -\frac{14736537}{2240000} & \frac{220167}{280000} & \frac{825511}{448000} & -\frac{366267}{280000} & -\frac{764799}{320000} & \frac{1295227}{140000} \\ 0 & -\frac{45990701}{3360000} & \frac{41053}{2240000} & -\frac{10138257}{2240000} & \frac{294061}{840000} & \frac{1920013}{1344000} & -\frac{226787}{280000} & -\frac{2117717}{960000} & \frac{3200041}{420000} \\ 0 & -\frac{4590713}{1120000} & \frac{445467}{2240000} & -\frac{3295623}{2240000} & \frac{10393}{280000} & \frac{230969}{448000} & -\frac{52293}{280000} & -\frac{327521}{320000} & \frac{436133}{140000} \\ 0 & -\frac{67927}{140000} & \frac{12543}{280000} & -\frac{51567}{280000} & -\frac{103}{35000} & \frac{3601}{56000} & \frac{3}{35000} & -\frac{7609}{40000} & \frac{8857}{17500} \end{pmatrix}.$$

$$T^{-1}B^{(1)}=\begin{pmatrix} \frac{32}{63} & \frac{23}{112} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{88}{7} & \frac{2277}{448} & -12 \\ \frac{722}{21} & \frac{24909}{1792} & -\frac{117}{4} \\ \frac{274}{7} & \frac{28359}{1792} & -\frac{501}{16} \\ \frac{1426}{63} & \frac{16399}{1792} & -\frac{279}{16} \\ \frac{46}{7} & \frac{4761}{1792} & -\frac{79}{16} \\ \frac{16}{21} & \frac{69}{224} & -\frac{9}{16} \end{pmatrix}, \quad T^{-1}B^{(2)}=\begin{pmatrix} \frac{8}{315} & -\frac{1}{80} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ \frac{22}{35} & -\frac{99}{320} & \frac{9}{2} \\ \frac{361}{210} & -\frac{1083}{1280} & \frac{33}{4} \\ \frac{137}{70} & -\frac{1233}{1280} & \frac{63}{8} \\ \frac{713}{630} & -\frac{713}{1280} & \frac{33}{8} \\ \frac{23}{70} & -\frac{207}{1280} & \frac{9}{8} \\ \frac{4}{105} & -\frac{3}{160} & \frac{1}{8} \end{pmatrix},$$

7. APPENDIX D

$$UT = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{11}{64} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & -\frac{13}{32} & \frac{17}{16} & -2 \\ 1 & \frac{685}{783} & \frac{2125}{783} & -\frac{497}{261} & \frac{1841}{783} & -\frac{1987}{783} & \frac{643}{261} & -\frac{1667}{783} & \frac{1201}{783} \\ 1 & \frac{289}{1008} & \frac{17}{630} & \frac{1}{560} & -\frac{1}{630} & \frac{1}{1008} & 0 & -\frac{1}{720} & \frac{1}{315} \end{pmatrix},$$

$$T^{-1}VT = \begin{pmatrix} 1 & \frac{289}{1008} & \frac{17}{630} & \frac{1}{560} & -\frac{1}{630} & \frac{1}{1008} & 0 & -\frac{1}{720} & \frac{1}{315} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2533}{448} & -\frac{163}{280} & -\frac{2701}{2240} & \frac{59}{280} & \frac{123}{448} & -\frac{1}{4} & -\frac{91}{320} & \frac{93}{70} \\ 0 & -\frac{102311}{5376} & -\frac{5413}{3360} & -\frac{40917}{8960} & \frac{2369}{3360} & \frac{6073}{5376} & -\frac{15}{16} & -\frac{4921}{3840} & \frac{4643}{840} \\ 0 & -\frac{42391}{1792} & -\frac{1381}{1120} & -\frac{59247}{8960} & \frac{913}{1120} & \frac{3273}{1792} & -\frac{21}{16} & -\frac{3017}{1280} & \frac{2571}{280} \\ 0 & -\frac{231415}{16128} & -\frac{1109}{10080} & -\frac{40727}{8960} & \frac{3697}{10080} & \frac{22889}{16128} & -\frac{13}{16} & -\frac{25193}{11520} & \frac{19099}{2520} \\ 0 & -\frac{7689}{1792} & \frac{181}{1120} & -\frac{13233}{8960} & \frac{47}{1120} & \frac{919}{1792} & -\frac{3}{16} & -\frac{1303}{1280} & \frac{869}{280} \\ 0 & -\frac{341}{672} & \frac{17}{420} & -\frac{207}{1120} & -\frac{1}{420} & \frac{43}{672} & 0 & -\frac{91}{480} & \frac{53}{105} \end{pmatrix}.$$

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Received by the editors: November 24, 2017. Accepted: May 16, 2019. Published online: October 10, 2019.