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GENERALIZED GROWTH AND APPROXIMATION ERRORS OF ENTIRE HARMONIC FUNCTIONS IN $\mathbb{R}^n$, $n \geq 3$

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Abstract. In this paper we study the continuation of harmonic functions in the ball to the entire harmonic functions in space $\mathbb{R}^n, n \geq 3$. The generalized order introduced by M.N. Seremeta has been used to characterize the growth of such functions. Moreover, the generalized order, generalized lower order and generalized type have been characterized in terms of harmonic polynomial approximation errors. Our results apply satisfactorily for slow growth.

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1. INTRODUCTION

Since the entire functions form a simplest class of analytic functions which includes all polynomials several researchers like Varga [14], Batyrev [1], Reddy [7], Ibraginov and Shikhaliev [3], Vakarchuk [13], Kasana and Kumar [6] and others had obtained the characterization of growth parameters of an entire function $f(z)$ in terms of the sequences of polynomial approximation and interpolation errors taken over different domains in the complex plane. Similar characterizations had been investigated for entire harmonic functions in $\mathbb{R}^n, n \geq 3$ in terms of harmonic polynomial approximation errors. When we discuss time dependent problems in $\mathbb{R}^3$ it leads to study the entire harmonic functions in $\mathbb{R}^4$. Therefore, it is significant to mention here that the harmonic functions play an important role in theoretical mathematical research, physics and mechanics to describe different stationary processes. Hence, sometime it is reasonable to study generalized growth characteristics of harmonic functions in an n-dimensional space.

Let $x \in \mathbb{R}^n (n \geq 3)$ be an arbitrary point where $x = (x_1, x_2, \ldots, x_n)$ and put $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$. The set of all non-constant entire harmonic

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functions on $\mathbb{R}^n$ is denoted by H. For each $u \in H, r > 0$ we have an expansion into a Fourier-Laplace series [11]

$$u(rx) = \sum_{k=0}^{\infty} Y^{(k)}(x;u)r^k, \tag{1}$$

where $x \in S^n = \{x \in \mathbb{R}^n : |x| = 1\}$ a unit sphere in $\mathbb{R}^n$ centered at the origin and

$$\begin{aligned} Y^{(k)}(x;u) =& a_1^{(k)}Y_1^{(k)}(x) + a_2^{(k)}Y_2^{(k)}(x) + \cdots + a_{\gamma_k}^{(k)}Y_{\gamma_k}^{(k)}(x), \\ a_j^{(k)} =& (u, Y_j^{(k)}) = \frac{\Gamma(\frac{n}{2})}{2(\pi)^{\frac{n}{2}}} \int_{S^n} u(x)Y_j^{(k)}(x)dS, \quad j = \overline{1, \gamma_k}, \\ \gamma_k =& \tfrac{(2k+n-2)(k+n-3)!}{k!(n-2)!}. \end{aligned}$$

Here dS is the element of the surface area on the sphere S^n, $(u, Y_j^{(k)})$ is the scalar product in $L^2(S^n)$ and $Y^{(k)}$ is a spherical harmonic of degree $k, k \in Z_+ = \{0, 1, 2, \ldots,\}$ on the unit sphere $S^n (n \geq 2)$ [10].

Let $B_R = \{y \in \mathbb{R}^n : |y| \leq R\}$ be the ball of radius R in space $\mathbb{R}^n, n \geq 3$ centered at the origin, and $\overline{B_R}$ be the closure of B_R. We denote H_R, the class of harmonic functions in B_R and continuous on $\overline{B_R}, 0 < R < \infty$.

An approximation error of function $u \in H_R$ by harmonic polynomials $P \in \Pi_k$ is defined as

$$E_R^k(u) = \inf_{P \in \Pi_k} \{\max_{y \in \overline{B_R}} |u(y) - P(y)|\}, \tag{2}$$

where Π_k be a set of harmonic polynomials of degree not exceeding k.

We state some results which will be useful in the sequel.

Lemma 1. [15] *If $u \in H_R$, then for all $k \in \mathbb{N}$ inequality*

$$\max_{x \in S^n} |Y^{(k)}(x;u)|R^k \leq \tfrac{4(k+2\nu)^{2\nu}}{(2\nu)!} E_R^{k-1}(u)$$

holds, where $\nu = \frac{n-2}{2}$.

Lemma 2. [15] *For an entire harmonic function $u \in \mathbb{R}^n, n \geq 3$ which is assigned by the series (1), the following inequality hold*

$$\sqrt{\tfrac{(2\nu)!}{2}} \tfrac{1}{(k+2\nu)^{\nu}} \max_{x \in S^n} |Y^{(k)}(x;u)| \leq M(r;u)r^{-k}, \quad \forall k \in \mathbb{Z}_+, r > 0,$$

where $M(r;u) = \max_{x \in S^n} |u(rx)|$.

Lemma 3. [15] *For an entire harmonic function $u \in \mathbb{R}^n, n \geq 3$, the following estimation holds*

$$E_R^k(u) \leq \sqrt{\tfrac{2}{(2\nu)!}}(2\nu + 1)(k + 2\nu)^{2\nu} M(r;u)(\tfrac{R}{r})^k, \quad \forall k \in \mathbb{Z}_+, r > eR.$$

2. GENERALIZED GROWTH PARAMETERS

Let ϕ be a real valued function defined and differentiable on $[a, +\infty)$ at some $a \geq 0$ such that $\phi(x)$ is positive, strictly monotonically increasing and tends to ∞. Then ϕ is said to belong to the class L^o if for every real valued function $\varphi(x)$ such that $\varphi(x) \to 0$ as $x \to \infty, \phi$ satisfies

$$\lim_{x\to\infty} \frac{\phi[(1+\varphi(x))x]}{\phi(x)} = 1 \tag{3}$$

and belongs to the class Δ if for all $c, 0 < c < \infty$, we have the stronger condition

$$\lim_{x\to\infty} \frac{\phi(cx)}{\phi(x)} = 1. \tag{4}$$

Using the functions from the classes L^o, Δ Seremeta [8] obtained the following characterizations for entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$:

Theorem A. *Let $\alpha(x) \in \Delta, \beta(x) \in L^o$. Set $G(x,c) = \beta^{-1}[c\alpha(x)]$. If $\frac{dG(x,c)}{d\log x} = \mathcal{O}(1)$, as $x \to \infty$ for all $c, 0 < c < \infty$, then the generalized order*

$$\rho(\alpha, \beta, f) = \limsup_{r\to\infty} \frac{\alpha(\log M(r;f))}{\beta(\log r)} = \limsup_{r\to\infty} \frac{\alpha(k)}{\beta(\log |a_k|^{-\frac{1}{k}})},$$

where $M(r; f) = \max_{|z|=r} |f(z)|$.

Theorem B. *Let $\alpha(x) \in L^o, \beta^{-1}(x) \in L^o, \varphi(x) \in L^o$. Let $\rho, 0 < \rho < \infty$, be a fixed number. Set $G(x, \sigma, \rho) = \varphi^{-1}\{[\beta^{-1}(\sigma\alpha(x))]^{\frac{1}{\rho}}\}$. Suppose that for all $\sigma,\ 0 < \sigma < \infty$,*

(1) *If $\varphi(x) \in \Delta$ and $\alpha(x) \in \Delta$, then $\frac{dG(x,\sigma,\rho)}{d\log x} = \mathcal{O}(1)$, as $x \to \infty$.*
(2) *If $\varphi(x) \in L^o - \Delta$ or $\alpha(x) \in L^o - \Delta$, then $\lim_{x\to\infty} \frac{dG(x,\sigma,\rho)}{d\log x} = \frac{1}{\rho}$.*

Then we have

$$\limsup_{r\to\infty} \frac{\alpha(\log M(r;f))}{\beta[(\varphi(r))^\rho]} = \limsup_{k\to\infty} \frac{\alpha(\frac{k}{\rho})}{\beta\{[\varphi(e^{\frac{1}{\rho}} |a_k|^{-\frac{1}{k}})]^\rho\}}.$$

S.M. Shah [9] introduced the generalized lower order $\lambda(\alpha, \beta, f)$ as

$$\lambda(\alpha, \beta, f) = \liminf_{r\to\infty} \frac{\alpha(\log M(r;f))}{\beta(\log r)}$$

and proved the following theorem:

Theorem C. *Let $f(z) = \Sigma_{k=0}^{\infty} a_k z^k$ be an entire function. Set $G(x) = \beta^{-1}(\alpha(x))$. For some function $\xi(x) \to \infty$ $\frac{\beta(x\xi(x))}{\beta(e^x)} \to 0$, $\frac{dG(x)}{d(\log x)} = \mathcal{O}(1)$, as $x \to \infty$,*

$|\frac{a_k}{a_{k+1}}|$ is ultimately a non decreasing function of k. Then

$$\lambda(\alpha, \beta, f) = \liminf_{k\to\infty} \frac{\alpha(k)}{\beta(\log |a_k|^{-\frac{1}{k}})}.$$

It has been observed that the functions $G(x,c)$ and $G(x)$ of Theorem A and Theorem C respectively, do not satisfy the conditions when $\alpha(x)=\beta(x)$ i.e., when the entire function $f(z)$ is of slow growth. To include these functions, Kapoor and Nautiyal [5] introduced a new class of functions and defined generalized order and generalized type as follows:

Let $\overline{\Omega}$ be the class of functions $\phi(x)$ satisfying the following conditions:

(H,i) $\phi(x)$ is defined on $[a,\infty)$ such that ϕ is positive, strictly increasing, differentiable and tends to ∞ as $x\to\infty$.

(H,ii)

$$\lim_{x\to\infty}\frac{d(\phi(x))}{d(\log x)}=K,\quad 0<K<\infty.$$

The generalized order $\rho(\alpha,\alpha,f)$, generalized lower order $\lambda(\alpha,\alpha,f)$ and generalized type of the entire function $f(z)$ were defined as

$$\begin{aligned}\rho(\alpha,\alpha,f)&=\limsup_{r\to\infty}\frac{\alpha(\log M(r;f))}{\alpha(\log r)},\\ \lambda(\alpha,\alpha,f)&=\liminf_{r\to\infty}\frac{\alpha(\log M(r;f))}{\alpha(\log r)},\quad 1\le\lambda(\alpha,\alpha,f)\le\rho(\alpha,\alpha,f)\le\infty,\\ T(\alpha,\alpha,f)&=\limsup_{r\to\infty}\frac{\alpha(\log M(r;f))}{[\alpha(\log r)]^{\rho}}.\end{aligned}$$

where $\alpha(x)\in\overline{\Omega}$.

N. Juhong and C. Qing [4] extended the range of $\alpha(x)$ by defining a new class Ω^* as the extension of $\overline{\Omega}$ and obtained some results concerning above generalized growth parameters of entire function $f(z)$.

Let Ω^* be the class of functions $\phi(x)$ satisfying the condition (H,i) and (H,iii),

(H,iii)

$$\lim_{x\to\infty}\frac{d(\phi(x))}{d(\log^{[p]}x)}=K,\quad 0<K<\infty, p\ge 1, p\in\mathbb{N}^+,$$

where $\log^{[0]}x=x, \log^{[1]}x=\log x, \log^{[p]}x=\log^{[p-1]}\log x$. Also $\phi(x)$ satisfies (3) and (4).

It is clear that $\alpha(x)\in\overline{\Omega}$ is a particular case of $\alpha(x)\in\Omega^*$ for $p=1$.

Theorem D. [4] *Let $\alpha(x)\in\Omega^*$, then some necessary and sufficient conditions of the entire function $f(z)$ with generalized order ρ is*

$$\limsup_{r\to\infty}\frac{\alpha(\log M(r;f))}{\alpha(\log r)}-1=\limsup_{k\to\infty}\frac{\alpha(k)}{\alpha(\log|a_k|^{-\frac{1}{k}})},\quad p=1,$$

$$\limsup_{k\to\infty}\frac{\alpha(k)}{\alpha(\log|a_k|^{-\frac{1}{k}})}\le\limsup_{r\to\infty}\frac{\alpha(\log M(r;f))}{\alpha(\log r)}\le\limsup_{k\to\infty}\frac{\alpha(k)}{\alpha(\log|a_k|^{-\frac{1}{k}})}+1,\ p=2,3,\dots$$

THEOREM E. [4]. *Let $\alpha(x) \in \Omega^*$, then the entire function $f(z)$ of generalized order $\rho, 1 < \rho < \infty$, is of generalized type T if, and only if*

$$\limsup_{r\to\infty} \frac{\alpha(\log M(r;f))}{[\alpha(\log r)]^\rho} = \limsup_{k\to\infty} \frac{\alpha(k)}{[\alpha(\log |a_k|^{-\frac{1}{k}})]^{\rho-1}}, \quad p = 1,$$
$$\limsup_{r\to\infty} \frac{\alpha(\log M(r;f))}{[\alpha(\log r)]^\rho} = \limsup_{k\to\infty} \frac{\alpha(k)}{[\alpha(\log |a_k|^{-\frac{1}{k}})]^{\rho}}, \quad p = 2, 3, \ldots$$

3. MAIN RESULTS

In this section we shall characterize the generalized growth parameters of entire harmonic functions in space $\mathbb{R}^n, n \geq 3$ in terms of harmonic polynomial approximation error defined by (2).

Let $u \in H_R$. Then the generalized order $\rho_R(\alpha, \beta, u)$ and generalized lower order $\lambda_R(\alpha, \beta, u)$ of u is defined as

$$(5) \quad \begin{aligned} \rho_R(\alpha,\beta,u) &= \limsup_{r\to\infty} \frac{\alpha(\log M(r;u))}{\alpha(\log r)}, \\ \lambda_R(\alpha,\beta,u) &= \liminf_{r\to\infty} \frac{\alpha(\log M(r;u))}{\alpha(\log r)}, \quad 1 \leq \lambda_R(\alpha,\beta,u) \leq \rho_R(\alpha,\beta,u) \leq \infty. \end{aligned}$$

Further, for $0 < \rho_R < \infty$, we define the generalized type $T_R(\alpha, \beta, u)$ of u as

$$(6) \qquad T(\alpha,\beta,u) = \limsup_{r\to\infty} \frac{\alpha(\log M(r;u))}{\beta((\varphi(r))^{\rho_R})}.$$

The functions α, β and φ satisfy the conditions stated in Theorem B.

Now we prove our main results.

THEOREM 4. *$\alpha(x) \in \Delta, \beta(x) \in L^o$. Set $G(x,c) = \beta^{-1}(c\alpha(x))$. If $\frac{dG(x,c)}{d(\log x)} = \mathcal{O}(1)$, as $x \to \infty$ for all $c, c \in (0,\infty)$, then for $u \in H_R$ can be continued to the entire harmonic function in space $\mathbb{R}^n, n \geq 3$, for which generalized order*

$$(7) \qquad \rho_R(\alpha,\beta,u) = \limsup_{k\to\infty} \frac{\alpha(k)}{\beta(\log R[E_R^k(u)]^{-\frac{1}{k}})}.$$

Proof. Consider the complex valued functions

$$f_1(z) = \sum_{k=0}^{\infty} \frac{\sqrt{(2\nu)!}}{\sqrt{2}(2\nu+1)!(k+2\nu)^{2\nu}} E_R^k(u)(\tfrac{z}{R})^k = \sum_{k=0}^{\infty} b_k z^k (say),$$
$$g(z) = \sum_{k=1}^{\infty} \tfrac{4}{(2\nu)!}(k+2\nu)^{2\nu} E_R^k(u)(\tfrac{z}{R})^k = \sum_{k=1}^{\infty} c_k z^k (say).$$

From Lemma 3 we have

$$(8) \qquad \lim_{k\to\infty} (b_k)^{-\frac{1}{k}} = \infty \;\; and \;\; \lim_{k\to\infty} (c_k)^{-\frac{1}{k}} = \infty,$$

therefore, $f_1(z)$ and $g(z)$ represents entire functions of complex variable z. Further, we have

$$\log(b_k)^{-1} \simeq \log R^k [E_R^k(u)]^{-1} \;\; as \;\; k \to \infty.$$

Using Theorem A, we get

$$\limsup_{r\to\infty} \frac{\alpha(\log M(r;f_1))}{\beta(\log r)} = \limsup_{k\to\infty} \frac{\alpha(k)}{\beta(\log R[E_R^k(u)]^{-\frac{1}{k}})}. \tag{9}$$

If $\mu(r;f_1)$ denotes the maximum term of $f_1(z)$ then by a result of Valiron [12, p. 34], we get $\log M(r;f_1) \simeq \log \mu(r;f_1)$ as $r \to \infty$. From Lemma 1 we have

$$M(r;u) \leq |Y^o(\xi;u)| + \tfrac{4}{(2\nu)!} \sum_{k=1}^{\infty} (k+2\nu)^{2\nu} E_R^{k-1}(u) (\tfrac{r}{R})^k$$

or

$$M(r;u) \leq |Y^o(\xi;u)| + M(r;g). \tag{10}$$

Now from Lemma 3 and (10) we have

$$\mu(r;f_1) \leq M(r;u) \leq |Y^o(\xi;u)| + M(r;g). \tag{11}$$

Thus (10) gives

$$\limsup_{r\to\infty} \frac{\alpha(\log M(r;f_1))}{\beta(\log r)} \leq \limsup_{r\to\infty} \frac{\alpha(\log M(r;u))}{\beta(\log r)} \leq \limsup_{r\to\infty} \frac{\alpha(\log M(r;g))}{\beta(\log r)}. \tag{12}$$

Now using (5), (9) and (12) we get (7). □

Theorem 5. *Let $u \in H_R$ be of generalized order $\rho_R(\alpha,\beta,u), 0 < \rho_R(\alpha,\beta,u) < \infty$. Let the functions α,β and φ satisfy the conditions of Theorem B, then the function u can be continued to the entire harmonic function in space $\mathbb{R}^n, n \geq 3$, for which generalized type*

$$T(\alpha,\beta,u) = \limsup_{r\to\infty} \frac{\alpha(\log M(r;u))}{\beta((\varphi(r))^{\rho_R})} = \limsup_{k\to\infty} \frac{\alpha(\frac{k}{\rho_R})}{\beta[\{\varphi(e^{\frac{1}{\rho_R}} R[E_R^k(u)]^{-\frac{1}{k}})\}^{\rho_R}]}. \tag{13}$$

Proof. Since $\alpha \in L^o$, we have from Theorem 4

$$\alpha(\log M(r;u)) \simeq \alpha(\log M(r;f_1)) \tag{14}$$

and $f_1(z)$ is also an entire function of generalized order $\rho_R(\alpha,\beta,f_1)$, now using Theorem B, we have

$$\limsup_{r\to\infty} \frac{\alpha(\log M(r;f_1))}{\beta((\varphi(r))^{\rho_R})} = \limsup_{k\to\infty} \frac{\alpha(\frac{k}{\rho_R})}{\beta[\{\varphi(e^{\frac{1}{\rho_R}} |b_k|^{-\frac{1}{k}})\}^{\rho_R}]}. \tag{15}$$

Since $\varphi(x) \in L^o$, we have

$$\limsup_{k\to\infty} \frac{\alpha(\frac{k}{\rho_R})}{\beta[\{\varphi(e^{\frac{1}{\rho_R}} |b_k|^{-\frac{1}{k}})\}^{\rho_R}]} = \limsup_{k\to\infty} \frac{\alpha(\frac{k}{\rho_R})}{\beta[\{\varphi(e^{\frac{1}{\rho_R}} R[E_R^k(u)]^{-\frac{1}{k}})\}^{\rho_R}]}. \tag{16}$$

The above relations (14), (15) and (16) with (6) together gives (13). □

Theorem 6. *Let $u \in H_R$. Set $G(x) = \beta^{-1}(\alpha(x))$. Then the function u can be continued to the entire harmonic function in space $\mathbb{R}^n, n \geq 3$, for which generalized lower order*

$$\lambda_R(\alpha,\beta,u) = \liminf_{r\to\infty} \frac{\alpha(\log M(r;u))}{\beta(\log r)} \geq \liminf_{k\to\infty} \frac{\alpha(k)}{\beta(\log R[E_R^k(u)]^{-\frac{1}{k}})}. \tag{17}$$

if the following condition satisfied:

For some function $\xi(x) \to \infty$ *as* $x \to \infty, \frac{\beta(x\xi(x))}{\beta(e^x)} \to 0, \frac{dG(x)}{d(\log x)} = \mathcal{O}(1)$, *as* $x \to \infty$,

If, in addition, ratio $|\frac{E_R^k(u)}{E_R^{k+1}(u)}|$ *is a non decreasing function of* k *then inequality in the right hand side of* (17) *transforms into the equality.*

Proof. We see that $f_1(z)$ is also of generalized lower order $\lambda_R(\alpha, \beta, f_1)$ and

$$\frac{b_k}{b_{k+1}} = [\frac{(k+1+2\nu)}{(k+2\nu)}]^{2\nu} \frac{E_R^k(u)}{E_R^{k+1}(u)}(\frac{R}{z}), \quad r > eR$$

is a non decreasing function of k. Now using (8) and applying Theorem C to the function $f_1(z)$, we obtain

$$\lambda_R(\alpha, \beta, f_1) = \liminf_{k\to\infty} \frac{\alpha(k)}{\beta(\log|b_k|^{-\frac{1}{k}})} = \liminf_{k\to\infty} \frac{\alpha(k)}{\beta(\log R[E_R^k(u)]^{-\frac{1}{k}})}.$$

□

Remark 7. If $\alpha(x) = \log x, \beta(x) = x$ in Theorem 4 and $\alpha(x) = \beta(x) = \varphi(x) = x$ in Theorem 5, we get the coefficient characterizations for the classical order and type of entire harmonic function u in space $\mathbb{R}^n, n \geq 3$, in terms of harmonic polynomial approximation errors.

Remark 8. If $\alpha(x) = x, \beta(x) = x^{\frac{1}{\rho_R}}, \varphi(x) = x^{\rho_R(x)}$ in Theorem 5, where $\rho_R(x)$ is the proximate order of harmonic function u such that $x = \theta(\tau)$ is the function inverse to $\tau = x^{\rho_R(x)}$, we obtain the formula for the generalized type with respect to proximate order $\rho_R(x)$.

Theorem 9. *Let* $\alpha(x) \in \Omega^*$, *then necessary and sufficient conditions for* $u \in H_R$ *to be continued to the entire harmonic function in space* $\mathbb{R}^n, n \geq 3$ *with generalized order* $\rho_R(\alpha, \alpha, u)$ *is*

$$\limsup_{r\to\infty} \frac{\alpha(\log M(r;u))}{\alpha(\log r)} - 1 = \limsup_{k\to\infty} \frac{\alpha(k)}{\alpha(\log R[E_R^k(u)]^{-\frac{1}{k}})}, for \quad p = 1,$$

$$\limsup_{k\to\infty} \frac{\alpha(k)}{\alpha(\log R[E_R^k(u)]^{-\frac{1}{k}})} \leq \limsup_{r\to\infty} \frac{\alpha(\log M(r;u))}{\alpha(\log r)}$$

$$\leq \limsup_{k\to\infty} \frac{\alpha(k)}{\alpha(\log R[E_R^k(u)]^{-\frac{1}{k}})} + 1, for \quad p = 2, 3, \ldots.$$

Proof follows on the lines of Theorem 4 and using Theorem D.

Theorem 10. *Let* $\alpha(x) \in \Omega^*$, *then the function* $u \in H_R$ *can be continued to the entire harmonic function in space* $\mathbb{R}^n, n \geq 3$, *with generalized order* $\rho_R(\alpha, \alpha, u), 1 < \rho_R(\alpha, \alpha, u) < \infty$, *is of generalized type* $T_R(\alpha, \alpha, u)$ *if, and only if*

$$\limsup_{r\to\infty} \frac{\alpha(\log M(r;u))}{[\alpha(\log r)]^{\rho_R}} = \limsup_{k\to\infty} \frac{\alpha(k)}{[\alpha(\log R[E_R^k(u)]^{-\frac{1}{k}})]^{\rho_R - 1}}, \quad p = 1,$$

$$\limsup_{r\to\infty} \frac{\alpha(\log M(r;u))}{[\alpha(\log r)]^{\rho_R}} = \limsup_{k\to\infty} \frac{\alpha(k)}{[\alpha(\log R[E_R^k(u)]^{-\frac{1}{k}})]^{\rho_R}}, \quad p = 2, 3, \ldots$$

Proof follows on the lines of Theorem 5 and using Theorem E.

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