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SHAPE PRESERVING PROPERTIES AND MONOTONICITY PROPERTIES OF THE SEQUENCES OF CHOQUET TYPE INTEGRAL OPERATORS

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Abstract. In this paper, for the univariate Bernstein-Kantorovich-Choquet, Szász-Kantorovich-Choquet, Baskakov-Kantorovich-Choquet and Bernstein-Durrmeyer-Choquet operators written in terms of the Choquet integrals with respect to monotone and submodular set functions, we study the preservation of the monotonicity and convexity of the approximated functions and the monotonicity of some approximation sequences.

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1. INTRODUCTION

Qualitative results and quantitative uniform, pointwise and L^p results in approximation by Bernstein-Durrmeyer-Choquet, Bernstein-Kantorovich-Choquet, Szász-Kantorovich-Choquet and Baskakov-Kantorovich-Choquet operators defined in terms of the Choquet integral with respect to a family of monotone and submodular set functions, were obtained by the author in a series of very recent papers [7]-[11]. As it was pointed out in some of these papers, for large classes of functions, the Choquet type operators approximate better than their classical correspondents.

By analogy with what happens in the case of the classical positive and linear operators, it is a natural question to look for shape preserving properties of these Choquet type operators and for monotonicity of the sequences of approximation.

The aim of the present paper is to give answers to this question.

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The plan of the paper goes as follows. Section 2 contains some preliminaries on the Choquet integral. In Section 3 we prove monotonicity and convexity preserving properties for the Bernstein-Kantorovich-Choquet, Szász-Kantorovich-Choquet, Baskakov-Kantorovich-Choquet operators and we discuss these properties for the Bernstein-Durrmeyer-Choquet type operators. Section 4 proves the monotonicity property of the sequences of Baskakov-Kantorovich-Choquet and of Bernstein-Kantorovich-Choquet type operators.

2. PRELIMINARIES

In this section we present some concepts and results on the Choquet integral which will be used in the main section.

Definition 2.1. *Let Ω be a nonempty set and $\mathcal{C}$ be a σ-algebra of subsets in Ω.*

(i) (see, e.g., [18, p. 63]*) Let $\mu : \mathcal{C} \to [0, +\infty)$. If $\mu(\emptyset) = 0$ and $A, B \in \mathcal{C}$, with $A \subset B$, implies $\mu(A) \leq \mu(B)$, then μ is called a monotone set function (or capacity). Also, if*

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B), \text{ for all } A, B \in \mathcal{C},$$

then μ is called submodular. Finally, if $\mu(\Omega) = 1$, then μ is called normalized.

(ii) (see [4]*, or* [18, p. 233]*) Let μ be a monotone set function on $\mathcal{C}$.*

If $f : \Omega \to \mathbb{R}$ is $\mathcal{C}$-measurable, i.e. for any Borel subset $B \subset \mathbb{R}$ we have $f^{-1}(B) \in \mathcal{C}$, then for any $A \in \mathcal{C}$, the Choquet integral is defined by

$$(C)\int_A f d\mu = \int_0^{+\infty} \mu(F_\beta(f) \cap A) d\beta + \int_{-\infty}^0 [\mu(F_\beta(f) \cap A) - \mu(A)] d\beta,$$

where $F_\beta(f) = \{\omega \in \Omega; f(\omega) \geq \beta\}$. If $(C)\int_A f d\mu \in \mathbb{R}$, then f is called Choquet integrable on A. Notice that if $f \geq 0$ on A, then in the above formula we get $\int_{-\infty}^0 = 0$.

If μ is the Lebesgue measure, then the Choquet integral $(C)\int_A f d\mu$ reduces to the Lebesgue integral.

In what follows, we list some known properties of the Choquet integral.

Remark 2.2. If $\mu : \mathcal{C} \to [0, +\infty)$ is a monotone set function, then the following properties hold :

(i) For all $a \geq 0$ we have $(C)\int_A a f d\mu = a \cdot (C)\int_A f d\mu$ (if $f \geq 0$ then see, e.g., [18, Theorem 11.2, (5), p. 228] and if f is of arbitrary sign, then see, e.g., [5, Proposition 5.1, (ii), p. 64]).

(ii) If μ is submodular too, then for all f, g of arbitrary sign and lower bounded we have (see, e.g., [5, Theorem 6.3, p. 75])

$$(C)\int_A (f + g) d\mu \leq (C)\int_A f d\mu + (C)\int_A g d\mu,$$

that is the Choquet integral is sublinear.

However, in particular, the comonotonic additivity holds, that is if μ is a monotone set function and f, g are $\mathcal{C}$-measurable and comonotone on A (that is $(f(\omega) - f(\omega')) \cdot (g(\omega) - g(\omega')) \geq 0$, for all $\omega, \omega' \in A$), then by, e.g. [5, Proposition 5.1, (vi), p. 65], we have

$$(C)\int_A (f+g)d\mu = (C)\int_A f d\mu + (C)\int_A g d\mu.$$

(iii) If $f \leq g$ on A then $(C)\int_A f d\mu \leq (C)\int_A g d\mu$ (see, e.g., [18, Theorem 11.2, (3), p. 228] if $f, g \geq 0$ and [18, Theorem 11.2, (3), p. 232] if f, g are of arbitrary sign). Also, $(C)\int_A 1 d\mu = \mu(A)$.

(iv) The formula $\mu(A) = \gamma(M(A))$, where $\gamma : [0,1] \to [0, L]$ is an increasing and concave function, with $\gamma(0) = 0$ and M is a σ-additive measure (or only finitely additive) on a σ-algebra on Ω (that is, $M(\emptyset) = 0$ and M is countably additive), gives simple examples of monotone and submodular set functions (see, e.g., [5, Example 2.1, pp. 16–17]). For a simple example, we can take $\gamma(t) = \sqrt{t}$.

Such of set functions μ are also called distorsions of countably additive measures (or distorted measures). If M is the Lebesgue measure, then μ defined as above will be called distorted Lebesgue measure. □

3. SHAPE PRESERVING PROPERTIES

Firstly, we deal with the Kantorovich-Choquet type operators.

Denoting by $\mathcal{B}_I$ the sigma algebra of all Borel measurable subsets in $\mathcal{P}(I)$, everywhere in this section, $(\Gamma_{n,x})_{n\in\mathbb{N}, x\in I}$, will be a collection of families $\Gamma_{n,x} = \{\mu_{n,k,x}\}_{k=0}^{n}$, of monotone, submodular and strictly positive set functions $\mu_{n,k,x}$ on $\mathcal{B}_I$, with $I = [0,1]$ in the case of Bernstein-Kantorovich polynomials and $I = [0, +\infty)$ in the cases of Szász-Mirakjan-Kantorovich and Baskakov-Kantorovich operators.

Suggested by the classical forms of the linear and positive operators of Bernstein-Kantorovich (see, e.g., [12]), Szász-Kantorovich (see, e.g., [3], [2]) and Baskakov-Kantorovich (see, e.g., [17]), in the paper [8] were introduced and studied the approximation properties of the following Choquet type operators.

Definition 3.1. *The Bernstein-Kantorovich-Choquet, Szász-Kantorovich-Choquet and Baskakov-Kantorovich-Choquet operators with respect to $\Gamma_{n,x} = \{\mu_{n,k,x}\}_{k=0}^{n}$, are defined by the formulas*

$$K_{n,\Gamma_{n,x}}(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \cdot \frac{(C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) d\mu_{n,k,x}(t)}{\mu_{n,k,x}([\frac{k}{n+1}, \frac{k+1}{n+1}])}, \tag{3.1}$$

$$S_{n,\Gamma_{n,x}}(f)(x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\cdot\frac{(C)\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)d\mu_{n,k,x}(t)}{\mu_{n,k,x}([\frac{k}{n},\frac{k+1}{n}])}, \tag{3.2}$$

$$V_{n,\Gamma_{n,x}}(f)(x) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\cdot\frac{(C)\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)d\mu_{n,k,x}(t)}{\mu_{n,k,x}([\frac{k}{n},\frac{k+1}{n}])}. \tag{3.3}$$

We note that in order to be well defined these operators, it is good enough if, for example, we suppose that $f : I \to \mathbb{R}_+$ is a $\mathcal{B}_I$-measurable function, bounded on I, where $I = [0,1]$ for $K_{n,\Gamma_{n,x}}(f)(x)$ and $I = [0,+\infty)$ for $S_{n,\Gamma_{n,x}}(f)(x)$ and $V_{n,\Gamma_{n,x}}(f)(x)$.

Since in general, the change of variable does not work for the Choquet integral, we also can introduce the following different Choquet operators, given formally by

$$\overline{K}_{n,\Gamma_n}(f)(x) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\cdot(C)\int_0^1 f\left(\frac{k+t}{n+1}\right)d\mu_{n,k,x}(t), \tag{3.4}$$

$$\overline{S}_{n,\Gamma_n}(f)(x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\cdot(C)\int_0^1 f\left(\frac{k+t}{n}\right)d\mu_{n,k,x}(t), \tag{3.5}$$

$$\overline{V}_{n,\Gamma_n}(f)(x) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\cdot(C)\int_0^1 f\left(\frac{k+t}{n}\right)d\mu_{n,k,x}(t), \tag{3.6}$$

different from K_{n,Γ_n}, S_{n,Γ_n} and V_{n,Γ_n}, correspondingly.

Remark 3.2. It is known that if all the set functions in the family Γ_n one reduce to the Lebesgue measure denoted here by M (which is independent of n too), then $\overline{K}_{n,M} = K_{n,M}$, $\overline{S}_{n,M} = S_{n,M}$ and $\overline{V}_{n,M} = V_{n,M}$. But as we will show later, these equalities also hold for some monotone and submodular function different from the Lebesgue measure. □

Everywhere in this paper the shape preserving properties will be considered in the case when the set functions in the collection of families $\Gamma_{n,x}$ are independent of x and and k.

The main result of the paper is the following.

Theorem 3.3. *Let $\Gamma_n = \{\mu_n\}$, $n \in \mathbb{N}$, be a family of monotone set functions on $\mathcal{B}_I$, where $I = [0,1]$ for $K_{n,\Gamma_n}(f)(x)$, $\overline{K}_{n,\Gamma_n}(f)(x)$ and $I = [0,+\infty)$ for $S_{n,\Gamma_n}(f)(x)$, $\overline{S}_{n,\Gamma_n}(f)(x)$ and $V_{n,\Gamma_n}(f)(x)$, $\overline{V}_{n,\Gamma_n}(f)(x)$. Suppose that $f : I \to \mathbb{R}_+$ is bounded on I.*

(i) If f is nondecreasing on I, then for all $n \in \mathbb{N}$, $K_{n,\Gamma_n}(f)$, $S_{n,\Gamma_n}(f)$ and $V_{n,\Gamma_n}(f)$ are nondecreasing on I;

(ii) If f is nondecreasing on I, then for all $n \in \mathbb{N}$, $\overline{K}_{n,\Gamma_n}(f)$, $\overline{S}_{n,\Gamma_n}(f)$ and $\overline{V}_{n,\Gamma_n}(f)$ are nondecreasing on I;

(iii) Suppose that, in addition, $\Gamma_n = \{\mu_n\}$, $n \in \mathbb{N}$, is a family of submodular set functions on $\mathcal{B}_I$. If f is nonconcave on I, then for all $n \in \mathbb{N}$, $\overline{K}_{n,\Gamma_n}(f)$, $\overline{S}_{n,\Gamma_n}(f)$ and $\overline{V}_{n,\Gamma_n}(f)$ are nonconcave on I.

(iv) If f is nonconcave on I and all $\mu_n(A) = \gamma_n(M(A)), n \in \mathbb{N}$ are distorted Lebesgue measures with all the γ_n, increasing, concave and continuous on $[0,1]$, then for all $n \in \mathbb{N}$, $K_{n,\Gamma_n}(f)$, $S_{n,\Gamma_n}(f)$ and $V_{n,\Gamma_n}(f)$ are nonconcave on I.

Proof. (i) Denoting

$$A_{n,k} = \frac{(C)\displaystyle\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)d\mu_n(t)}{\mu_n([\frac{k}{n+1}, \frac{k+1}{n+1}])}, B_{n,k} = \frac{(C)\displaystyle\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)d\mu_n(t)}{\mu_n([\frac{k}{n}, \frac{k+1}{n}])},$$

by simple calculation (for Bernstein fundamental polynomials, for Szász fundamental polynomials and for Baskakov fundamental polynomials, also see, e.g., [1, pp. 83–84, pp. 169–170] and [6, p. 125], correspondingly) we get

$$K'_{n,\Gamma_n}(f)(x) = n \cdot \sum_{k=0}^{n-1} p_{n-1,k}(x) \left[A_{n,k+1} - A_{n,k}\right]$$

$$S'_{n,\Gamma_n}(f)(x) = ne^{-nx} \sum_{k=0}^{\infty} \tfrac{(nx)^k}{k!} \left[B_{n,k+1} - B_{n,k}\right],$$

$$V'_{n,\Gamma_n}(f)(x) = n \sum_{k=0}^{\infty} v_{n+1,k}(x) \left[B_{n,k+1} - B_{n,k}\right].$$

Since f is nondecreasing on I, by applying the properties in Remark 2.2, (iii), we get

$$\begin{aligned} A_{n,k} &\le f\left(\tfrac{k+1}{n+1}\right) \cdot \frac{(C)\displaystyle\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} d\mu_n(t)}{\mu_n([\frac{k}{n+1}, \frac{k+1}{n+1}])} = f\left(\tfrac{k+1}{n+1}\right) \\ &= \frac{(C)\displaystyle\int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}} f\left(\tfrac{k+1}{n+1}\right) d\mu_n(t)}{\mu_n([\frac{k+1}{n+1}, \frac{k+2}{n+1}])} \le \frac{(C)\displaystyle\int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}} f(t)d\mu_n(t)}{\mu_n([\frac{k+1}{n+1}, \frac{k+2}{n+1}])} \\ &= A_{n,k+1}, \end{aligned}$$

since $f(\frac{k+1}{n+1}) \le f(t)$, for all $t \in [\frac{k+1}{n+1}, \frac{k+2}{n+1}]$. This implies that $K'_{n,\Gamma_n}(f)(x) \ge 0$, for all $x \in I$, that is $K_{n,\Gamma_n}(f)(x)$ is nondecreasing on I, for all $n \in \mathbb{N}$. The proofs in the cases of $B_{n,k}$ and the corresponding $S_{n,\Gamma_n}(f)$ and $V_{n,\Gamma_n}(f)$ operators are similar.

(ii) Since f is nondecreasing on I (therefore is $\mathcal{B}_I$-measurable) and for any k, n, $f\left(\frac{k+t}{n+1}\right)$ remains nondecreasing on I, it follows that $f\left(\frac{k+t}{n+1}\right)$ is $\mathcal{B}_I$-measurable as function of t. Now, by simple calculations we immediately obtain

$$\overline{K}'_{n,\Gamma_n}(f)(x) =$$
$$= n \cdot \sum_{k=0}^{n-1} p_{n-1,k}(x) \left[(C)\int_0^1 f\left(\tfrac{k+1+t}{n+1}\right) d\mu_n(t) - (C)\int_0^1 f\left(\tfrac{k+t}{n+1}\right) d\mu_n(t)\right],$$

$$\overline{S}'_{n,\Gamma_n}(f)(x) =$$
$$= ne^{-nx} \sum_{k=0}^{\infty} \tfrac{(nx)^k}{k!} \left[(C)\int_0^1 f\left(\tfrac{k+1+t}{n}\right) d\mu_n(t) - (C)\int_0^1 f\left(\tfrac{k+t}{n}\right) d\mu_n(t)\right],$$

$$\overline{V}'_{n,\Gamma_n}(f)(x) =$$
$$= n \sum_{k=0}^{\infty} v_{n+1,k}(x) \left[(C)\int_0^1 f\left(\tfrac{k+1+t}{n}\right) d\mu_n(t) - (C)\int_0^1 f\left(\tfrac{k+t}{n}\right) d\mu_n(t)\right].$$

Since f is nondecreasing on I, it follows that (for all $t \in [0, 1]$, n, k)

$$f\left(\tfrac{k+1+t}{n+1}\right) \geq f\left(\tfrac{k+t}{n+1}\right) \text{ and } f\left(\tfrac{k+1+t}{n}\right) \geq f\left(\tfrac{k+t}{n}\right).$$

Applying the property in Remark 2.2, (iii), we get

$$(C)\int_0^1 f\left(\tfrac{k+1+t}{n+1}\right) d\mu_n(t) - (C)\int_0^1 f\left(\tfrac{k+t}{n+1}\right) d\mu_n(t) \geq 0$$

and

$$(C)\int_0^1 f\left(\tfrac{k+1+t}{n}\right) d\mu_n(t) - (C)\int_0^1 f\left(\tfrac{k+t}{n}\right) d\mu_n(t) \geq 0.$$

This implies that the first derivatives of these operators are positive, that is the operators $\overline{K}_{n,\Gamma_n}(f)$, $\overline{S}_{n,\Gamma_n}(f)$ and $\overline{V}_{n,\Gamma_n}(f)$ also are nondecreasing for any $n \in \mathbb{N}$.

(iii) Since f is nonconcave on I (therefore is $\mathcal{B}_I$-measurable) and for any k, n, $f(\frac{k+t}{n+1})$ remains nonconcave on I, it follows that $f(\frac{k+t}{n+1})$ is $\mathcal{B}_I$-measurable as function of t. Now, by the calculations for the classical Kantorovich variants of the operators (see, e.g. again [1, pp. 83–84, pp. 169–170] and [6, p. 125], correspondingly), we immediately obtain

$$\begin{aligned}\overline{K}''_{n,\Gamma_n}(f)(x) =& n(n-1)\cdot\sum_{k=0}^{n-2} p_{n-2,k}(x)\\ &\cdot\left[(C)\int_0^1 f\left(\tfrac{k+2+t}{n+1}\right)d\mu_n(t)-2(C)\int_0^1 f\left(\tfrac{k+1+t}{n+1}\right)d\mu_n(t)\right.\\ &\left.+(C)\int_0^1 f\left(\tfrac{k+t}{n+1}\right)d\mu_n(t)\right],\\ \overline{S}''_{n,\Gamma_n}(f)(x) =& n^2e^{-nx}\sum_{k=0}^{\infty}\tfrac{(nx)^k}{k!}\\ &\cdot\left[(C)\int_0^1 f\left(\tfrac{k+2+t}{n}\right)d\mu_n(t)-2(C)\int_0^1 f\left(\tfrac{k+1+t}{n}\right)d\mu_n(t)\right.\\ &\left.+(C)\int_0^1 f\left(\tfrac{k+t}{n}\right)d\mu_n(t)\right],\\ \overline{V}''_{n,\Gamma_n}(f)(x) =& n(n+1)\sum_{k=0}^{\infty} v_{n+2,k}(x)\\ &\cdot\left[(C)\int_0^1 f\left(\tfrac{k+2+t}{n}\right)d\mu_n(t)-2(C)\int_0^1 f\left(\tfrac{k+1+t}{n}\right)d\mu_n(t)\right.\\ &\left.+(C)\int_0^1 f\left(\tfrac{k+t}{n}\right)d\mu_n(t)\right].\end{aligned}$$

Since f is nonconcave on I, it follows that (for all $t\in[0,1]$, $n,k\in\mathbb{N}, 0\le k\le n$) we have

$$f\left(\tfrac{k+2+t}{n+1}\right)+f\left(\tfrac{k+t}{n+1}\right)\ge 2f\left(\tfrac{k+1+t}{n+1}\right).$$

Since every μ_n is submodular, applying consecutively the properties in Remark 2.2, (iii) and (ii), we obtain

$$\begin{aligned}2\cdot(C)\int_0^1 f\left(\tfrac{k+1+t}{n+1}\right)d\mu_n(t) \le& (C)\int_0^1\left[f\left(\tfrac{k+2+t}{n+1}\right)+f\left(\tfrac{k+t}{n+1}\right)\right]d\mu_n(t)\\ \le& (C)\int_0^1 f\left(\tfrac{k+2+t}{n+1}\right)d\mu_n(t)\\ &+(C)\int_0^1 f\left(\tfrac{k+t}{n+1}\right)d\mu_n(t).\end{aligned}$$

This implies that the second derivatives $\overline{K}''_{n,\Gamma_n}(f)(x)$ is ≥ 0 on I, that is the operator $\overline{K}_{n,\Gamma_n}(f)(x)$ is nonconcave on I. The proof in the cases of the other two operators is similar.

(iv) We use the notations for $A_{n,k}$ and $B_{n,k}$ from the point (i) and the ideas of calculation from the point (iii). We present here only the proof in the case of $K_{n,\Gamma_n}(f)$, because the proofs in the case of $B_{n,k}$ and of the operators $S_{n,\Gamma_n}(f)$

and $V_{n,\Gamma_n}(f)$ are similar. Thus, we have

$$K''_{n,\Gamma_n}(f)(x) = n(n-1)\cdot\sum_{k=0}^{n-2} p_{n-2,k}(x)\cdot\left[A_{n,k+2}-2A_{n,k+1}+A_{n,k}\right].$$

Since the Lebesgue measure M is invariant at translations, it immediately follows that

$$\mu_n([\tfrac{k}{n+1},\tfrac{k+1}{n+1}]) = \mu_n([\tfrac{k+1}{n+1},\tfrac{k+2}{n+1}]) = \mu_n([\tfrac{k+2}{n+1},\tfrac{k+3}{n+1}]) := C_n, \quad n\in\mathbb{N},$$

which implies

$$K''_{n,\Gamma_n}(f)(x) = \tfrac{n(n-1)}{C_n}\cdot\sum_{k=0}^{n-2} p_{n-2,k}(x)\cdot\left[a_{n,k+2}-2a_{n,k+1}+a_{n,k}\right].$$

with $a_{n,k} = (C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)d\mu_n(t)$. But we can write

$$\begin{aligned}
a_{n,k+1} &= (C)\int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}} f(t)d\mu_n(t)\\
&= \int_0^\infty \mu_n(\{t\in[\tfrac{k+1}{n+1},\tfrac{k+2}{n+1}]; f(t)\ge\alpha\})d\alpha\\
&= \int_0^\infty \mu_n(\{t-\tfrac{1}{n+1}\in[\tfrac{k}{n+1},\tfrac{k+1}{n+1}]; f(t)\ge\alpha\})d\alpha\\
&= \int_0^\infty \mu_n(\{w\in[\tfrac{k}{n+1},\tfrac{k+1}{n+1}]; f(w+\tfrac{1}{n+1})\ge\alpha\})d\alpha\\
&= (C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t+\tfrac{1}{n+1})d\mu_n(t).
\end{aligned}$$

Since in the similar way we get $a_{n,k+2} = (C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t+\frac{2}{n+1})d\mu_n(t)$, it follows that

$$\begin{aligned}
a_{n,k+2}-2a_{n,k+1}+a_{n,k} =& (C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)d\mu_n(t) + (C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t+\tfrac{2}{n+1})d\mu_n(t)\\
&-2(C)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t+\tfrac{1}{n+1})d\mu_n(t).
\end{aligned}$$

But since f is nonconcave on $I=[0,1]$, we have

$$f(t)+f(t+\tfrac{2}{n+1}) \ge 2f(t+\tfrac{1}{n+1}), \text{ for all } t\in[\tfrac{k}{n+1},\tfrac{k+1}{n+1}].$$

Since every μ_n is submodular, applying the Choquet integral on $[\frac{k}{n+1},\frac{k+1}{n+1}]$ to the previous inequality and applying consecutively the properties in Remark 2.2, (iii) and (ii), reasoning exactly as at the above point (iii), we arrive at $a_{n,k+2}-2a_{n,k+1}+a_{n,k}\ge 0$, which leads to the desired conclusion. □

REMARK 3.4. From the proofs, it easily follows that Theorem 3.3, (i), (ii) hold if we replace in their statements the word nondecreasing with the word nonincreasing. But if we replace in Theorem 3.3, (iii) and (iv) the word nonconcave with the word nonconvex, it is easy to see that their proofs do not work since the subaditivity of the Choquet integral is not helpful. However, under some additional hypothesis, we can prove the shape preserving properties concerning the nonconvexity, as follows. □

COROLLARY 3.5. *Let $\Gamma_n = \{\mu_n\}$, $n \in \mathbb{N}$, be a family of monotone set functions on $\mathcal{B}_I$, where $I = [0,1]$ for $K_{n,\Gamma_n}(f)(x)$, $\overline{K}_{n,\Gamma_n}(f)(x)$ and $I = [0,+\infty)$ for $S_{n,\Gamma_n}(f)(x)$, $\overline{S}_{n,\Gamma_n}(f)(x)$ and $V_{n,\Gamma_n}(f)(x)$, $\overline{V}_{n,\Gamma_n}(f)(x)$. Suppose that $f : I \to \mathbb{R}_+$ is bounded on I.*

(i) If f is nonconvex and monotone on I, then $\overline{K}_{n,\Gamma_n}(f)(x)$, $\overline{S}_{n,\Gamma_n}(f)(x)$ and $\overline{V}_{n,\Gamma_n}(f)(x)$ are nonconvex and of the same monotonicity with f on I, for any $n \in \mathbb{N}$.

(ii) If f is nonconvex and monotone on I and all $\mu_n(A) = \gamma_n(M(A)), n \in \mathbb{N}$ are distorted Lebesgue measures with all the γ_n, increasing, concave and continuous on $[0,1]$, then for all $n \in \mathbb{N}$, $K_{n,\Gamma_n}(f)$, $S_{n,\Gamma_n}(f)$ and $V_{n,\Gamma_n}(f)$ are nonconvex and of the same monotonicity on I with f.

Proof. The preservation of monotonicity of f in both cases (i) and (ii), follows from Theorem 3.3. For the preservation of the nonconvexity of f, we deal only with the case (i), since the proof in the case (ii) is similar. Indeed, the nonconvexity of f implies that (for all $t \in [0,1]$, $n, k \in \mathbb{N}, 0 \le k \le n$) we have

$$f\left(\tfrac{k+2+t}{n+1}\right) + f\left(\tfrac{k+t}{n+1}\right) \le 2f\left(\tfrac{k+1+t}{n+1}\right).$$

Since as functions of t, $f(\frac{k+2+t}{n+1})$ and $f(\frac{k+t}{n+1})$ are of the same monotonicity, they are comonotonic and applying to the previous inequality the Choquet integral and the property in Remark 2.2, (ii), we obtain

$$\begin{aligned}2 \cdot (C)\int_0^1 f\left(\tfrac{k+1+t}{n+1}\right) d\mu_n(t) \ge&(C)\int_0^1 \left[f\left(\tfrac{k+2+t}{n+1}\right) + f\left(\tfrac{k+t}{n+1}\right)\right] d\mu_n(t)\\ =&(C)\int_0^1 f\left(\tfrac{k+2+t}{n+1}\right) d\mu_n(t)\\ &+(C)\int_0^1 f\left(\tfrac{k+t}{n+1}\right) d\mu_n(t).\end{aligned}$$

Using the relationship for $\overline{K}''_{n,\Gamma_n}(f)(x)$ in the proof of Theorem 3.3, (iii), this immediately implies that the second derivatives $\overline{K}''_{n,\Gamma_n}(f)(x)$ is ≤ 0 on I, that is the operator $\overline{K}_{n,\Gamma_n}(f)(x)$ is nonconvex on I. The proof in the cases of the other two operators is similar. □

REMARK 3.6. If in Theorem 3.3, f is of arbitrary sign on I, then the statement of Theorem 3.3 can be restated for the slightly modified operator defined by

$$L_n^*(f)(x) = L_n(f - m)(x) + m,$$

where L_n is any from the operators in the statement and $m \in \mathbb{R}$ is a lower bound for f, that is $f(x) \geq m$, for all $x \in I$.

Indeed, this is immediate from the fact that if f is nondecreasing (nonconcave), then $f - m$ remains nondecreasing (nonconcave, respectively). $\square$

In continuation to the comments in Remark 3.2, we can prove the following result.

LEMMA 3.7. *Suppose that $\mu_{n,k,x} = \mu := \sqrt{M}$, for all n, k and x, where M is the Lebesgue measure and $f : I \to \mathbb{R}_+$ is bounded and $\mathcal{B}_I$-measurable. Then we have*

$$\frac{(C)\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)d\mu(t)}{\mu([\frac{k}{n}, \frac{k+1}{n}])} = (C)\int_0^1 f\left(\frac{k+t}{n}\right) d\mu(t).$$

Proof. For any fixed $\alpha \geq 0$, let us make the notations

$$A_{n,k}(\alpha) = \left\{t \in [0,1]; f\left(\frac{k+t}{n}\right) \geq \alpha\right\} \text{ and } B_{n,k}(\alpha) = \left\{w \in \left[\frac{k}{n}, \frac{k+1}{n}\right]; f(w) \geq \alpha\right\}.$$

It is clear that $B_{n,k}(\alpha)$ is obtained by applying to $A_{n,k}(\alpha)$ the linear transform $w(t) = \frac{t}{n} + \frac{k}{n} = \frac{1}{n}(t + k)$, $t \in [0,1]$. By the well-known properties of the Lebesgue measure, we get $M(B_{n,k}(\alpha)) = \frac{1}{n}M(A_{n,k}(\alpha))$, which evidently implies $\mu(B_{n,k}(\alpha)) = \frac{1}{\sqrt{n}}\mu(A_{n,k}(\alpha))$. Therefore, we get

$$\begin{aligned}(C)\int_0^1 f\left(\frac{k+t}{n}\right) d\mu(t) &= \int_0^{\infty} \mu(A_{n,k}(\alpha))d\alpha = \sqrt{n}\int_0^{\infty} \mu(B_{n,k}(\alpha))d\alpha \\ &= \sqrt{n} \cdot (C)\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(w)d\mu(w) = \frac{(C)\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(w)d\mu(w)}{\mu(\frac{k}{n}, \frac{k+1}{n}]},\end{aligned}$$

which proves our assertion. Evidently that the above relationship remains valid by replacing n with $n + 1$. $\square$

REMARK 3.8. Lemma 3.7 shows that for the monotone and submodular set function $\mu = \sqrt{M}$, with M the Lebesgue measure, the operators given by (3.4), (3.5) and (3.6), coincide with the operators given by (3.1), (3.2) and resp. (3.3). $\square$

REMARK 3.9. In the papers [9]-[11], were introduced and studied the qualitative and quantitative approximation properties of the multivariate Bernstein-Durrmeyer-Choquet polynomials, which in the univariate are given by the formula

$$D_{n,\Gamma_{n,x}}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(C)\displaystyle\int_0^1 f(t)t^k(1-t)^{n-k}d\mu_{n,k,x}(t)}{(C)\displaystyle\int_0^1 t^k(1-t)^{n-k}d\mu_{n,k,x}(t)},$$

where $\{\mu_{n,k,x}\}$, $n \in \mathbb{N}$, $k \in \{0,1,...,n\}$, $x \in [0,1]$, is a family of monotone, submodular and strictly positive set functions on $\mathcal{B}_{[0,1]}$.

It is well-known that the proof of the shape preserving properties for the classical Bernstein-Durrmeyer operators is based on the integration by parts, rule which does not hold for the general Choquet integral. This fact induces much difficulty in any attempt to prove these properties for Bernstein-Durrmeyer-Choquet polynomials and for this reason, it remains as an open question under which conditions still they hold.

However, we can show that, in general, the shape preserving properties for these polynomials do not hold. Indeed, for example, let us consider the Bernstein-Durrmeyer-Choquet polynomials introduced by [10, Example 5.2], given by

$$D_{n,\Gamma_n}(f)(x) = B_n(f)(x) + x^n \left[\frac{(C)\displaystyle\int_0^1 f(t)t^n d\mu(t)}{(C)\displaystyle\int_0^1 t^n d\mu(t)} - f(1)\right],$$

with respect to the strictly positive, monotone and submodular set function $\mu(A) = \sqrt{M(A)}$, where $M(A)$ denotes the classical Lebesgue measure. Note here that if f is positive and increasing (but not necessarily constant function), then the above quantity between the right brackets is, in general, < 0.

Now, suppose that f is positive, continuous and nondecreasing on $[0,1]$, such that it is constant equal to $f(1)$ in a small left neighbourhood of 1 and strictly increasing on the rest of $[0,1]$. Then, since $B_n'(f)(1) = n\left[f(1) - f\left(\frac{n-1}{n}\right)\right]$, we obtain

$$\begin{aligned}
D_{n,\Gamma_n}'(f)(1) &= n\left[f(1) - f\left(\tfrac{n-1}{n}\right)\right] + n\left[\frac{(C)\displaystyle\int_0^1 f(t)t^n d\mu(t)}{(C)\displaystyle\int_0^1 t^n d\mu(t)} - f(1)\right] \\
&= n\left[\frac{(C)\displaystyle\int_0^1 f(t)t^n d\mu(t)}{(C)\displaystyle\int_0^1 t^n d\mu(t)} - f\left(1-\tfrac{1}{n}\right)\right] \\
&< n\left[f(1)\cdot\frac{(C)\displaystyle\int_0^1 t^n d\mu(t)}{(C)\displaystyle\int_0^1 t^n d\mu(t)} - f\left(1-\tfrac{1}{n}\right)\right] = f(1) - f\left(1-\tfrac{1}{n}\right) = 0,
\end{aligned}$$

for all sufficiently large $n \in \mathbb{N}$.

From the continuity of the polynomial $D_{n,\Gamma_n}(f)(x)$, the last inequality implies that for any sufficiently large $n \in \mathbb{N}$, there exists a small neighborhood of 1, such that $D'_{n,\Gamma_n}(f)(x) < 0$ for all x in that neighborhood, contradicting a possible preservation of the nondecreasing monotonicity of f. □

4. MONOTONICITY OF THE APPROXIMATION SEQUENCES

In this section we present two samples concerning the monotonicity of the sequences of Choquet type operators, the rest of the cases being leaved as open questions to the readers.

In this sense, we can state the following.

THEOREM 4.1. *Suppose that Γ_n one reduces to a single monotone and submodular function μ.*

(i) Let $f : [0, +\infty) \to \mathbb{R}_+$ be differentiable on $[0, +\infty)$. If f is nonconcave, $f' > 0$ and f' is nonconvex on $[0, +\infty)$, then the sequence $(\overline{V}_{n,\mu}(f)(x))_{n\in\mathbb{N}}$ is nonincreasing, i.e. $\overline{V}_{n,\mu}(f)(x) \geq \overline{V}_{n+1,\mu}(f)(x)$, for all $n \in \mathbb{N}$, $x \in [0, +\infty)$;

(ii) Let f be differentiable on $[0, 1]$. If f is nonconcave, $f' > 0$ and f' nonconvex on $[0, 1]$, then the sequence (different a bit from $\overline{K}_{n,\mu}$)

$$\tilde{K}_{n,\mu}(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \cdot (C)\int_0^1 f\left(\frac{k+t}{n}\right) d\mu(t), n \in \mathbb{N},$$

is decreasing, i.e. $\tilde{K}_{n,\mu}(f)(x) \geq \tilde{K}_{n+1,\mu}(f)(x)$, for all $n \in \mathbb{N}$, $x \in [0, 1]$.

Proof. (i) For the classical Baskakov operators

$$V_n(f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f(k/n),$$

it is known the formula (see, [14], or also [1, pp. 176–177]),

$$V_n(f)(x) - V_{n+1}(f)(x) = \frac{1}{n(n+1)} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \left[\frac{k+1}{n}, \frac{k+1}{n+1}, \frac{k}{n+1}; f\right]$$
$$= n(n+1) \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \cdot E(f),$$

where

$$E(f) = \left(\frac{f\left(\frac{k+1}{n}\right)}{(k+1)(n+k+1)} - \frac{f\left(\frac{k+1}{n+1}\right)}{n(k+1)} + \frac{f\left(\frac{k}{n+1}\right)}{n(n+k+1)} \right).$$

Denoting $F(\frac{k}{n}) = (C)\int_0^1 f\left(\frac{k+t}{n}\right) d\mu(t)$, by the above calculations we get

$$\overline{V}_{n,\mu}(f)(x) - \overline{V}_{n+1,\mu}(f)(x) = n(n+1) \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \cdot E(F),$$

where

$$E(F)=\left(\frac{F\left(\frac{k+1}{n}\right)}{(k+1)(n+k+1)}-\frac{F\left(\frac{k+1}{n+1}\right)}{n(k+1)}+\frac{F\left(\frac{k}{n+1}\right)}{n(n+k+1)}\right)$$
$$=\frac{1}{n(k+1)(n+k+1)}\cdot\left((C)\int_0^1 nf(\tfrac{k+1+t}{n})d\mu(t)+(C)\int_0^1(k+1)f(\tfrac{k+t}{n+1})d\mu(t)\right.$$
$$\left.-(C)\int_0^1(n+k+1)f'(\tfrac{k+1+t}{n+1})d\mu(t)\right).$$

If we prove that for all $t\in[0,1]$, $n,k\in\mathbb{N}$ we have

$$G(t):=nf\left(\frac{k+1+t}{n}\right)+(k+1)f\left(\frac{k+t}{n+1}\right)-(n+k+1)f\left(\frac{k+1+t}{n+1}\right)\geq 0,$$

then applying here the Choquet integral and taking into account its properties in Remark 2.2, (iii) and (ii), it follows $E(F)\geq 0$ and the required conclusion. In this sense, let us observe that from the above considerations, $G(0)\geq 0$ means exactly the nonconcavity of f. Let n and k arbitrary fixed. It follows that if we prove that $G'(t)\geq 0$ for all $t\geq 0$, then we arrive at the desired conclusion. Indeed, we have

$$G'(t)=f'(\tfrac{k+1+t}{n})+\tfrac{k+1}{n+1}f'(\tfrac{k+t}{n+1})-\tfrac{n+k+1}{n+1}f'(\tfrac{k+1+t}{n+1})$$
$$=f'(\tfrac{k+1+t}{n})-f'(\tfrac{k+1+t}{n+1})+\tfrac{k+1}{n+1}f'(\tfrac{k+t}{n+1})-\tfrac{k}{n+1}f'(\tfrac{k+1+t}{n+1}).$$

Since $\frac{k+1+t}{n}>\frac{k+1+t}{n+1}>\frac{k+t}{n+1}$ and by hypothesis f' is nondecreasing, firstly it follows

$$f'(\tfrac{k+1+t}{n})-f'(\tfrac{k+1+t}{n+1})\geq 0.$$

Then, since f' is nonconvex, by e.g. [13, pp. 44], it follows that $\frac{f'(x)}{x}$ is nonincreasing on $(0,+\infty)$, which will imply that $\frac{k+1}{n+1}f'(\frac{k+t}{n+1})-\frac{k}{n+1}f'(\frac{k+1+t}{n+1})\geq 0$, finishing the proof. Indeed, we get

$$\frac{f'(\frac{k+1+t}{n+1})}{f'(\frac{k+t}{n+1})}\leq\frac{k+1+t}{k+t}\leq\frac{k+1}{k},$$

which leads to the desired conclusion.

(ii) For the classical Bernstein operators

$$B_n(f)(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f(\tfrac{k}{n}),$$

it is known the formula ([16, Theorem 1], or also [1, pp. 88–89])

$$B_n(f)(x)-B_{n+1}(f)(x)=\frac{x(1-x)}{n(n+1)}\sum_{k=0}^{n-1}\binom{n-1}{k}x^k(1-x)^{n-1-k}\cdot\left[\frac{k}{n},\frac{k+1}{n+1},\frac{k+1}{n};f\right],$$

where by easy calculation we get

$$\left[\frac{k}{n},\frac{k+1}{n+1},\frac{k+1}{n};f\right]=$$
$$=\frac{n^2(n+1)}{(k+1)(n-k)}\cdot\left[n(f(\tfrac{k+1}{n})-f(\tfrac{k+1}{n+1})-k(f(\tfrac{k+1}{n})-f(\tfrac{k}{n}))+f(\tfrac{k}{n})-f(\tfrac{k+1}{n+1})\right].$$

Denoting $F(\frac{k}{n}) = (C)\int_0^1 f\left(\frac{k+t}{n}\right) d\mu(t)$ and

$$\tilde{K}_{n,\mu}(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \cdot (C)\int_0^1 f\left(\frac{k+t}{n}\right) d\mu(t)$$

and reasoning as at the point (i), by the above calculations we get

$$\tilde{K}_{n,\mu}(f)(x) - \tilde{K}_{n+1,\mu}(f)(x) = \frac{x(1-x)}{n(n+1)} \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-1-k} E,$$

where

$$E = \frac{n^2(n+1)}{(k+1)(n-k)} \left[(C)\int_0^1 (n-k) f\left(\frac{k+1+t}{n}\right) d\mu(t) \right.$$
$$\left. - (C)\int_0^1 (n+1) f\left(\frac{k+1+t}{n+1}\right) d\mu(t) + (C)\int_0^1 (k+1) f\left(\frac{k+t}{n}\right) d\mu(t) \right].$$

By using similar reasoning with those from the point (i), here it remains to prove that for all $t \in [0,1]$ and $n \in \mathbb{N}$, $k = 0, 1, ..., n-1$, we have

$$G(t) := (n-k) f(\tfrac{k+1+t}{n}) + (k+1) f(\tfrac{k+t}{n}) - (n+1) f(\tfrac{k+1+t}{n+1}) \geq 0.$$

From this point, the proof is identical with that for the above point (i). $\square$

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