JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY

J. Numer. Anal. Approx. Theory, vol. 47 (2018) no. 2, pp. 167–176

ictp.acad.ro/jnaat

APPROXIMATIONS OF OBJECTIVE FUNCTION AND CONSTRAINTS IN BI-CRITERIA OPTIMIZATION PROBLEMS

IONUT TRAIAN LUCA* and DOREL I. DUCA†

Abstract. In this paper we study approximation methods for solving bi-criteria optimization problems. Initial problem is approximated by a new one which has the components of the objective and the constraints replaced by their approximation functions. Components of the objective function are first and second order approximated and constraints are first order approximated. Conditions such that efficient solution of the approximate problem will remain efficient for initial problem and reciprocally are studied. Numerical examples are developed to emphasize the importance of these conditions.

MSC 2010. 90C46, 90C59.

Keywords. efficient solution, bi-criteria optimization, η-approximation, invex and incave function.

1. INTRODUCTION

Bi-criteria optimization problems are quite often used as mathematical models for all kind of phenomena generated by real-world and theoretical situations. As examples we might mention portfolio theory [4], energy field [5], data analysis [3], logistics [6].

Among methods widely used to solve bi-criteria optimization problems are "scalarization" methods [2] (weighting problem, k^{th} objective Lagrangian problem, k^{th} objective ε-constrained problem). Sometimes mathematical models are highly complex and thus using approximation problems might be a more efficient method to solve bi-criteria optimization problems.

This article is analyzing conditions such that efficient solution of a certain approximate problem will remain efficient for the initial problem and reciprocally. Approximate problem consists of replacing components of objective function and also constraints with their approximate functions.

*Faculty of Business, Babes-Bolyai University, e-mail: ionut.luca@tbs.ubbcluj.ro.

†Faculty of Mathematics and Computer Science, Babes-Bolyai University, e-mail: dduca@math.ubbcluj.ro.

2. BASIC CONCEPTS

Let X be a set in $\mathbb{R}^n$, x_0 an interior point of X, $\eta : X \times X \to X$ and $f : X \to \mathbb{R}$. If f is differentiable at x_0 then we denote:

$$F^1(x) = f(x_0) + \nabla f(x_0)\,\eta(x, x_0)$$

and call it first η-approximation of f, while if f is twice differentiable at x_0 then we denote:

$$F^2(x) = f(x_0) + \nabla f(x_0)\,\eta(x, x_0) + \tfrac{1}{2}\eta(x, x_0)^T\,\nabla^2 f(x_0)\,\eta(x, x_0).$$

and call it second η-approximation of f.

Definition 1. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of X, $f : X \to \mathbb{R}$ a function differentiable at x_0 and $\eta : X \times X \to X$. Then function f is:* ***invex*** *at x_0 with respect to η if for all $x \in X$ we have:*

$$f(x) - f(x_0) \geq \nabla f(x_0)\,\eta(x, x_0)$$

or equivalently:

$$f(x) \geq F^1(x);$$

incave *at x_0 with respect to η if for all $x \in X$ we have:*

$$f(x) - f(x_0) \leq \nabla f(x_0)\,\eta(x, x_0)$$

or equivalently

$$f(x) \leq F^1(x);$$

avex *at x_0 with respect to η if it is both invex and incave at x_0 w.r.t. η.*

If function f is invex, respectively incave or avex we denote invex^1, respectively incave^1 or avex^1.

Definition 2. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of X, $f : X \to \mathbb{R}$ a function twice differentiable at x_0 and $\eta : X \times X \to X$. Then function f is:*
second order invex *at x_0 with respect to η if for all $x \in X$ we have:*

$$f(x) - f(x_0) \geq \nabla f(x_0)\,\eta(x, x_0) + \tfrac{1}{2}\eta(x, x_0)^T\,\nabla^2 f(x_0)\,\eta(x, x_0)$$

or equivalently:

$$f(x) \geq F^2(x);$$

second order incave *at x_0 with respect to η if for all $x \in X$ we have:*

$$f(x) - f(x_0) \leq \nabla f(x_0)\,\eta(x, x_0) + \tfrac{1}{2}\eta(x, x_0)^T\,\nabla^2 f(x_0)\,\eta(x, x_0)$$

or equivalently:

$$f(x) \leq F^2(x);$$

second order avex *at x_0 with respect to η if it is both second order invex and second order incave at x_0 w.r.t. η.*

If function f is second order invex, respectively second order incave or second order avex we denote invex^2, respectively incave^2 or avex^2.

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

We consider the bi-criteria optimization problem $(P_0^{0,0})$, defined as:

$$\begin{cases} \min \left(f_1, f_2\right)(x) \\ x = (x_1, x_2, ..., x_n) \in X \\ g_t\left(x\right) \leq 0,\ t \in T \\ h_s\left(x\right) = 0,\ s \in S. \end{cases}$$

Assuming that functions f_1, f_2, are differentiable of order $i,\ j \in \{1,\ 2\}$ and functions $g_t,\ \ (t \in T)\,,\ h_s,\ \ (s \in S)$ are first order differentiable, we will approximate original problem $(P_0^{0,0})$ by problems $(P_1^{i,j})$:

$$\begin{cases} \min \left(F_1^i, F_2^j\right)(x) \\ x = (x_1, x_2, ..., x_n) \in X \\ G_t^1\left(x\right) \leq 0,\ t \in T \\ H_s^1\left(x\right) = 0,\ s \in S \end{cases}$$

where $(i, j) \in \{(1,0)\,,\ (1,1)\,,\ (2,0)\,,\ (2,1)\,,\ (2,2)\}$ and $F_1^0 = f_1,\ F_2^0 = f_2$.

We denote by

$$\mathcal{F}^k = \left\{x \in X :\ G_t^k\left(x\right) \leq 0,\ t \in T,\ H_s^k\left(x\right) = 0,\ s \in S,\ k \in \{0,1\}\right\}$$

the set of feasible solutions for bi-criteria optimization problem $(P_k^{i,j})$, where $(i, j) \in \{(1,0)\,,\ (1,1)\,,\ (2,0)\,,\ (2,1)\,,\ (2,2)\}$ and $k \in \{0,1\}$.

3. APPROXIMATE PROBLEMS AND RELATION TO INITIAL PROBLEM

In this section we will study the conditions such that efficient solution of approximated problems $(P_1^{1,0})$, $(P_1^{2,0})$, $(P_1^{2,1})$ and $(P_1^{2,2})$ will remain efficient also for initial problem $(P_0^{0,0})$ and reciprocally.

Conditions for the relation $(P_0^{0,0})$ *vs.* $(P_1^{1,1})$ have been studied in [1] so we will not analyze them anymore.

By approximating also the feasible set it is important to determine conditions such that $\mathcal{F}^0 \subseteq \mathcal{F}^1$ and $\mathcal{F}^1 \subseteq \mathcal{F}^0$. These inclusions were studied in [1]. We will use them in our work, so we will briefly present the Theorems stating these inclusions.

THEOREM 3. [1]. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.*

Assume that:

a) *for each $t \in T$, the function g_t is differentiable at x_0 and invex^1 at x_0 with respect to η,*

b) *for each $s \in S$, the function h_s is differentiable at x_0 and avex*^{1} *at x_0 with respect to η,*

then

$$\mathcal{F}^0 \subseteq \mathcal{F}^1.$$

THEOREM 4. [1]. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.*

Assume that

a) *for each $t \in T$, the function g_t is differentiable at x_0 and incave*^{1} *at x_0 with respect to η,*

b) *for each $s \in S$, the function h_s is differentiable at x_0 and avex*^{1} *at x_0 with respect to η,*

then

$$\mathcal{F}^1 \subseteq \mathcal{F}^0.$$

THEOREM 5. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each $t \in T$, the function g_t is differentiable at x_0 and invex*^{1} *at x_0 with respect to η,*

c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex*^{1} *at x_0 with respect to η,*

d) *f_1 is twice differentiable at x_0 and invex*^{2} *at x_0 with respect to η,*

e) $\eta\,(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $(P_1^{2,0})$, then x_0 is an efficient solution for $(P_0^{0,0})$.

Proof. x_0 being an efficient solution for $(P_1^{2,0})$, implies that

$$\nexists x \in \mathcal{F}^1 \ s.t.\ \Big(F_1^2\,(x)\,, f_2\,(x)\Big) \leq \Big(F_1^2\,(x_0)\,, f_2\,(x_0)\Big)\,.$$

Conditions *b*) and *c*) imply that

$$\mathcal{F}^0 \subseteq \mathcal{F}^1$$

and thus

$$\nexists x \in \mathcal{F}^0 \ s.t.\ \Big(F_1^2\,(x)\,, f_2\,(x)\Big) \leq \Big(F_1^2\,(x_0)\,, f_2\,(x_0)\Big)\,. \tag{1}$$

Let's assume that x_0 is not an efficient solution for $(P_0^{0,0})$. Then

$$\exists y \in \mathcal{F}^0 \ s.t.\ \left(f_1\,(y)\,, f_2\,(y)\right) \leq \left(f_1\,(x_0)\,, f_2\,(x_0)\right)$$

which implies that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\begin{cases} f_1\,(y) < f_1\,(x_0) \\ f_2\,(y) \leqq f_2\,(x_0) \end{cases} \tag{2}$$

or

$$\begin{cases} f_1(y) \leqq f_1(x_0) \\ f_2(y) < f_2(x_0). \end{cases} \tag{3}$$

Because f_1 is invex^2 at x_0 with respect to η we get $F_1^2(y) \le f_1(y)$, $\forall y \in \mathcal{F}^0$. Because $\eta(x_0,\ x_0) = 0$ we get $f_1(x_0) = F_1^2(x_0)$. Thus from (2) we get that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\begin{cases} F_1^2(y) < F_1^2(x_0) \\ f_2(y) \leqq f_2(x_0) \end{cases}$$

which contradicts (1) and from (3) we get that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\begin{cases} F_1^2(y) \leqq F_1^2(x_0) \\ f_2(y) < f_2(x_0) \end{cases}$$

which contradicts (1).
In conclusion x_0 is an efficient solution for $(P_0^{0,0})$. □

Theorem 6. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*
Assume that:

a) $x_0 \in \mathcal{F}^1$,
b) *for each $t \in T$, the function g_t is differentiable at x_0 and incave^1 at x_0 with respect to η,*
c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex^1 at x_0 with respect to η,*
d) *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
e) $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $(P_0^{0,0})$, then x_0 is an efficient solution for $(P_1^{2,0})$.

Proof. x_0 being an efficient solution for $(P_0^{0,0})$, implies that

$$\nexists x \in \mathcal{F}^0 \ s.t.\ (f_1(x), f_2(x)) \le (f_1(x_0), f_2(x_0)).$$

Conditions *b*) and *c*) imply that

$$\mathcal{F}^1 \subseteq \mathcal{F}^0$$

and thus

$$\nexists x \in \mathcal{F}^1 \ s.t.\ (f_1(x), f_2(x)) \le (f_1(x_0), f_2(x_0)). \tag{4}$$

Let's assume that x_0 is not an efficient solution for $(P_1^{2,0})$. Then

$$\exists y \in \mathcal{F}^1 \ s.t.\ \left(F_1^2(y), f_2(y)\right) \le \left(F_1^2(x_0), f_2(x_0)\right)$$

which implies that $\exists y \in \mathcal{F}^1$ *s.t.*

$$\left\{ \begin{array}{l} F_1^2\left(y\right) < F_1^2\left(x_0\right) \\ f_2\left(y\right) \leqq f_2\left(x_0\right) \end{array} \right. \tag{5}$$

or

$$\left\{ \begin{array}{l} F_1^2\left(y\right) \leqq F_1^2\left(x_0\right) \\ f_2\left(y\right) < f_2\left(x_0\right). \end{array} \right. \tag{6}$$

Because f_1 is incave[2] at x_0 with respect to η we get $f_1\left(y\right) \leq F_1^2\left(y\right),\ \forall y \in \mathcal{F}^1$. Because $\eta\left(x_0,\ x_0\right) = 0$ we get $f_1\left(x_0\right) = F_1^2\left(x_0\right)$. Thus from (5) we get that $\exists y \in \mathcal{F}^1$ *s.t.*

$$\left\{ \begin{array}{l} f_1\left(y\right) < f_1\left(x_0\right) \\ f_2\left(y\right) \leqq f_2\left(x_0\right) \end{array} \right.$$

which contradicts (4) and from (6) we get that $\exists y \in \mathcal{F}^1$ *s.t.*

$$\left\{ \begin{array}{l} f_1\left(y\right) \leqq f_1\left(x_0\right) \\ f_2\left(y\right) < f_2\left(x_0\right) \end{array} \right.$$

which contradicts (4).

In conclusion x_0 is an efficient solution for $(P_1^{2,0})$. □

THEOREM 7. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

a) $x_0 \in \mathcal{F}^0$,
b) *for each $t \in T$, the function g_t is differentiable at x_0 and invex[1] at x_0 with respect to η,*
c) *for each $s \in S$, the function h_s is differentiable at x_0 and avex[1] at x_0 with respect to η,*
d) *f_1 is differentiable at x_0 and invex[1] at x_0 with respect to η,*
e) $\eta\left(x_0,\ x_0\right) = 0$.

If x_0 is an efficient solution for $(P_1^{1,0})$, then x_0 is an efficient solution for $(P_0^{0,0})$.

Proof. Proof is similar to Theorem 5. □

THEOREM 8. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

a) $x_0 \in \mathcal{F}^1$,
b) *for each $t \in T$, the function g_t is differentiable at x_0 and incave[1] at x_0 with respect to η,*

c) *for each* $s \in S$, *the function* h_s *is differentiable at* x_0 *and avex*^{1} *at* x_0 *with respect to* η,

d) f_1 *is differentiable at* x_0 *and incave*^{1} *at* x_0 *with respect to* η,

e) $\eta(x_0,\ x_0) = 0$.

If x_0 *is an efficient solution for* $(P_0^{0,0})$, *then* x_0 *is an efficient solution for* $(P_1^{1,0})$.

Proof. Proof is similar to Theorem 6. □

THEOREM 9. *Let* X *be a nonempty set of* $\mathbb{R}^n$, x_0 *an interior point of* $X, \eta : X \times X \to X$, T *and* S *index sets,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *and* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ *functions.*

Assume that:

a) $x_0 \in \mathcal{F}^0$,

b) *for each* $t \in T$, *the function* g_t *is differentiable at* x_0 *and invex*^{1} *at* x_0 *with respect to* η,

c) *for each* $s \in S$, *the function* h_s *is differentiable at* x_0 *and avex*^{1} *at* x_0 *with respect to* η,

d) f_1 *is twice differentiable at* x_0 *and invex*^{2} *at* x_0 *with respect to* η,

e) f_2 *is differentiable at* x_0 *and invex*^{1} *at* x_0 *with respect to* η,

f) $\eta(x_0,\ x_0) = 0$.

If x_0 *is an efficient solution for* $(P_1^{2,1})$, *then* x_0 *is an efficient solution for* $(P_0^{0,0})$.

Proof. Proof is similar to Theorem 5. □

THEOREM 10. *Let* X *be a nonempty set of* $\mathbb{R}^n$, x_0 *an interior point of* $X, \eta : X \times X \to X$, T *and* S *index sets,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *and* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ *functions.*

Assume that:

a) $x_0 \in \mathcal{F}^1$,

b) *for each* $t \in T$, *the function* g_t *is differentiable at* x_0 *and incave*^{1} *at* x_0 *with respect to* η,

c) *for each* $s \in S$, *the function* h_s *is differentiable at* x_0 *and avex*^{1} *at* x_0 *with respect to* η,

d) f_1 *is twice differentiable at* x_0 *and incave*^{2} *at* x_0 *with respect to* η,

e) f_2 *is differentiable at* x_0 *and incave*^{1} *at* x_0 *with respect to* η,

f) $\eta(x_0,\ x_0) = 0$.

If x_0 *is an efficient solution for* $(P_0^{0,0})$, *then* x_0 *is an efficient solution for* $(P_1^{2,1})$.

Proof. Proof is similar to Theorem 6. □

THEOREM 11. *Let* X *be a nonempty set of* $\mathbb{R}^n$, x_0 *an interior point of* $X, \eta : X \times X \to X$, T *and* S *index sets,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *and* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ *functions.*

Assume that:

a) $x_0 \in \mathcal{F}^0$,
b) *for each* $t \in T$, *the function* g_t *is differentiable at* x_0 *and invex*^{1} *at* x_0 *with respect to* η,
c) *for each* $s \in S$, *the function* h_s *is differentiable at* x_0 *and avex*^{1} *at* x_0 *with respect to* η,
d) f_1 *is twice differentiable at* x_0 *and invex*^{2} *at* x_0 *with respect to* η,
e) f_2 *is twice differentiable at* x_0 *and invex*^{2} *at* x_0 *with respect to* η,
f) $\eta(x_0,\ x_0) = 0$.

If x_0 *is an efficient solution for* $(P_1^{2,2})$, *then* x_0 *is an efficient solution for* $(P_0^{0,0})$.

Proof. Proof is similar to Theorem 5. □

THEOREM 12. *Let* X *be a nonempty set of* $\mathbb{R}^n$, x_0 *an interior point of* $X, \eta : X \times X \to X$, T *and* S *index sets,* $f = (f_1, f_2) : X \to \mathbb{R}^2$ *and* $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ *functions.*
Assume that:

a) $x_0 \in \mathcal{F}^1$,
b) *for each* $t \in T$, *the function* g_t *is differentiable at* x_0 *and incave*^{1} *at* x_0 *with respect to* η,
c) *for each* $s \in S$, *the function* h_s *is differentiable at* x_0 *and avex*^{1} *at* x_0 *with respect to* η,
d) f_1 *is twice differentiable at* x_0 *and incave*^{2} *at* x_0 *with respect to* η,
e) f_2 *is twice differentiable at* x_0 *and incave*^{2} *at* x_0 *with respect to* η,
f) $\eta(x_0,\ x_0) = 0$.

If x_0 *is an efficient solution for* $(P_0^{0,0})$, *then* x_0 *is an efficient solution for* $(P_1^{2,2})$.

Proof. Proof is similar to Theorem 6. □

4. NUMERICAL EXAMPLES

In the above theorems, conditions referring to invexity, incavity or avexity of functions are essential to ensure that efficient solution of the initial problem remains efficient for the approximate problem and reciprocally. If those conditions are not fulfill it is possible either that efficient solution of initial problem remains efficient for the approximate problem (and reciprocally) or it does not remain efficient.

EXAMPLE 1. Let the initial bi-criteria optimization problem $(P_0^{0,0})$ be:

$$\begin{cases} \min\left(x_1 - 2x_2; x_1 + x_2\right) \\ -x_1 x_2 + 1 \leq 0 \\ x_1; x_2 \geq 0 \end{cases}$$

An efficient solution of problem $(P_0^{0,0})$ is $x_0 = (1,\ 1) \in \mathcal{F}^0$ and the value of the objective function in x_0 is $f(1,\ 1) = (-1,\ 2)$. First and second approximate functions for the components of the objective function in $x_0 = (1,\ 1)$ are:

$$F_p^1(x) = f_p(x_0) + \nabla f_p(x_0)\,\eta(x, x_0),\, p \in \{1, 2\}$$

and

$$F_p^2(x) = f_p(x_0) + \nabla f_p(x_0)\,\eta(x, x_0) + \tfrac{1}{2}\eta(x, x_0)^T\,\nabla^2 f_p(x_0)\,\eta(x, x_0),\, p \in \{1, 2\},$$

while first approximate functions for the constraint is:

$$G_t^1(x) = g_t(x_0) + \nabla g_t(x_0)\,\eta(x, x_0),\, t \in \{1, 2, 3\}.$$

Considering $\eta(x, x_0) = x - x_0$ we get:

$$F_1^i(x) = F_1^i(x) = x_1 - 2x_2,\ i \in \{0, 1, 2\}$$

$$F_2^j(x) = F_2^j(x) = x_1 + x_2,\ j \in \{0, 1, 2\}$$

and

$$G_1^1(x) = -x_1 - x_2 + 2, G_2^1(x) = x_1, G_3^1(x) = x_2$$

Consequently, the approximate problems $(P_1^{i,j})$, with $(i, j) \in \{(1, 0), (1, 1), (2, 0), (2, 1), (2, 2)\}$ are:

$$\begin{cases} \min(x_1 - 2x_2; x_1 + x_2) \\ -x_1 - x_2 + 2 \le 0 \\ x_1; x_2 \ge 0 \end{cases}$$

Calculating the value of objective function for problem $(P_1^{i,j})$ in $x = (0,\ 2) \in \mathcal{F}^1$ we obtain:

$$\left(F_1^i, F_2^j\right)(0,\ 2) = (-4,\ 2) < (-1,\ 2) = \left(F_1^i, F_2^j\right)(1,\ 1)$$

where $(i, j) \in \{(1, 0),\ (1, 1),\ (2, 0),\ (2, 1),\ (2, 2)\}$, which proves that $x_0 = (1,\ 1) \in \mathcal{F}^1$ is not an efficient solution for approximate problem $(P_1^{i,j})$. □

Example 2. Let the initial bi-criteria optimization problem $(P_0^{0,0})$ be:

$$\begin{cases} \min\left(x_1^2 + (x_2 - \pi - 1)^2;\ (x_1 + \frac{1}{10})^2 - \frac{1}{2}(x_2 + 1)^2\right) \\ -x_1 - \sin x_1 + x_2 \le 0 \\ x_1 - \frac{5\pi}{2} \le 0 \\ x_1; x_2 \ge 0 \end{cases}$$

An efficient solution of problem $(P_0^{0,0})$ is $x_0 = (\frac{\pi}{2},\ 1 + \frac{\pi}{2}) \in \mathcal{F}^0$ and the value of the objective function in x_0 is $f(\frac{\pi}{2},\ 1 + \frac{\pi}{2}) = (\frac{\pi^2}{2};\ \frac{\pi^2}{8} - \frac{9\pi}{10} - \frac{199}{100})$.

To compute the approximate problem $(P_1^{1,1})$ in x_0 we have to calculate:

$$F_p^1(x) = f_p(x_0) + \nabla f_p(x_0)\,\eta(x, x_0),\ p \in \{1, 2\}$$

and

$$G_t^1(x) = g_t(x_0) + \nabla g_t(x_0)\,\eta(x, x_0),\ t \in \{1, 2, 3, 4\}$$

Considering $\eta\left(x, x_0\right) = x - x_0$ we get:

$$F_1^1\left(x\right) = \pi x_1 - \pi x_2 + \pi + \tfrac{\pi^2}{2},$$
$$F_2^1\left(x\right) = (\pi + \tfrac{1}{5})x_1 - (\tfrac{\pi}{2} + 2)x_2 - \tfrac{\pi^2}{8} + \tfrac{\pi}{2} + \tfrac{1}{100},$$
$$G_1^1\left(x\right) = -x_1 + x_2 - 1,$$
$$G_2^1\left(x\right) = x_1 - \tfrac{5\pi}{2},\ G_3^1\left(x\right) = x_1,\ G_4^1\left(x\right) = x_2$$

Thus, the approximate problem $(P_1^{1,1})$ is:

$$\begin{cases} \min\left(\pi x_1 - \pi x_2 + \pi + \tfrac{\pi^2}{2};\ (\pi + \tfrac{1}{5})x_1 - (\tfrac{\pi}{2} + 2)x_2 - \tfrac{\pi^2}{8} + \tfrac{\pi}{2} + \tfrac{1}{100}\right) \\ -x_1 + x_2 - 1 \leq 0 \\ x_1 - \tfrac{5\pi}{2} \leq 0 \\ x_1; x_2 \geq 0 \end{cases}$$

Calculating the value for the objective function of problem $(P_1^{1,1})$ in $x = (\tfrac{5\pi}{2},\ 1 + \tfrac{5\pi}{2}) \in \mathcal{F}^1$ we get:

$F^1(\tfrac{5\pi}{2},\ 1 + \tfrac{5\pi}{2}) = (\tfrac{\pi^2}{2},\ \tfrac{9\pi^2}{8} - \tfrac{9\pi}{2} - \tfrac{199}{100}) < (\tfrac{\pi^2}{2};\ \tfrac{\pi^2}{8} - \tfrac{9\pi}{10} - \tfrac{199}{100}) = F^1(\tfrac{\pi}{2},\ 1 + \tfrac{\pi}{2})$ and thus we have proved that $x_0 = (\tfrac{\pi}{2},\ 1 + \tfrac{\pi}{2})$ is not an efficient solution for problem $(P_1^{1,1})$. $\square$

REFERENCES

[1] D.I. Duca, A. Ratiu, *Semi-infinite optimization problems and their first order approximations*, Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity, **11** (2013), pp. 87–94,

[2] V. Chankong, Y. Haimes, *Multiobjective Decision Making Theory and Methodology*, North-Holland, (1983).

[3] I. Chikalov, S. Hussain, M. Moshkov, *Bi-criteria optimization of decision trees with applications to data analysis*, European journal of Operational Research, **266** (2018) no. 2, pp. 689–701.

[4] H. Konno, H. Yamazaki, *Mean absolute deviation portfolio optimization model and its applications to Tokyo Stock Market*, Management Science, **37** (1991) no. 5, pp. 519–531.

[5] A. Mahalov, T.I. Luca, *Minimax rule for energy optimization*, Computers and Fluids, **151** (2017), pp. 35–45.

[6] A. Palacio, B. Adenso-Diaz, S. Lozano, S. Furio, *Bi-criteria optimization model for locating maritime container depots: application to the port of Valencia*, Networks and Spatial Economics, **16** (2016) no. 1, pp. 331–348.

Received by the editors: June 14, 2018. Accepted: October 23, 2018. Published online: February 7, 2019.