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EXTENDED CONVERGENCE OF TWO-STEP ITERATIVE METHODS FOR SOLVING EQUATIONS WITH APPLICATIONS

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Abstract. The convergence of two-step iterative methods of third and fourth order of convergence are studied under weaker hypotheses than in earlier works using our new idea of the restricted convergence region. This way, we obtain a finer semilocal and local convergence analysis, and under the same or weaker hypotheses. Hence, we extend the applicability of these methods in cases not covered before. Numerical examples are used to compare our results favorably to earlier ones.

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1. INTRODUCTION

Let B_1,B_2 stand for Banach spaces and $\Omega \subseteq B_1$ be a nonempty, convex and open set. By $LB(B_1, B_2)$ we denote the space of bounded linear operators from B_1 to B_2.

There is a plethora of problems in various disciplines that can be written using mathematical modeling like

$$F(x) = 0 \tag{1}$$

where $F : \Omega \to B_2$ is differentiable in the sense of Fréchet. Therefore finding a solution x_* of equation (1) is of great importance and challenge [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. One wishes x_* to be found in closed form but this is done only in special cases. This is why, we resort to iterative methods approximating x_*. Numerous studies have been published on the local as well as the semilocal convergence of iterative methods. Among these methods is the single step Newton's method defined by

$$z_0 \in \Omega, z_{n+1} = z_n - F'(z_{n-1})F(z_n) \tag{2}$$

for each $n = 0, 1, 2, \ldots$, which is considered the most popular.

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Iterative methods converge under certain hypotheses. However, their convergence region is small in general. Finding a more precise than Ω set D containing the iterates is very important, since the Lipschitz constants in D will be at least as tight as in Ω. This will in turn lead to a finer convergence analysis of these methods. We pursue this goal in the present study by studying the two-step fourth convergence order Newton's method defined as

$$(3)\qquad \begin{aligned} x_0 \in \Omega, y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - F'(y_n)^{-1}F(y_n) \end{aligned}$$

as well as the two-step third order Traub method [20]

$$(4)\qquad \begin{aligned} \bar{x_0} \in \Omega, \bar{y_n} &= \bar{x_n} - F'(\bar{x_n})^{-1}F(\bar{x_n}) \\ \bar{x_{n+1}} &= \bar{y_n} - F'(\bar{x_n})^{-1}F(\bar{y_n}) \end{aligned}$$

The local as well as the semilocal convergence of method (3) and (4) is carried out under the same set of hypotheses.

The layout of the rest of the article involves the semilocal and local convergence of these methods in Section 2 and Section 3, respectively. Numerical examples are given in Section 4.

2. SEMILOCAL CONVERGENCE ANALYSIS

We present first the semilocal convergence analysis for method (3). First, we need to show an auxiliary result on majorizing sequences for method (3). The proof is an extension of the corresponding one given by us in [8].

Let $l > 0, l_0 > 0$ and $\eta > 0$ be given parameters. Define parameters γ, s_0 and t_1 by

$$(5)\qquad \gamma = \frac{2l}{l+\sqrt{l^2+8l_0l}}, s_0 = \eta, \text{ and } t_1 = s_0[1 + \frac{l_0 s_0}{2(1-l_0s_0)}], \text{ for } l_0s_0 \neq 1,$$

and the scalar sequence t_n for each n=1,2,... by

$$\begin{aligned} t_0 = 0, t_1 &= s_0 + \frac{ls_0^2}{2(1-l_0s_0)}, \\ s_n &= t_n + \frac{l(t_n - s_{n-1})^2}{2(1-l_0t_n)}, \\ t_{n+1} &= s_n + \frac{l(s_n-t_n)^2}{2(1-l_0s_n)}. \end{aligned} \qquad (6)$$

Lemma 1. *Let $> 0, l_0 > 0$ and $\eta > 0$ be given parameters. Suppose that*

$$(7)\qquad 0 < \max\{\frac{l(s_0-t_0)}{2(1-l_0s_0)}, \frac{l(t_1-s_0)}{2(1-l_0t_1)}\} \leq \gamma < 1 - l_0s_0.$$

Then, the sequence t_n is nondecreasing, bounded from above by $t_{**} = \frac{\eta}{1-\gamma}$, and converges to its unique least upper bound t_* satisfying $\eta \leq t_* \leq t_{**}$. Moreover, the following items hold for each $n = 1, 2, \ldots$

$$(8)\qquad 0 < t_{n+1} - s_n \leq \gamma(s_n - t_n) \leq \gamma^{2n+1}\eta,$$

$$(9)\qquad 0 < s_n - t_n \leq \gamma(t_n - s_{n-1}) \leq \gamma^{2n}\eta$$

and

$$t_n \leq s_n \leq t_{n+1}. \tag{10}$$

Proof. Estimations (8)–(10) hold true, if

$$0 < \frac{l(s_m - t_m)}{2(1 - l_0 s_m)} \leq \gamma, \tag{11}$$

$$0 < \frac{l(t_{m+1} - s_m)}{2(1 - l_0 t_{m+1})} \leq \gamma \tag{12}$$

and

$$0 < t_m \leq s_m \leq t_{m+1}. \tag{13}$$

Estimations (11)–(13) hold true for $m = 0$, by (5)–(7). Suppose (11)–(13) hold true for $m = 1, 2, \ldots, n$.

By (8) and (9), we get

$$\begin{aligned} s_m &\leq t_m + \gamma^{2m}\eta \leq s_{m-1} + \gamma^{2m-1}\eta + \gamma^{2m}\eta \\ &\leq \eta + \ldots \gamma^{2m}\eta = \frac{1-\gamma^{2m+1}}{1-\gamma}\eta < \frac{\eta}{1-\gamma} = t_{**} \end{aligned} \tag{14}$$

and

$$\begin{aligned} t_{m+1} &\leq t_m + \gamma^{2m+1}\eta \leq t_m + \gamma^{2m}\eta + \gamma^{2m+1}\eta \\ &\leq \eta + \ldots + \gamma^{2m+1}\eta = \frac{1-\gamma^{2m+2}}{1-\gamma}\eta < \frac{\eta}{1-\gamma} = t_{**}. \end{aligned} \tag{15}$$

Then, (11) shall hold, if

$$\frac{l\gamma^{2m}\eta}{2(1 - l_0(\frac{1-\gamma^{2m+1}}{1-\gamma})\eta)} \leq \gamma \tag{16}$$

or

$$\frac{l}{2}\gamma^{2m}\eta + l_0\gamma\frac{1-\gamma^{2m+1}}{1-\gamma}\eta \leq \gamma \tag{17}$$

or

$$\frac{l}{2}\gamma^{2m-1}\eta + l_0(1 + \gamma + \ldots + \gamma^{2m})\eta - 1 \leq 0. \tag{18}$$

We are motivated by (18) to define recurrent polynomials f_m defined on the interval $[0, 1)$ by

$$f_m(t) = \frac{l}{2}t^{2m-1}\eta + l_0(1 + t + \ldots + t^{2m})\eta - 1, \tag{19}$$

which satisfies

$$f_{m+1}(t) = f_m(t) + p(t)t^{2m-1}\eta, \tag{20}$$

where, polynomial p is given by

$$p(t) = l_0 t^3 + (\tfrac{l}{2} + l_0)t^2 - 1 = (t + 1)(l_0 t^2 + \tfrac{l}{2}t - 1), \tag{21}$$

so $p(\gamma) = 0$.

Notice, in particular from (20) that

$$f_{m+1}(\gamma) = f_m(\gamma) = \lim_{m\to\infty} f_m(\gamma) = f_\infty(\gamma). \tag{22}$$

Evidently by (21), (18) is true, if

$$f_\infty(\gamma) \le 0 \tag{23}$$

But

$$f_\infty(\gamma) = \frac{l_0\eta}{1-\gamma} - 1 \tag{24}$$

Then, by (19) and (24), we see that (23) is satisfied.
Similarly, to show (12), we must have

$$\frac{l\gamma^{2m+1}\eta}{2(1-l_0\frac{1-\gamma^{2m+2}}{1-\gamma}\eta)} \le \gamma \tag{25}$$

or

$$\frac{l}{2}\gamma^{2m}\eta + l_0(1+\gamma+\ldots+\gamma^{2m+1})\eta - 1 \le 0 \tag{26}$$

leading to the definition of recurrent functions g_m defined on the interval $[0, 1)$ by

$$g_m(t) = \frac{l}{2}t^{2m}\eta + l_0(1+t+\ldots+t^{2m+1})\eta - 1,$$

which satisfies

$$g_{m+1}(t) = g_m(t) + p(t)t^{2m}\eta,$$

so again

$$g_{m+1}(\gamma) = g_m(\gamma) == \lim_{m\to\infty} g_m(\gamma) = g_\infty(\gamma)$$

Item (26) holds, if

$$g_\infty(\gamma) \le 0. \tag{27}$$

But

$$g_\infty(\gamma) = \frac{l_0\eta}{1-\gamma} - 1, \tag{28}$$

so (27) holds true by (7) and (28). Item (13) also holds true by (6),(11) and (12). Hence, the induction for (11)–(13) is completed, and items (8)–(10) hold for each $n = 1, 2, \ldots$. It follows from (13)-(15) that sequence t_n is nondecreasing, bounded from above by t_{**} and as such it converges to t_*. □

REMARK 2. *It is worth noticing that if*

$$l_0\eta \le 1/2$$

then

$$\frac{l(t_1-s_0)}{2(1-l_0t_1)} \le \frac{l(s_0-t_0)}{2(1-l_0s_0)}$$

That is (7) *holds, if*

$$0 < \frac{l\eta}{2(1-l_0\eta)} \le \gamma < 1 - l_0\eta$$

or equivalently, if

$$h_A = l_1\eta \le \frac{1}{2}, \tag{29}$$

where

$$l_1 = \frac{1}{8}(l + 4l_0 + \sqrt{l^2 + 8l_0l}) \tag{30}$$

Hence, (29) *and* (30) *can replace* (7) *in Lemma 1 and in what follows from now on. The sufficient convergence criterion* (29) *is similar to the corresponding one for Newton's method given by us in* [10]*, if l replaces l_1. But $l \le l_1$, where l_1 is the Lipschitz constant on Ω. Hence, the sufficient convergence criteria for Newton's method in* [10] *are also improved.*

Let $S(x,a)$ stand for the open ball in B_1 with center $x \in B_1$ and of radius $a > 0$. By $\bar{S}(x,a)$, we denote the closure of $S(x,a)$.

The semilocal convergence of method (3) uses the conditions (A):

(a1) $F : \Omega \subset B_1 \to B_2$ is a continuously differentiable operator in the sense of, Fréchet, and there exists $x_0 \in \Omega$ such that $F'(x_0)^{-1} \in LB(B_2, B_1)$ with $\|F'(x_0)^{-1}F(x_0)\| = \eta$

(a2) There exists $l_0 > 0$ such that for each $x \in \Omega$

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le l_0\|x - x_0\|.$$

Define $S_0 = \Omega \cap S(x_1, \frac{1}{l_0} - \eta)$, where $x_1 = x_0 - F'(x_0)^{-1}F(x_0)$, and $l_0\eta < 1$ by (29) and (30).

(a3) There exists $l > 0$ such that for each $x, y \in S_0$

$$\|F'(x_0)^{-1}(F'(y) - F'(x))\| \le l\|y - x\|.$$

(a4) Hypotheses of Lemma 1 hold with (7) replaced by (29) and (30).

(a5) $\bar{S}(x_0, t_*) \subset \Omega$, where t_* is given in Lemma 1.

(a6) There exists $t_*^1 \ge t_*$ such that $l_0(t_* + t_*^1) < 2$.

Set $S_1 = \Omega \cap \bar{S}(x_0, t_*^1)$. Next, we can show the semilocal convergence result for methos (3)

THEOREM 3. *Assume that the conditions* (A) *hold. Then, $x_n \subset S(y_0, \frac{1}{l_0} - \eta), n = 1, 2\ldots$ and converges to some x_* which is the only solution of equation $F(x) = 0$ in the set S_1.*

Proof. We must prove using mathematical induction that

$$\|x_{m+1} - y_m\| \le t_{m+1} - s_m \tag{31}$$

and

$$\|y_m - x_m\| \le s_m - t_m \tag{32}$$

By (a1) and (29), we have

$$\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \le \eta \le \tfrac{1}{l_0} - \eta,$$

so $x_0 \in S(y_0, \frac{1}{l_0} - \eta)$, and (32) holds for $\eta = 0$. By method (3) for $\eta = 0$, (a1),(6) and (a2) we have in turn that

$$\begin{aligned}
\|x_1 - y_0\| &= \|F'(x_0)^{-1}[F(y_0) - F(x_0) - F'(x_0)(y_0 - x_0)]\| \\
&= \Big\| \int_0^1 F'(x_0)^{-1}(F'(x_0 + \tau(y_0 - x_0)) - F'(x_0))d\tau(y_0 - x_0)\Big\| \\
&\leq \tfrac{l_0}{2}\|y_0 - x_0\|^2 \\
&\leq \tfrac{l_0}{2}(s_0 - t_0)^2 \\
&= t_1 - s_0 < \tfrac{1}{l_0} - \eta
\end{aligned}$$

so $x_1 \in \bar{S}(y_0, \frac{1}{l_0} - \eta)$.

Then, by (a2), we have

$$\|F'(x_0)^{-1}(F'(x_1) - F'(x_0)\| \leq l_0\|x_1 - x_0\| \leq l_0 t_1 < 1 \tag{33}$$

so by (33) and the Banach lemma on invertible operators [17] $F'(x_1)^{-1} \in LB(B_2, B_1)$, and

$$\|F'(x_1)^{-1}F'(x_0)\| \leq \tfrac{1}{1 - l_0\|x_1 - x_0\|}. \tag{34}$$

In view of method (3) and (a3), we get

$$\begin{aligned}
\|F'(x_0)^{-1}F(x_1)\| &= \|F'(x_0)^{-1}[F(x_1) - F(y_0) - F'(y_0)(x_1 - y_0)]\| \\
&= \| \int_0^1 F'(x_0)^{-1}(F'(y_0 + \tau(x_1 - y_0) - F'(y_0)))\|d\tau\|x_1 - y_0\| \\
&\leq \tfrac{l_0}{2}\|x_1 - y_0\|^2 \leq \tfrac{l}{2}(t_1 - s_0)^2
\end{aligned}$$

so

$$\begin{aligned}
\|y_1 - x_1\| &= \|[F'(x_1)^{-1}F'(x_0)][F'(x_0)^{-1}F(x_1)]\| \\
&\leq \|[F'(x_1)^{-1}F'(x_0)]\|\|[F'(x_0)^{-1}F(x_1)]\| \\
&\leq \tfrac{l(t_1 - s_0)^2}{2(1 - l_0 t_1)} = s_1 - t_1,
\end{aligned}$$

and

$$\|y_1 - y_0\| \leq \|y_1 - x_1\| + \|x_1 - y_0\| \leq s_1 - t_1 + t_1 - s_0 = s_1 - s_0 < \tfrac{1}{l_0} - \eta,$$

so (32) holds for $m = 1$ and $y_1 \in \bar{S}(y_0, \frac{1}{l_0} - \eta)$.

Using method (3) as above, we have

$$\begin{aligned}
\|x_2 - y_1\| &\leq \|F'(y_0)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F'(y_1)\| \\
&\leq \tfrac{l\|y_1 - x_1\|^2}{2(1 - l_0\|y_1 - x_0\|)} \\
&\leq \tfrac{l(s_1 - t_1)^2}{2(1 - l_0 s_1)} = t_2 - s_1,
\end{aligned}$$

and

$$\|x_2 - y_0\| \leq \|x_2 - y_1\| + \|y_1 - y_0\| \leq t_2 - s_1 + s_1 - s_0 = t_2 - s_0 < \tfrac{1}{l_0} - \eta,$$

so (31) holds for $m=1$ and $x_2 \in \bar{S}(y_0, \frac{1}{l_0}-\eta)$.

Then, replace x_1, y_1, x_2 by x_m, y_m, x_{m+1} to complete the induction for (31) and (32). Moreover, we have

(35)
$$\|x_{m+1}-x_m\| \leq \|x_{m+1}-y_m\|+\|y_m-x_m\| \leq t_{m+1}-s_m+s_m-t_m = t_{m+1}-t_m,$$

and

(36)
$$\|x_m-y_0\| \leq \|x_m-y_{m-1}\|+\|y_{m-1}-y_0\| \leq t_m-s_{m-1}+s_{m-1}-s_0 \leq t_m-s_0 < \frac{1}{l_0}-\eta.$$

It then follows from (35), (36) and Lemma 1 that sequence t_m is complete in B_1, so there exists $x_* \in \bar{S}(y_0, \frac{1}{l_0}-\eta)$ with $\lim_{m\to\infty} x_m = x_*$.

Furthermore, by the second substep of method (3), we have

$$\|F'(x_0)^{-1}F(x_{m+1})\| \leq \frac{l}{2}\|x_{m+1}-y_m\| \leq \frac{l}{2}(t_{m+1}+s_m)^2,$$

so $F(x_*)=0$, by the continuity of F and Lemma 1. The uniqueness of the solution part is given in [10]. □

Next, we study the semilocal convergence of method (4) in an analogous way.

REMARK 4. *Let p_1 be a cubic polynomial defined by*

$$p_1(t) = l_0 t_3 + \frac{l}{2}t_2 + \frac{l}{2}t - l$$

We have $p_1(0) = -l < 0$ and $p_1(1) = l_0 > 0$.

It follows by the intermediate value theorem that P_1 has at least one root in $(0,1)$.

But $p_1'(t) = 3l_0 t_2 + lt + \frac{l}{2} > 0$, so p_1 increasing, so $p_1(t)=0$ has a unique root in $(0,1)$. Denote by δ this root.

The following estimate is needed:

(37)
$$0 < \frac{l\eta}{2(1-l_0\eta)} \leq \frac{l\eta(2+\frac{l}{2}\eta)}{2(1-l_0\eta(1+\frac{l_0}{2}))}$$

Evidently, (37) holds, if

$$0 \leq 1 + (\frac{l}{2}-l_0)\eta + \frac{l_0}{2}\eta^2(l_0+l)$$

Estimations (37) is true, if $l \geq 2l_0$. Moreover, if $l < 2l_0$, then $\frac{l_0}{2}(l_0+l)t^2 + (\frac{l}{2}-l_0)t+1=0$, since it has two negative roots by the Descarte's rule of signs, so (33) holds in this case too.

We need an auxiliary result on majorizing sequences for method (4) similar to Lemma 1.

Define sequence $\bar{t}_n$ for each $n=1,2,\ldots$ by $\bar{t}_0=0, \bar{t}_1=\eta(1+l_0\eta)$

$$\bar{s}_n = \bar{t}_n + \frac{l(\bar{t}_n+\bar{s}_{n-1}-2\bar{t}_{n-1})(\bar{t}_n-\bar{s}_{n-1})}{1-l_0 t_n}$$
$$\bar{t}_{n+1} = \bar{s}_n + \frac{l(\bar{s}_n-\bar{t}_n)}{2(1-l_0\bar{t}_n)}(\bar{s}_n-\bar{t}_n)$$

Lemma 5. *Let $l_0 > 0, l > 0$ and $\eta > 0$ be positive parameters. Assume that*

(38) $$0 < \frac{l\eta(2+\frac{l}{2}\eta)}{2(1-l_0\eta(1+\frac{l_0}{2}\eta))} \leq \delta \leq 1 - l_0\eta.$$

Then, the conclusions of Lemma 1 hold with sequence $\bar{t}_n$ defined by (38) replacing sequence t_n given by (6).

Proof. As in Lemma 1, we must show

$$0 < \frac{l(\bar{s}_m - \bar{t}_m)}{2(1-l_0\bar{t}_m)} \leq \delta,$$

(39) $$0 < \frac{l(\bar{t}_{m+1}+\bar{s}_m-2\bar{t}_m)}{2(1-l_0\bar{t}_{m+1})} \leq \delta$$

and

$$0 \leq \bar{t}_m \leq \bar{s}_m \leq \bar{s}_{m+1}.$$

But

$$0 < \frac{l(\bar{s}_m - \bar{t}_m)}{2(1-l_0\bar{t}_m)} \leq \frac{l(\bar{t}_{m+1}+\bar{s}_m-2\bar{t}_m)}{2(1-l_0\bar{t}_{m+1})}$$

and

$$0 < \frac{1}{1-l_0\bar{t}_1} \leq \frac{1}{1-l_0\bar{t}_m}.$$

Notice that

$$0 < \frac{l(\bar{s}_m - \bar{t}_m)}{2(1-l_0\bar{t}_m)} \leq \frac{l(\bar{t}_{m+1}+\bar{s}_m-2\bar{t}_m)}{2(1-l_0\bar{t}_{m+1})}$$

and

$$0 < \frac{1}{1-l_0\bar{t}_1} \leq \frac{1}{1-l_0\bar{t}_m}$$

so it suffices to show only (39), or

$$\frac{l(\frac{1-\delta^{2m+2}}{1-\delta} - \frac{1-\delta^{2m}}{1-\delta})\eta}{1-l_0(1+\ldots+\delta^{2m+1})\eta} \leq \delta$$

or

$$f_m(t) = \frac{l}{2}t^{2m-1}(t+2)\eta + l_0(1+\ldots+t^{2m+1})\eta - 1 \leq 0$$

But

$$f_{m+1}(t) = f_m(t) + p_1(t)t^{2m-1}(t+1)\eta,$$

so

$$f_{m+1}(\delta) = f_m(\delta) = \lim_{m\to\infty} f_m(\delta) = f_\infty(\delta) = \frac{l\eta}{1-\delta} - 1$$

The rest follows as in Lemma 1 with $\bar{t}_* = \lim_{n\to\infty} \bar{t}_n$ and $\bar{t}_{**} = \frac{\eta}{1-\delta}$.

Replace $x_m, y_m, t_*, t_{**}, t_*^1, t_n, s_n$ by $\bar{x}_m, \bar{y}_m, \bar{t}_*, \bar{t}_{**}, \bar{t}_*^1, \bar{t}_n, \bar{s}_n$, hypotheses of Lemma 1, method (3) by Lemma 5, method (4) respectively. Call the resulting hypotheses (A)'. Then in an analogous to Theorem 3 way, we arrive at:

Theorem 6. *Suppose that the conditions* (A)' *hold. Then, the conclusions of Theorem 3 hold but with method* (4) *replacing method* (3).

Proof. Notice that the only difference in the proof is that we use

$$\|\bar{y}_m - \bar{x}_m\| \le \tfrac{l(\bar{t}_m+\bar{s}_{m-1}-2\bar{s}_{m-1})}{1-l_0\bar{t}_m}(\bar{t}_m - \bar{s}_{m-1}),$$

$$\begin{aligned}\|\bar{x}_{m+1} - \bar{y}_m\| &= \|(F'(\bar{x}_m)^{-1}F'(x_0))(F'(x_0)^{-1}F(\bar{y}_m))\| \le \\ &\le \|(F'(\bar{x}_m)^{-1}F'(x_0))\|\|(F'(x_0)^{-1}F(\bar{y}_m))\| \\ &\le \tfrac{l(\bar{s}_m-\bar{t}_m)^2}{2(1-l_0\bar{t}_m)}\end{aligned}$$

instead of

$$\|x_{m+1} - y_m\| \le \tfrac{l(s_m-t_m)^2}{2(1-l_0s_m)}$$

and

$$\|y_m - x_m\| \le \tfrac{l(t_m-s_{m-1})^2}{2(1-l_0t_m)},$$

respectively. □

3. LOCAL CONVERGENCE ANALYSIS

The local convergence analysis for both methods uses the hypotheses (H):

(h1) $F : \Omega \subset B_1 \to B_2$ is differentiable in the sense of Fr'echet and there exist $x_* \in \Omega$ such that $F'(x_*)^{-1} \in LB(B_2, B_1)$ and $F(x_*) = 0$.

(h2) There exists $L_0 > 0$ such that for each $x \in \Omega$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \le L_0\|x - x_*\|.$$

Set $S_2 = \Omega \cap S(x, \frac{1}{L_0})$.

(h3) There exists $L > 0$ such that for each $x, y \in S_2$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \le L\|y - x\|.$$

(h4) $S(x_*, \rho) \subseteq \Omega$, where

$$\rho = \begin{cases} \mu_A = \frac{2}{2L_0+L}, & \text{if method (3) is used} \\ R = \frac{4}{4L_0+(1+\sqrt{5})L}, & \text{if method (4) is used} \end{cases}$$

(h5) There exists $\bar{r} \ge \rho$ such that $L_0\bar{r} < 1$.
Set $S_3 = \Omega \cap \bar{S}(x_*, \bar{r})$.

The proofs of the following two results are omitted, since they follow as the corresponding ones for single step Newton's method (2) given in [8, 10].

Theorem 7. *Under the hypotheses* (H) *starting from* $x_0 \in S(x_*, \mu_A) - x_*$ *sequence* x_n *produced by method* (3) *converges to* x_* *which is the only solution of equation* $F(x) = 0$ *in the set* S_3. *Moreover, the following items hold:*

$$\|y_n - x_*\| \le \tfrac{L\|x_n-x_*\|^2}{2(1-L_0\|x_n-x_*\|)} \tag{40}$$

and

$$\|x_{n+1} - x_*\| \le \tfrac{L\|y_n-x_*\|^2}{2(1-L_0\|y_n-x_*\|} \tag{41}$$

THEOREM 8. *Under the hypotheses* (H) *starting from* $x_0 \in S(x_*, R) - x_*$ *sequence* $\bar{x}_n$ *produced by method* (4) *converges to* x_* *which is the only solution of equation* $F(x) = 0$ *in the set* S_3. *Moreover the following items hold:*

(42) $$\|y_n - x_*\| \leq \frac{L\|x_n - x_*\|^2}{2(1 - L_0\|x_n - x_*\|)}$$

and

(43) $$\|y_{n+1} - x_*\| \leq \frac{L(2\|x_n - x_*\| + \|y_n - x_*)}{2(1 - L_0\|x_n - x_*\|)}\|y_n - x_*\|$$

LEMMA 9. *The radius of convergence in* [8, 10] *for the single step Newton's method was given by*

(44) $$\bar{\mu}_A = \frac{2}{2L_0 + L_1},$$

where L_1 *is the Lipschitz constant on* Ω. *Then, since* $s_0 \subseteq \Omega$, *we get*

(45) $$L \leq L_1$$

so

(46) $$\bar{\mu}_A \leq \mu_A$$

The error bounds are tighter too, since L_1 *is used in* (40) *and* (41).

4. NUMERICAL EXAMPLES

EXAMPLE 10. *Let* $B_1 = B_2 = \mathbb{R}$, $\Omega = S(x_0, 1 - \alpha)$, $x_0 = 1$ *and* $\alpha \in I = [0, \frac{1}{2})$. *Define function* f *on* Ω *by* $f(x) = x^3 - \alpha$.

Then, using hypotheses (a1)-(a3), *we get* $l_0 = 3 - \alpha, l = \frac{2(6+5\alpha-2\alpha^2)}{3(3-\alpha)}$ *and* $l_1 = 2(2-\alpha)$. *Method* (2) *For* $\alpha \in I_0 = [0.371269, 0.5]$ *has solutions under our approach but no solutions according to Kantorovich, since* $h_K = l_1\eta > \frac{1}{2}$ *for each* $\alpha \in [0, 0.5]$. *Method* (3) *has no solution in* $[0, 0.5]$.

EXAMPLE 11. *Let* $B_1 = B_2 = C[0,1]$, *where* $C[0,1]$ *stands for the space of continuous function on* $[0,1]$. *We shall use the mimum norm. Let* $\Omega_0 = \{x \in C[0,1] : \|x\| \leq d\}$.

Define operator G *on* Ω_0 *by*

$$G(x)(s) = x(s) - g(s) - b\int_0^1 K(s,t)x(t)^3 dt, x \in C[0,1], s \in [0,1]$$

where $g \in C[0,1]$ *is a given function,* ξ *is a real constant and the kernel* K *is the Green's function. In this case, for each* $x \in D^*$, $F'(x)$ *is a linear operator defined on* D^* *by the following expression:* $[F'(x)(v)](s) = v(s) - 3\xi\int_0^1 K(s,t)x(t)^2 v(t)dt, v \in C[0,1], s \in [0,1]$ *If we choose* $x_0(s) = f(s) = 1$, *it follows* $\|I - F'(x_0)\| \leq \frac{3|\xi|}{8}$. *Thus, if* $|\xi| < \frac{8}{3}$, $f'(x_0)^{-1}$ *is defined and*

$$\|F'(x_0)^{-1}\| \leq \frac{8}{8-3|\xi|}, \|F(x_0)\| \leq \frac{|\xi|}{8}, \eta = \|F'(x_0)^{-1}F(x_0)\| \leq \frac{|\xi|}{8-3|\xi|}.$$

Choosing $\xi = 1.00$ *and* $x = 3$, *we have* $\eta = 0.2, T = 3.8, b = 2.6, L_1 = 2.28$, *and* $l = 1.38154\ldots$

Using this values we obtain that conditions (28)-(30) *are not satisfied, since the Kantorovich condition* $h_K = l_1\eta \le \frac{1}{2}$*, gives* $h_K = 0.76 > \frac{1}{2}$*. but condition* (29) *is satisfied since* $0.485085 < \frac{1}{2}$ *The convergence of the Newton's method follows by Theorem 3.*

Example 12. *Let* $B_1 = B_2 = \mathbb{R}^3$, $\Omega = S(0,1), x^* = (0,0,0)^T$ *and define* G *on* Ω *by*

$$G(x) = F(x_1, x_2, x_3) = (e^{x_1} - 1, \tfrac{e-1}{2}{x_2}^2 + x_2, x_3)^T. \tag{47}$$

For the points $u = (u_1, u_2, u_3)^T$*, the Fréchet derivative is given by*

$$G'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, $G'(x^*) = diag(1,1,1)$*, we have* $L_0 = e - 1, L = e^{\frac{1}{e-1}}, L_1 = e.$

Then, we obtain that

$$\rho = \begin{cases} \mu_A = 0.3827, & \text{if method (3) is used} \\ R = 0.3158, & \text{if method (4) is used.} \end{cases}$$

Example 13. *Let* $B_1 = B_2 = C[0,1]$*, the space of continuous functions defined on* $[0,1]$ *and be equipped with the max norm. Let* $\Omega = \overline{S}(0,1)$*. Define function* G *on* Ω *by*

$$G(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{48}$$

We have that

$$G'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \quad \text{for each } \xi \in \Omega.$$

Then, we get that $x^* = 0$, $L_0 = 7.5, L = 15 = L_1$*. This way, we have that*

$$\rho = \begin{cases} \mu_A = 0.0667, & \text{if method (3) is used,} \\ R = 0.0509, & \text{if method (4) is used.} \end{cases}$$

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