

ON THE NUMERICAL PICARD ITERATIONS WITH
COLLOCATIONS FOR THE INITIAL VALUE PROBLEM

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Abstract. Some variants of the numerical Picard iterations method are presented to solve an IVP for an ordinary differential system. The term numerical emphasizes that a numerical solution is computed. The method consists in replacing the right hand side of the differential system by Lagrange interpolation polynomials followed by successive approximations. In the case when the number of interpolation point is fixed a convergence result is given. For stiff problems, starting from a stabilization principle, it is given a variant of the Picard iteration method.

Finally, some numerical experiments are reported.

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1. INTRODUCTION

This paper presents variants of the Picard iterations to solve an *initial value problem* (IVP) for ordinary differential equations. On subintervals the right hand side of the differential system is replaced by Lagrange interpolation polynomials and then successive approximations are used. The interpolation nodes are the images of a set of reference points. The number of these reference points can be fixed or variable, i.e. increasing in number [8].

When the number of reference nodes is fixed the approximations of the solution of the IVP are computed by collocations. A convergence result is given. This case appears in [7, p. 211]. In [3] the spectral deferred correction is defined adding a term to the iteration formula and the convergence of that method is proved.

If the number of reference points increases then the values of the unknown function are determined iteratively [8].

The idea to replace the right hand side of a differential equation with an interpolation polynomial occurs in the multi-step methods, but instead of using the points of the main mesh a reference set of points is used. The usage of an iterative process to obtain some convergence is present in predictor-corrector methods.

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We use the terminology numerical Picard iterations to emphasize that the method builds a numerical solution. For an IVP the usual Picard iterations are exemplified with Computer Algebra code in [12].

For stiff problems, starting from a *stabilization principle*, [4], [2], we derived a variant of the Picard iteration method.

There is another approach to the numerical Picard iterations for an IVP, where the approximations are a linear form of Chebyshev polynomials [6], [1], [3].

Sometimes to verify a numerical computation, it is a practical rule to use two different tools or methods. The presented methods offer an alternative to solve an IVP.

The paper is organized as follows. After introducing the numerical Picard iterations method in Section 2, two cases are presented in the next sections. In Section 3 the Picard iteration method is studied when the reference set contains a fixed number of points, while in Section 4 the Picard iteration method is considered with an increasing in number of points of the reference set. In Section 5 the case of a stiff problem is treated. In the last section some results of our computational experiments are presented.

2. NUMERICAL PICARD ITERATIONS

Let the IVP be given by

$$(2.1) \quad \dot{y}(x) = f(x, y(x)), \quad x \in [x_0, x_f],$$

$$(2.2) \quad y(x_0) = y^0,$$

where the function $f : [x_0, x_f] \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ has the components $f = (f^1, \dots, f^N)$.

In \mathbb{R}^N for $y = (y^1, \dots, y^N)$ we shall use the norm $\|y\| = \max_{1 \leq j \leq N} |y^j|$.

We assume that f is Lipschitz continuous, i.e. there exists $L > 0$ such that

$$|f^\mu(x, y_1) - f^\mu(x, y_2)| \leq L \sum_{j=1}^N |y_1^j - y_2^j| \quad \forall y_1, y_2 \in \mathbb{R}^N, \mu \in \{1, 2, \dots, N\}$$

and consequently

$$\|f(x, y_1) - f(x, y_2)\| \leq \tilde{L} \|y_1 - y_2\|,$$

where $\tilde{L} = NL$.

The IVP (2.1)-(2.2) may be reformulated as the integral equation

$$(2.3) \quad y(x) = y^0 + \int_{x_0}^x f(s, y(s)) ds.$$

Within these hypotheses the problem (2.1)-(2.2) or (2.3) has a unique solution. This solution may be obtained with the Picard iterations

$$\begin{aligned} y^{(n+1)}(x) &= y^0 + \int_{x_0}^x f(s, y^{(n)}(s)) ds, \quad n \in \mathbb{N}, \\ y^{(0)}(x) &= y^0, \end{aligned}$$

for $x \in [x_0, x_f]$. The sequence $(y^{(n)}(x))_{k \in \mathbb{N}}$ converges uniformly in $[x_0, x_f]$ to the solution of IVP.

Let M be a positive integer, $h = \frac{x_f - x_0}{M}$ and the mesh be defined as $x_i = x_0 + ih$, $i \in \{0, 1, \dots, M\}$. The numerical solution is given by the sequence $u_h = (u_0, u_1, \dots, u_M)$, where each $u_i = u(x_i)$ is an approximation of $y(x_i)$.

If u_i was computed, on the interval $[x_i, x_{i+1}]$ the function $f(s, y(s))$ under the integral in

$$(2.4) \quad y(x) = y(x_i) + \int_{x_i}^x f(s, y(s)) ds$$

will be replaced by a Lagrange interpolation polynomial

$$(2.5) \quad u(x) = u(x_i) + \int_{x_i}^x L(\mathbb{P}_{m-1}; x_{i,1}, x_{i,2}, \dots, x_{i,m}; f(\cdot, u(\cdot)))(s) ds, \quad x \in [x_i, x_{i+1}].$$

The interpolation nodes $x_i \leq x_{i,1} < x_{i,2} < \dots < x_{i,m} \leq x_{i+1}$ are fixed by a certain rule. The used notation states the interpolation constraints

$$L(\mathbb{P}_{m-1}; x_{i,1}, x_{i,2}, \dots, x_{i,m}; f(\cdot, u(\cdot)))(x_{i,j}) = f(x_{i,j}, u(x_{i,j})), \quad j \in \{1, 2, \dots, m\}.$$

From (2.5) we deduce

$$(2.6) \quad u(x) = u(x_i) + \sum_{j=1}^m \left(\int_{x_i}^x l_j(s) ds \right) f(x_{i,j}, u(x_{i,j})),$$

where $(l_j)_{1 \leq j \leq m}$ are the Lagrange fundamental polynomials

$$(2.7) \quad l_j(x) = \frac{(x-x_{i,1}) \dots (x-x_{i,j-1})(x-x_{i,j+1}) \dots (x-x_{i,m})}{(x_{i,j}-x_{i,1}) \dots (x_{i,j}-x_{i,j-1})(x_{i,j}-x_{i,j+1}) \dots (x_{i,j}-x_{i,m})}.$$

3. PICARD ITERATIONS WITH A FIXED REFERENCE SET

In (2.6) the values

$$u(x_{i,1}), u(x_{i,2}), \dots, u(x_{i,m})$$

are unknown. To compute these vectors the collocation method will be used.

Choosing $x := x_{i,k}$ in (2.6) we get

$$(3.1) \quad u(x_{i,k}) = u(x_i) + \sum_{j=1}^m \left(\int_{x_i}^{x_{i,k}} l_j(s) ds \right) f(x_{i,j}, u(x_{i,j})), \quad k \in \{1, 2, \dots, m\}.$$

The relations (3.1) form a nonlinear system with the unknowns $u(x_{i,1}), \dots, u(x_{i,m}) \in \underbrace{\mathbb{R}^N \times \dots \times \mathbb{R}^N}_m \simeq \mathbb{R}^{mN}$.

In order to simplify and provides a unitary approach to the computation of the integrals from (3.1) we fix the nodes $\xi_1 < \xi_2 < \dots < \xi_m$ within an interval $[a, b]$. We call these nodes the reference interpolation nodes. The function

$$\varphi_i(\xi) = x_i + \frac{h}{b-a}(\xi - a)$$

maps the interval $[a, b]$ into $[x_i, x_{i+1}]$. For any $i \in \{0, 1, \dots, M-1\}$ the nodes $x_{i,j}$ will be defined as

$$x_{i,j} = \varphi_i(\xi_j), \quad \forall j \in \{1, 2, \dots, m\}.$$

If $s = \varphi_i(\xi)$ then

$$l_j(s) = \frac{(\xi - \xi_1) \dots (\xi - \xi_{j-1})(\xi - \xi_{j+1}) \dots (\xi - \xi_m)}{(\xi_j - \xi_1) \dots (\xi_j - \xi_{j-1})(\xi_j - \xi_{j+1}) \dots (\xi_j - \xi_m)} = \tilde{l}_j(\xi)$$

and

$$\int_{x_i}^{x_{i,k}} l_j(s) ds = \frac{h}{b-a} \int_a^{\xi_k} \tilde{l}_j(\xi) d\xi.$$

Denoting

$$w_{j,k} = \frac{1}{b-a} \int_a^{\xi_k} \tilde{l}_j(\xi) d\xi$$

the nonlinear system (3.1) becomes

$$(3.2) \quad u(x_{i,k}) = u(x_i) + h \sum_{j=1}^m w_{j,k} f(x_{i,j}, u(x_{i,j})), \quad k \in \{1, 2, \dots, m\}.$$

In order to prove the existence of a solution of the nonlinear system we shall use a simplified notation $u(x_{i,k}) = u_k$, $k \in \{1, 2, \dots, m\}$. Then the system (3.2) is written as

$$(3.3) \quad u_k = u(x_i) + h \sum_{j=1}^m w_{j,k} f(x_{i,j}, u_j), \quad k \in \{1, 2, \dots, m\}.$$

The operator

$$\Phi = (\Phi_k)_{1 \leq k \leq m}, \quad \text{where} \quad \Phi_k : \underbrace{\mathbb{R}^N \times \dots \times \mathbb{R}^N}_m \rightarrow \mathbb{R}^N$$

is defined by

$$\Phi_k(u) = u(x_i) + h \sum_{j=1}^m w_{j,k} f(x_{i,j}, u_j), \quad u = (u_1, \dots, u_m).$$

The used norm in $\underbrace{\mathbb{R}^N \times \dots \times \mathbb{R}^N}_m$ will be

$$\|u\| = \|(u_1, \dots, u_m)\| = \sum_{j=1}^m \|u_j\|.$$

If $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_m)$ then following equality is valid

$$\Phi_k(u) - \Phi_k(v) = h \sum_{j=1}^m w_{j,k} (f(x_{i,j}, u_j) - f(x_{i,j}, v_j)), \quad k \in \{1, 2, \dots, m\}.$$

Then

$$\|\Phi_k(u) - \Phi_k(v)\| \leq h \tilde{L} \sum_{j=1}^m |w_{j,k}| \|u_j - v_j\|$$

and

$$\sum_{k=1}^m \|\Phi_k(u) - \Phi_k(v)\| \leq h\tilde{L} \sum_{k,j=1}^m |w_{j,k}| \|u_j - v_j\|.$$

If $\omega = \max_{1 \leq j \leq m} \sum_{k=1}^m |w_{j,k}|$ the above inequality gives

$$(3.4) \quad \|\Phi(u) - \Phi(v)\| \leq h\omega\tilde{L}\|u - v\|.$$

Thus, if $h\omega\tilde{L} < 1$ then Φ is a contraction. Following theorem is a consequence of the above:

THEOREM 3.1. *For h small enough ($h < \frac{1}{\omega\tilde{L}}$) the nonlinear system (3.2) has a unique solution.*

In the hypothesis of the above theorem, the nonlinear system (3.2) may be solved using the successive approximation method

$$(3.5) \quad u^{(n+1)}(x_{i,k}) = u(x_i) + h \sum_{j=1}^m w_{j,k} f(x_{i,j}, u^{(n)}(x_{i,j})), \quad n \in \mathbb{N}$$

$$(3.6) \quad u^{(0)}(x_{i,k}) = u(x_i)$$

for $k \in \{1, 2, \dots, m\}$. Because Φ is contraction, (3.4), the sequences

$$u_{i,j}^{(n)} \stackrel{\text{def}}{=} u^{(n)}(x_{i,j}), \quad n \in \mathbb{N}, \quad j \in \{1, 2, \dots, m\}$$

will converge to the solution of the system (3.2).

The iterative relations (3.5) can be written in matrix form

$$(3.7) \quad [u_{i,1}^{(n+1)} \quad u_{i,2}^{(n+1)} \quad \dots \quad u_{i,m}^{(n+1)}] = \underbrace{[u_i \quad u_i \quad \dots \quad u_i]}_m + h[f_{i,1} \quad f_{i,2} \quad \dots \quad f_{i,m}] \begin{pmatrix} w_{1,1} & \dots & w_{1,m} \\ w_{2,1} & \dots & w_{2,m} \\ \vdots & & \vdots \\ w_{m,1} & \dots & w_{m,m} \end{pmatrix},$$

where $f_{i,j} = f(x_{i,j}, u_{i,j}^{(n)})$, $j \in \{1, \dots, m\}$. Denoting $u_i^{(n)} = (u_{i,j}^{(n)})_{1 \leq j \leq m}$ the iterations stop when the following condition is fulfilled $\|u_i^{(n)} - u_i^{(n-1)}\| < \varepsilon$, where $\varepsilon > 0$ is a tolerance. The initial approximations are chosen as $u_{i,j}^{(0)} = u(x_i)$ for any $j \in \{1, 2, \dots, m\}$.

This method to solve the nonlinear system (3.2) leads to an approximation to the solution of the IVP in the most right node which may differ from x_{i+1} . We point out two variants of the computations:

- We change the initial mesh such that x_{i+1} will be the most right node ($x_{i+1} = \varphi_i(\xi_m)$) and the computation continue in the interval $[x_{i+1}, x_{i+1} + h]$. In this case we have

$$u_{i+1} \stackrel{\text{def}}{=} u(x_{i+1}) = u_{i,m}^{(n)}.$$

- In (2.5) we set $x := x_{i+1} = \varphi_i(b)$ and

$$u_{i+1} \stackrel{\text{def}}{=} u(x_{i+1}) = u(x_i) + \frac{h}{b-a} \sum_{j=1}^m \left(\int_a^b \tilde{l}_j(\xi) d\xi \right) f(x_{i,j}, u_{i,j}^{(n)}).$$

In this way m new integrals must be computed additionally.

With the new notations we have $u_0 \stackrel{\text{def}}{=} u(x_0) = y^0$.

The coefficients $w_{j,k}$ do not depend on the computation interval. We highlight some cases when these coefficients may be easily computed.

Some particular cases.

- (1) Equidistant nodes. If $\xi_j = \frac{j-1}{m-1}$, $j \in \{1, 2, \dots, m\}$, then

$$w_{j,k} = \int_0^{\xi_k} \tilde{l}_j(\xi) d\xi = \frac{(-1)^{m-j}}{(j-1)!(m-j)!} \int_0^{\frac{k-1}{m-1}} \prod_{\substack{\mu=0 \\ \mu \neq j-1}}^{m-1} ((m-1)\xi - \mu) d\xi.$$

The following Mathematica code computes these coefficients:

```
Wcoeff[j_, k_, m.]:=
Module[{x, w},
w = Integrate[Product[If[i ≠ j - 1, (m - 1)x - i, 1], {i, 0, m - 1}],
{x, 0, (k - 1)/(m - 1)}]; (-1)^(m - j)w/((j - 1)!(m - j)!)]
```

The results obtained for $m = 2$ are

```
MatrixForm[Table[Wcoeff[j, k, 2], {k, 1, 2}, {j, 1, 2}]]
```

$$\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Because $x_{i,1} = x_i$ și $x_{i,2} = x_{i+1}$ the recurrence formula (3.5)-(3.6) becomes

$$(3.8) \quad \begin{aligned} u_{i+1}^{(n+1)} &= u_i + \frac{h}{2} \left(f(x_i, u_i) + f(x_{i+1}, u_{i+1}^{(n)}) \right); \\ u_{i+1}^{(0)} &= u_i. \end{aligned}$$

For $m = 3$ the results are

```
MatrixForm[Table[Wcoeff[j, k, 3], {k, 1, 3}, {j, 1, 3}]]
```

$$\begin{pmatrix} 0 & 0 & 0 \\ \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix}$$

In this case $x_{i,1} = x_i, x_{i,2} = \frac{1}{2}(x_i + x_{i+1}) \stackrel{\text{def}}{=} x_{i+\frac{1}{2}}, x_{i,3} = x_{i+1}$ and the recurrence formulas (3.5)-(3.6) become

$$(3.9) \quad \begin{aligned} u_{i+\frac{1}{2}}^{(n+1)} &= u_i + h \left(\frac{5}{24} f(x_i, u_i) + \frac{1}{3} f(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}^{(n)}) - \frac{1}{24} f(x_{i+1}, u_{i+1}^{(n)}) \right); \\ u_{i+1}^{(n+1)} &= u_i + \frac{h}{6} \left(f(x_i, u_i) + 4f(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}^{(n)}) + f(x_{i+1}, u_{i+1}^{(n)}) \right); \\ u_{i+\frac{1}{2}}^{(0)} &= u_i; \\ u_{i+1}^{(0)} &= u_i. \end{aligned}$$

In matrix form the above relations are

$$\begin{bmatrix} u_i \\ u_{i+\frac{1}{2}}^{(n+1)} \\ u_{i+1}^{(n+1)} \end{bmatrix} = \begin{bmatrix} u_i \\ u_i \\ u_i \end{bmatrix} + h \begin{pmatrix} 0 & 0 & 0 \\ \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix} \begin{bmatrix} f(x_i, u_i) \\ f(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}^{(n)}) \\ f(x_{i+1}, u_{i+1}^{(n)}) \end{bmatrix}.$$

Transposing the above equality we get the form corresponding to (3.7).

To compute u_{i+1} we observe that for $m = 2$ the trapezoidal rule (3.8), while for $m = 3$ the Simpson integration formula (3.9) are used.

- (2) Chebyshev points of second kind $\xi_j = \cos \frac{(j-1)\pi}{m-1}$, $j \in \{1, \dots, m\}$. Then

$$w_{j,k} = \frac{1}{2} \int_{-1}^{\xi_k} \tilde{l}_j(\xi) d\xi = \frac{(-1)^{j-1} 2^{m-3} \gamma_j}{m-1} \int_{-1}^{\xi_k} \prod_{k=1, k \neq j}^m (\xi - \xi_k) d\xi$$

$$\text{with } \gamma_j = \begin{cases} 0.5 & \text{if } j \in \{1, m\} \\ 1 & \text{if } j \in \{2, \dots, m-1\} \end{cases}.$$

- (3) The nodes are the roots of an orthogonal polynomial. Now we suppose that the polynomial $p_m(\xi) = \prod_{j=1}^m (\xi - \xi_j)$ is orthogonal to \mathbb{P}_{m-1} , the set of polynomials of degree at most $m-1$, with the weight $\rho(\xi)$ on the interval $I = [a, b]$. In this case the Lagrange fundamental polynomials $\tilde{l}_j(\xi)$, $j \in \{1, \dots, m\}$ are orthogonal.

- If $\rho(\xi) = 1, I = [a, b]$ then $p_m(\xi) = \frac{m!}{(2m)!} \frac{d^m}{d\xi^m} (\xi - a)^m (\xi - b)^m$ is the Laguerre polynomial. For $a = 0, b = 1$ and $m = 1$ following results are obtained

$$\begin{aligned} p_1(\xi) &= \xi - \frac{1}{2} \quad \Rightarrow \quad \xi_1 = \frac{1}{2} \\ w_{1,1} &= \int_0^{\frac{1}{2}} d\xi = \frac{1}{2} \\ u_{i+\frac{1}{2}}^{(n+1)} &= u_i + \frac{h}{2} f(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}^{(n)}) \end{aligned}$$

Again we observe that $u(x_{i+\frac{1}{2}})$ is computed using the rectangle rule in the right hand side of (2.5).

- The Chebyshev polynomials $p_m(\xi) = \frac{1}{2^{m-1}} \cos(m \arccos \xi)$, $m \in \mathbb{N}$, are orthogonal with the weight $\rho(\xi) = \frac{1}{\sqrt{1-\xi^2}}$ in $I = [-1, 1]$.

The nodes will be

$$\xi_j = \cos \frac{(2j-1)\pi}{2m} \Rightarrow x_{i,j} = x_i + \frac{h}{2}(\xi_j + 1), \quad j \in \{1, 2, \dots, m\}.$$

The biggest node is $x_{i,1}$. The Lagrange fundamental polynomials are

$$\tilde{l}_j(\xi) = \frac{2^{m-1}}{m} (-1)^{j-1} \sin \frac{(2j-1)\pi}{2m} \prod_{\substack{\mu=1 \\ \mu \neq j}}^m \left(\xi - \cos \frac{(2\mu-1)\pi}{2m} \right)$$

and

$$w_{j,k} = \frac{2^{m-2}}{m} (-1)^{j-1} \sin \frac{(2j-1)\pi}{2m} \int_{-1}^{\cos \frac{(2k-1)\pi}{2m}} \prod_{\substack{\mu=1 \\ \mu \neq j}}^m \left(\xi - \cos \frac{(2\mu-1)\pi}{2m} \right) d\xi$$

The integral can be analytically computed but it involves rounding errors.

3.1. The convergence of the method. When h , the distance between two mesh points x_i and $x_{i+1} = x_i + h$, is small enough the convergence of Picard iterations is provided by the contraction condition. We recall that m , the number of the reference set is fixed. The convergence problem refers to the behavior of the numerical solution (u_0, u_1, \dots, u_M) to the analytical solution $(y(x_0), y(x_1), \dots, y(x_M))$.

We suppose that:

- A_1 The function $f(x, y(x))$ is continuous and then there exists a constant $K_1 > 0$ such that

$$\max_{1 \leq \mu \leq N} \max_{x \in [x_0, x_f]} |f^\mu(x, y(x))| \leq K_1.$$

- A_2 The function $f(x, y)$ have continuous partial derivatives of order m for any $x \in [x_0, x_f], y \in \mathbb{R}^N$. There exists $K_m > 0$ such that

$$\max_{1 \leq \mu \leq N} \max_{x \in [x_0, x_f]} \left| \frac{d^m f^\mu(x, y(x))}{dx^m} \right| \leq K_m.$$

In any interval $[x_i, x_{i+1}]$ the following equality is valid, [10, Th. 2.1.4.1],

$$\begin{aligned} & f^\mu(x, y(x)) - L(\mathbb{P}_{m-1}; x_{i,1}, x_{i,2}, \dots, x_{i,m}; f^\mu(\cdot, y(\cdot)))(x) = \\ & = \frac{1}{m!} \prod_{j=1}^m (x - x_{i,j}) \left. \frac{d^m f^\mu(x, y(x))}{dx^m} \right|_{x=\eta_\mu} \end{aligned}$$

where $\eta_\mu \in [x_i, x_{i+1}]$.

We denote by $R^\mu(x)$ the right hand side of the above equality and then $\max_{x \in [x_i, x_{i+1}]} |R^\mu(x)| \leq \frac{K_m}{m!} h^m$. If $R(x) = (R^1(x), \dots, R^N(x))$ then (2.4) implies the vectorial relation

(3.10)

$$y(x) = y(x_i) + \int_{x_i}^x L(\mathbb{P}_{m-1}; x_{i,1}, x_{i,2}, \dots, x_{i,m}; f(\cdot, y(\cdot)))(s) ds + \int_{x_i}^x R(s) ds$$

and $\| \int_{x_i}^x R(s) ds \| \leq \frac{K_m}{m!} h^{m+1}$.

We make the following notations

$$\begin{aligned} e_i &= \|y(x_i) - u_i\|, & i \in \{0, 1, \dots, M\}; \\ r_{i,j}^{(n)} &= \|y(x_{i,j}) - u_{i,j}^{(n)}\|, & j \in \{1, 2, \dots, m\}; \\ r_i^{(n)} &= \max_{1 \leq j \leq m} r_{i,j}^{(n)}. \end{aligned}$$

and additionally

$$w = \max \left\{ \max_{1 \leq j, k \leq m} |w_{j,k}|, \max_{1 \leq j \leq m} \frac{1}{b-a} \left| \int_a^b \tilde{l}_j(\xi) d\xi \right| \right\}, \quad \tilde{w} = mw.$$

We emphasize that n represents the number of iterations on an interval $[x_i, x_{i+1}]$. This number differs from one interval to another. For simplicity we omitted the index i when n is written.

Several times the following theorem will be used

THEOREM 3.2. *If $(z_k)_{k \in \mathbb{N}}$ is a sequence of nonnegative numbers such that*

$$z_{k+1} \leq az_k + b \quad \forall k \in \mathbb{N} \quad \text{si} \quad a, b > 0, \quad a \neq 1,$$

then

$$z_k \leq a^k z_0 + b \frac{a^k - 1}{a - 1}, \quad \forall k \in \mathbb{N}.$$

The above inequality implies: if $a > 1$ then $z_k \leq a^k \left(z_0 + \frac{b}{a-1} \right)$ and if $a < 1$ then $z_k \leq a^k z_0 + \frac{b}{1-a}$.

The following result of convergence occurs:

THEOREM 3.3. *If the above assumptions take place then*

$$\lim_{h \searrow 0} \max_{i \in \{0, 1, \dots, M\}} \|y(x_i) - u_i\| = 0,$$

that is, the convergence of the method.

Proof. In the beginning we determine an evaluation for $r_i^{(n)}$.

For $n = 0$ the equalities hold:

$$\begin{aligned} y(x_{i,j}) - u_{i,j}^{(0)} &= y(x_{i,j}) - u_i = (y(x_{i,j}) - y(x_i)) + (y(x_i) - u_i) \\ &= \int_{x_i}^{x_{i,j}} f(s, y(s)) ds + (y(x_i) - u_i) \end{aligned}$$

and then we deduce

$$r_{i,j}^{(0)} \leq K_1 h + e_i, \quad \forall j \in \{1, 2, \dots, m\} \quad \Rightarrow \quad r_i^{(0)} \leq K_1 h + e_i.$$

If $n > 0$, for $x = x_{i,k}$ the equality (3.10) may be written as

$$(3.11) \quad y(x_{i,k}) = y(x_i) + h \sum_{j=1}^m w_{j,k} f(x_i, j, y(x_{i,j})) + \int_{x_i}^{x_{i,k}} R(s) ds.$$

Subtracting (3.5) from (3.11) we obtain

$$\begin{aligned} y(x_{i,k}) - u_{i,k}^{(n+1)} &= \\ &= y(x_i) - u_i + h \sum_{j=1}^m w_{j,k} \left(f(x_{i,j}, y(x_{i,j})) - f(x_{i,j}, u_{i,j}^{(n)}) \right) + \int_{x_i}^{x_{i,k}} R(s) ds. \end{aligned}$$

It follows that

$$r_{i,k}^{(n+1)} \leq e_i + h\tilde{L}\tilde{w}r_i^{(n)} + \frac{K_m}{m!}h^{m+1} \quad \Rightarrow \quad r_i^{(n+1)} \leq e_i + h\tilde{L}\tilde{w}r_i^{(n)} + \frac{K_m}{m!}h^{m+1}$$

If h is small enough ($h\tilde{L}\tilde{w} < 1$) then

$$\begin{aligned} r_i^{(n)} &\leq (h\tilde{L}\tilde{w})^n r_i^{(0)} + \frac{1}{1-h\tilde{L}\tilde{w}} \left(e_i + \frac{K_m}{m!}h^{m+1} \right) \\ &\leq (h\tilde{L}\tilde{w})^n (K_1 h + e_i) + \frac{1}{1-h\tilde{L}\tilde{w}} \left(e_i + \frac{K_m}{m!}h^{m+1} \right) \\ (3.12) \quad &= \left((h\tilde{L}\tilde{w})^n + \frac{1}{1-h\tilde{L}\tilde{w}} \right) e_i + h^{n+1} (\tilde{L}\tilde{w})^n K_1 + \frac{K_m h^{m+1}}{m!(1-h\tilde{L}\tilde{w})}. \end{aligned}$$

Evaluating e_i we distinguish two cases depending on the definition of u_{i+1} :

$$u_{i+1} = u_{i,m}^{(n)} = u_i + h \sum_{j=1}^m w_{j,m} f(x_{i,j}, u_{i,j}^{(n-1)}), \quad (x_{i+1} = \varphi_i(\xi_m))$$

or

$$u_{i+1} = u_i + \frac{h}{b-a} \sum_{j=1}^m \left(\int_a^b \tilde{l}_j(\xi) d\xi \right) f(x_{i,j}, u_{i,j}^{(n)}), \quad (x_{i+1} = \varphi_i(b)).$$

Corresponding to the two cases, from (3.10) we obtain the equalities

$$y(x_{i+1}) = y(x_i) + h \sum_{j=1}^m w_{j,m} f(x_{i,j}, y(x_{i,j})) + \int_{x_i}^{x_{i,m}} R(s) ds$$

and respectively

$$y(x_{i+1}) = y(x_i) + \frac{h}{b-a} \sum_{j=1}^m \left(\int_a^b \tilde{l}_j(\xi) d\xi \right) f(x_{i,j}, y(x_{i,j})) + \int_{x_i}^{x_{i+1}} R(s) ds.$$

Computing $y(x_{i+1}) - u_{i+1}$ it results

$$y(x_{i+1}) - u_{i+1} = y(x_i) - u_i + h \sum_{j=1}^m w_{j,m} \left(f(x_{i,j}, y(x_{i,j})) - u_{i,j}^{(n-1)} \right) + \int_{x_i}^{x_{i,m}} R(s) ds,$$

respectively

$$\begin{aligned} y(x_{i+1}) - u_{i+1} &= y(x_i) - u_i + \frac{h}{b-a} \sum_{j=1}^m \left(\int_a^b \tilde{l}_j(\xi) d\xi \right) \left(f(x_{i,j}, y(x_{i,j})) - u_{i,j}^{(n)} \right) + \\ &+ \int_{x_i}^{x_{i+1}} R(s) ds. \end{aligned}$$

It follows that

$$e_{i+1} \leq e_i + h\tilde{L}\tilde{w}r_i^{(n-1)} + \frac{K_m}{m!}h^{m+1}$$

and

$$e_{i+1} \leq e_i + h\tilde{L}\tilde{w}r_i^{(n)} + \frac{K_m}{m!}h^{m+1}.$$

We remark that between the two estimates only the upper index of r_i differs. This justifies that in the second case m additional integrals must be computed.

From hereon it is sufficient to consider only the first case. Using (3.12) we obtain

$$\begin{aligned} e_{i+1} &\leq \\ &\leq e_i + h\tilde{L}\tilde{w} \left(\left((h\tilde{L}\tilde{w})^{n-1} + \frac{1}{1-h\tilde{L}\tilde{w}} \right) e_i + h^n (\tilde{L}\tilde{w})^{n-1} K_1 + \frac{K_m h^{m+1}}{m!(1-h\tilde{L}\tilde{w})} \right) + \frac{K_m}{m!} h^{m+1} \\ &= e_i \left(1 + (h\tilde{L}\tilde{w})^n + \frac{h\tilde{L}\tilde{w}}{1-h\tilde{L}\tilde{w}} \right) + h^{n+1} (\tilde{L}\tilde{w})^n K_1 + \frac{K_m h^{m+1}}{m!(1-h\tilde{L}\tilde{w})}. \end{aligned}$$

Because $h\tilde{L}\tilde{w} < 1 \Rightarrow (h\tilde{L}\tilde{w})^n \leq h\tilde{L}\tilde{w}$ the above inequality becomes

$$e_{i+1} \leq e_i \left(1 + h\tilde{L}\tilde{w} + \frac{h\tilde{L}\tilde{w}}{1-h\tilde{L}\tilde{w}} \right) + h^2 \tilde{L}\tilde{w} K_1 + \frac{K_m h^{m+1}}{m!(1-h\tilde{L}\tilde{w})}.$$

Consequently

$$\begin{aligned} e_i &\leq \left(1 + h\tilde{L}\tilde{w} + \frac{h\tilde{L}\tilde{w}}{1-h\tilde{L}\tilde{w}} \right)^i \left(e_0 + \frac{h^2 \tilde{L}\tilde{w} K_1 + \frac{K_m h^{m+1}}{m!(1-h\tilde{L}\tilde{w})}}{h\tilde{L}\tilde{w} + \frac{h\tilde{L}\tilde{w}}{1-h\tilde{L}\tilde{w}}} \right) \\ &\leq e^{\left(h\tilde{L}\tilde{w} + \frac{h\tilde{L}\tilde{w}}{1-h\tilde{L}\tilde{w}} \right) i} \left(e_0 + \frac{h\tilde{L}\tilde{w} K_1 + \frac{K_m h^m}{m!(1-h\tilde{L}\tilde{w})}}{\tilde{L}\tilde{w} + \frac{\tilde{L}\tilde{w}}{1-h\tilde{L}\tilde{w}}} \right). \end{aligned}$$

Because $e_0 = 0$, from the above inequality it results that:

$$\max_{1 \leq i \leq M} e_i \leq e^{(x_f - x_0)\tilde{L}\tilde{w} \left(1 + \frac{1}{1-h\tilde{L}\tilde{w}} \right)} \left(\frac{h\tilde{L}\tilde{w} K_1 + \frac{K_m h^m}{m!(1-h\tilde{L}\tilde{w})}}{\tilde{L}\tilde{w} + \frac{\tilde{L}\tilde{w}}{1-h\tilde{L}\tilde{w}}} \right) \rightarrow 0,$$

for $h \searrow 0 \Leftrightarrow M \rightarrow \infty$. □

4. PICARD ITERATIONS WITH A VARIABLE REFERENCE SET

We shall keep some of the above introduced notations and we shall define those that differ.

Let $a \leq \xi_1^m < \xi_2^m < \dots < \xi_m^m \leq b$ be the roots of the polynomial $p_m(x)$, where $(p_m)_{m \in \mathbb{N}}$ is a sequence of orthogonal polynomials with the weight $\rho \in L_2[a, b]$ on the interval $[a, b]$. It is assumed that $\frac{1}{\rho} \in L_2[a, b]$, too. These are requirements of the convergence theorem [8].

If φ_i is the affine function transforming $[a, b]$ onto $[x_i, x_{i+1}]$ then the nodes are introduced

$$x_{i,j}^m = \varphi_i(\xi_j^m), \quad j \in \{1, 2, \dots, m\}, \quad m \in \mathbb{N}^*.$$

For $x \in [x_i, x_{i+1}]$, we define

$$\begin{aligned} u^{m+1}(x) &= u_i + \int_{x_i}^x L(\mathbb{P}_{m-1}; x_{i,1}^m, x_{i,2}^m, \dots, x_{i,m}^m; f(\cdot, u^m(\cdot)))(s) ds = \\ &= u_i + \sum_{j=1}^m \left(\int_{x_i}^x l_j^m(s) ds \right) f(x_{i,j}^m, u(x_{i,j}^m)) \\ &= u_i + \frac{h}{b-a} \sum_{j=1}^m \left(\int_a^\zeta \tilde{l}_j^m(\xi) d\xi \right) f(x_{i,j}^m, u(x_{i,j}^m)), \end{aligned}$$

where $\zeta = \varphi_i^{-1}(x)$ and

$$\tilde{l}_j^m(\xi) = \frac{(\xi - \xi_1^m) \dots (\xi - \xi_{j-1}^m) (\xi - \xi_{j+1}^m) \dots (\xi - \xi_m^m)}{(\xi_j^m - \xi_1^m) \dots (\xi_j^m - \xi_{j-1}^m) (\xi_j^m - \xi_{j+1}^m) \dots (\xi_j^m - \xi_m^m)}.$$

The vectors $u_{i,j}^m$ are defined iteratively

$$\begin{aligned} u_{i,1}^1 &= u_i, \\ u_{i,1}^2 &= u_i + \frac{h}{b-a} \left(\int_a^{\xi_1^2} \tilde{l}_1^1(\xi) d\xi \right) f(x_{i,1}^1, u_{i,1}^1) \\ &= u_i + \frac{h}{b-a} (\xi_1^2 - a) f(x_{i,1}^1, u_{i,1}^1); \\ u_{i,2}^2 &= u_i + \frac{h}{b-a} \left(\int_a^{\xi_2^2} \tilde{l}_1^1(\xi) d\xi \right) f(x_{i,1}^1, u_{i,1}^1) \\ &= u_i + \frac{h}{b-a} (\xi_2^2 - a) f(x_{i,1}^1, u_{i,1}^1). \end{aligned}$$

It was taken into account that $\tilde{l}_1^1(\xi) = 1$. As a rule

$$u_{i,k}^{m+1} = u^{m+1}(x_{i,k}^{m+1}) = u_i + \frac{h}{b-a} \sum_{j=1}^m \left(\int_a^{x_{i,k}^{m+1}} \tilde{l}_j^m(\xi) d\xi \right) f(x_{i,j}^m, u(x_{i,j}^m)),$$

for $k \in \{1, 2, \dots, m+1\}$ si $m \in \mathbb{N}^*$.

We must compute

$$u_{i+1}^{m+1} = u^{m+1}(x_{i+1}) = u_i + \frac{h}{b-a} \sum_{j=1}^m \left(\int_a^b \tilde{l}_j^m(\xi) d\xi \right) f(x_{i,j}^m, u(x_{i,j}^m)),$$

too.

The computation of the vectors $u_{i,k}^{m+1}$, $k \in \{1, 2, \dots, m+1\}$, u_{i+1}^{m+1} can be written in matrix form. For simplicity we denote

$$\begin{aligned} w_{j,k} &= \int_a^{x_{i,k}^{m+1}} \tilde{l}_j^m(\xi) d\xi, \quad j \in \{1, \dots, m\}, \quad k \in \{1, \dots, m+1\}, \\ w_j &= \int_a^b \tilde{l}_j^m(\xi) d\xi, \quad j \in \{1, \dots, m\} \end{aligned}$$

and the matrix

$$W = \frac{h}{b-a} \begin{pmatrix} w_{1,1} & w_{2,1} & \dots & w_{m,1} \\ w_{1,2} & w_{2,2} & \dots & w_{m,2} \\ \vdots & & & \vdots \\ w_{1,k+1} & w_{2,k+1} & \dots & w_{m,k+1} \\ w_1 & w_2 & \dots & w_m \end{pmatrix} \in M_{m+2,m}(\mathbb{R})$$

$$F = [f(x_{i,1}^m, u_{i,1}^m), f(x_{i,2}^m, u_{i,2}^m), \dots, f(x_{i,m}^m, u_{i,m}^m)] \in M_{N,m}(\mathbb{R})$$

The following equality holds

$$[u_{i,1}^{m+1}, u_{i,2}^{m+1}, \dots, u_{i,m+1}^{m+1}, u_{i+1}^{m+1}]^T = \underbrace{[u_i, \dots, u_i]}_{m+2}^T + W \cdot F^T.$$

For an imposed tolerance $\varepsilon > 0$, the iterations occurs until the condition $\|u_{i+1}^{m+1} - u_i^m\| < \varepsilon$ is fulfilled. The initial approximation is $u_{i+1}^1 = u_i$. If the above condition is fulfilled then we set $u_{i+1} = u_{i+1}^{m+1}$.

A convergence result is given in [8].

5. STIFF PROBLEMS

From (2.3), if $s = x_0 + h\sigma$ then

$$y(x) = y(x_0) + h \int_0^{\frac{x-x_0}{h}} f(x_0 + h\sigma, y(x_0 + h\sigma)) d\sigma,$$

with $x \in [x_0, x_0 + h] \Leftrightarrow \sigma \in [0, 1]$.

Setting

$$y(x_0 + h\sigma) = y_0 + hv(\sigma)$$

we derive that $v(0) = 0$ and

$$\frac{dv(\sigma)}{d\sigma} = f(x_0 + h\sigma, y_0 + hv(\sigma)) \Leftrightarrow v(s) = \int_0^s f(x_0 + h\sigma, y_0 + hv(\sigma)) d\sigma.$$

Following [4], [2], by the *stabilization principle*, the solution of the partial differential system

$$(5.1) \quad \frac{\partial w(\zeta, t)}{\partial t} = -w(\zeta, t) + \int_0^\zeta f(x_0 + h\sigma, y_0 + hw(\sigma, t)) d\sigma$$

has the property, cf. [4], [2],

$$(5.2) \quad \lim_{t \rightarrow \infty} \|w(\zeta, t) - v(\zeta)\| = 0, \quad \text{for } \zeta \in [0, 1].$$

We give a numerical solution to find an approximation of the solution of (5.1).

Let be $\tau > 0$ and the sequence $t^n = n\tau$, $n \in \mathbb{N}$. The equation (5.1) may be rewritten as

$$\frac{\partial e^t w(\zeta, t)}{\partial t} = e^t \int_0^\zeta f(x_0 + h\sigma, y_0 + hw(\sigma, t)) d\sigma$$

and integrating from $n\tau$ to $(n+1)\tau$ it results

$$(5.3) \quad w(\zeta, t^{n+1}) = e^{-\tau} w(\zeta, t^n) + e^{-(n+1)\tau} \int_{n\tau}^{(n+1)\tau} e^{\eta} \left(\int_0^{\zeta} f(x_0 + h\sigma, y_0 + hw(\sigma, \eta)) d\sigma \right) d\eta.$$

Without changing the notation for w , we substitute in (5.3) $f(x_0 + h\sigma, y_0 + hw(\sigma, \eta))$ by a Lagrange interpolation polynomial

$$(5.4) \quad \begin{aligned} w(\zeta, t^{n+1}) &= \\ &= e^{-\tau} w(\zeta, t^n) + \\ &\quad + e^{-(n+1)\tau} \int_{n\tau}^{(n+1)\tau} e^{\eta} \left(\int_0^{\zeta} L(\mathbb{P}_{m-1}; \xi_1, \dots, \xi_m; f(x_0 + h \cdot, y_0 + hw(\cdot, \eta))) d\sigma \right) d\eta \\ &= e^{-\tau} w(\zeta, t^n) + e^{-(n+1)\tau} \sum_{j=1}^m \int_{n\tau}^{(n+1)\tau} e^{\eta} \left(\int_0^{\zeta} f(x_0 + h\xi_j, y_0 + hw(\xi_j, \eta)) l_j(\sigma) d\sigma \right) d\eta, \end{aligned}$$

where $0 = \xi_1 < \xi_2 < \dots < \xi_m = 1$.

We denote $w^n(\zeta) = w(\zeta, t^n)$ and in the right hand side of (5.4) we take $w(\xi_j, \eta) = w^n(\xi_j)$, for any $j \in \{1, 2, \dots, m\}$ and $\eta \in [n\tau, (n+1)\tau]$. Then

$$w^{n+1}(\zeta) = e^{-\tau} w^n(\zeta) + (1 - e^{-\tau}) \sum_{j=1}^m f(x_0 + h\xi_j, y_0 + hw^n(\xi_j)) \int_0^{\zeta} l_j(\sigma) d\sigma.$$

Denoting $w_j^n = w^n(\xi_j)$, for $\zeta = \xi_k$, $k \in \{1, 2, \dots, m\}$ we obtain the iterative relations

$$w_k^{n+1} = e^{-\tau} w_k^n + (1 - e^{-\tau}) \sum_{j=1}^m f(x_0 + h\xi_j, y_0 + hw_j^n) \int_0^{\xi_k} l_j(\sigma) d\sigma.$$

The iterations occurs until the stopping condition $\max_{1 \leq j \leq m} \|w_j^{n+1} - w_j^n\| < \varepsilon$ is fulfilled. Here $\varepsilon > 0$ is a tolerance. According to (5.2) we consider $v(1) = w_j^{n+1}$ and the procedure continues with $u_{i+1} = u_i + hw_m^{n+1}$.

6. NUMERICAL EXPERIMENTS

Using computer programs based on these methods we solved the following IVPs:

(1) ([9, p. 234])

$$\begin{cases} \dot{y} &= y \frac{4(x+2)^3 - y}{(x+2)^4 - 1}, & x \in [0, 1], \\ y(0) &= 15 \end{cases}$$

with the solution $y(x) = 1 + (x+2) + (x+2)^2 + (x+2)^3$.

For $M = 5$ and the tolerance $\varepsilon = 10^{-5}$ the maximum error $\max_{0 \leq i \leq M} \|y(x_i) - u_i\|$ and N_f , the number of calling the function f , are given in Table 6.1.

Fixed equidistant reference set $m = 3$		Variable reference set	
Error	N_f	Error	N_f
1.82591e-08	75	8.94274e-08	99

Table 6.1. Results for Example (1).

(2) ([9, p. 244])

$$\begin{cases} \dot{y}_1 = y_2, & y_1(0) = 1, & x \in [0, x_f], \\ \dot{y}_2 = -\frac{y_1}{r^3}, & y_2(0) = 0, \\ \dot{y}_3 = y_4, & y_3(0) = 0, \\ \dot{y}_4 = -\frac{y_3}{r^3}, & y_4(0) = 1, \end{cases}$$

where $r = \sqrt{y_1^2 + y_3^2}$ and with the solution $y_1 = \cos x, y_2 = -\sin x, y_3 = \sin x, y_4 = \cos x$.

The results of our numerical experiments are listed in Table 6.2.

x_f	M	ε	Fixed equidistant reference set $m = 3$		Variable reference set	
			Error	N_f	Error	N_f
2π	10	10^{-5}	0.0247309	300	6.47998e-05	550
2π	10	10^{-9}	0.0246415	480	2.24345e-09	1050
4π	10	10^{-5}	0.888217	534	0.000142862	966
4π	20	10^{-9}	0.0496889	960	1.05491e-08	2100
6π	10	10^{-5}	14.4197	762	6.23799e-05	1530
6π	40	10^{-9}	0.0232977	1560	3.06542e-09	3640

Table 6.2. Results for Example (2).

Now we compare the results obtained using equidistant nodes and Chebyshev points of second kind for the reference set. For the same example the obtained results are given in Table 6.3.

x_f	M	ε	Fixed equidistant reference set $m = 5$		Chebyshev fixed	
			Error	N_f	Error	N_f
2π	10	10^{-5}	6.93002e-05	400	2.69646e-05	400
2π	10	10^{-9}	1.91509e-05	650	8.13527e-06	650
4π	10	10^{-5}	0.00215349	600	0.000338729	551
4π	20	10^{-9}	3.85763e-05	1300	1.6391e-05	1300
6π	10	10^{-5}	0.0275954	900	0.0164587	820
6π	40	10^{-9}	1.00764e-05	2200	4.18516e-06	2200

Table 6.3. Results for Example (2).

As expected, the results using Chebyshev points of second kind are better than that obtained using equidistant nodes, due to the better approximation property of Lagrange interpolation polynomial with Chebyshev points of second kind toward the equidistant points, [11].

- (3) ([9, p. 245]) Keeping the differential system as in the previous example but changing the initial value conditions to $y(0) = [0.4, 0, 0, 2]$, for $x_f = 2\pi$, $M = 20$ and $\varepsilon = 10^{-9}$ with the method based on variable reference set we obtained $\max_{0 \leq i \leq M} \|y(x_i) - u_i\| = 2.94126 \cdot 10^{-9}$ and $N_f = 1400$.

In this case the solution is

$$y(x) = [\cos u - 0.6, \frac{-\sin u}{1-0.6 \cos u}, 0.8 \sin u, \frac{0.8 \cos u}{1-0.6 \cos u}],$$

where $x = u - 0.6 \sin u$.

Based on the previous examples the method with variable number of reference points is more efficient than the method with fixed number reference points, but we cannot deduce theoretically such a conclusion.

Using the method for stiff problems presented above we solved:

(4)

$$\begin{cases} \dot{y}_1 = 998y_1 + 1998y_2, & y_1(0) = 1, & x \in [0, 1], \\ \dot{y}_2 = -999y_1 - 1999y_2, & y_2(0) = 0, \end{cases}$$

with the solution $y_1 = 2e^{-x} - e^{-1000x}$, $y_2 = -e^{-x} + e^{-1000x}$.

For $\tau = 10$ the results are given in Table 6.4. We recall that τ is the length of the step for the t variable of the stabilization principle.

M	ε	Fixed equidistant reference set $m = 5$		Chebyshev fixed	
		Error	N_f	Error	N_f
300	10^{-5}	0.00164977	8585	0.000402419	8435
500	10^{-7}	0.000128781	10700	4.35037e-05	10555

Table 6.4. Results for Example (4).

(5)

$$\dot{y} = -20y, \quad y(0) = 1, \quad x \in [0, 1].$$

For $\tau = 10$, $M = 20$ and $\varepsilon = 10^{-7}$ the results are given in Table 6.5.

Fixed equidistant reference set $m = 5$		Chebyshev fixed	
Error	N_f	Error	N_f
1.19382e-06	800	4.58431e-07	785

Table 6.5. Results for Example 5.

To make the results reproducible we provide some code at <https://github.com/e-scheiber/scilab-ivpsolvers.git>.

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