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EXTENDED CONVERGENCE ANALYSIS OF NEWTON-POTRA SOLVER FOR EQUATIONS

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Abstract. In the paper a local and a semi-local convergence of combined iterative process for solving nonlinear operator equations is investigated. This solver is built based on Newton solver and has R-convergence order $1.839\ldots$ The radius of the convergence ball and convergence order of the investigated solver are determined in an earlier paper. Modifications of previous conditions leads to extended convergence domain. These advantages are obtained under the same computational effort. Numerical experiments are carried out on the test examples with nondifferentiable operator.

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1. INTRODUCTION

Consider the operator equation

$$H(x) \equiv F(x) + Q(x) = 0, \tag{1}$$

where F and Q are nonlinear operators, defined on an subset D of a Banach space E_1 with values in a Banach space E_2. It is known, that F is a differentiable by Frèchet operator, Q is a continuous operator, whose differentiability in general is not required.

A plethora of problems from diverse disciplines can be converted to equation (1) *via* mathematical modelling [1–27]. Therefore, the task of computing a solution x_* is of extreme importance. We resort to iterative solvers, since closed form solutions can be obtained in rare cases.

The Newton solver [2] can not be used to find a solution of equation (1), because of the nondifferentiable Q. However, in this case the Newton-type solver [3], or one of combined iterative processes may be applicable [3]– [18].

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The special case of (1) is the equation $F(x) = 0$. Usually, to find the solution Newton's solver is used

$$x_{n+1} = x_n - [F'(x_n)]^{-1}F(x_n),\ n \geq 0,$$

whose convergence order is quadratic [19, 20]. Hence, one can use the difference solvers. These solvers use only a nonlinear operator, and do not require analytical derivatives. One of these solvers has R-convergence order $1.839\dots$

$$x_{n+1} = x_n - [F(x_n, x_{n-1}) + F(x_{n-2}, x_n) - F(x_{n-2}, x_{n-1})]^{-1}F(x_n),\ n \geq 0,$$

where $F(u, v)$ is a divided difference of order one. This solver was proposed by J. Traub for solving one nonlinear equation [21], later it was generalized to Banach spaces by F. Potra [22], and investigated under different conditions in [23, 24].

In the paper [25] a combined iterative process was proposed, which is built based Newton's and Potra's solvers

$$\begin{aligned} x_{n+1} &= x_n - A_n^{-1}H(x_n), \qquad n \geq 0, \\ A_n &= F'(x_n) + Q(x_n, x_{n-1}) + Q(x_{n-2}, x_n) - Q(x_{n-2}, x_{n-1}), \end{aligned} \tag{2}$$

where $Q(x, y)$ is a divided difference of order one, to be defined later.

This solver was studied in [26] under weak ω-conditions. In this work we continue the study of a local and a semi-local convergence of solver (2). It is established that the convergence order of the combined iterative process (2) is similar to the convergence order of the Potra solver. But also it is important to extend the convergence region in particular without requiring an additional hypotheses. This fact will extend the number of initial approximations. By applying a new approach we achieve fewer iterations to obtain a result with predetermined accuracy, at least as many initial points, and same or less computational cost.

The rest of the paper is structured as follows: In Section 2, we present the local convergence analysis of the solver (2) and a Corollary. Section 3 contains the proofs of semi-local convergence and uniqueness of solution. In Section 4, we provide the numerical example. The article ends with some conclusions.

2. LOCAL CONVERGENCE OF SOLVER (2)

Note that we used the classic Lipschitz conditions for the derivative of first order of operator F and for divided differences of order one and two of operator Q. The following theorem present the convergence radius and the convergence speed of iterative process (2). Although we assume, that Q is differentiable by Fréchet operator.

Set $U = U(x_*, r_*) = \{x : \|x - x_*\| < r_*\}$. Let $x, y, z \in D$.

DEFINITION 1. *The linear operator from E_1 to E_2 denoted as $Q(x, y)$ is called divided difference of order one of Q by points $x, y, (x \neq y)$ if it satisfies the condition*

$$Q(x, y)(x - y) = Q(x) - Q(y).$$

Definition 2. *The operator $Q(x,y,z)$ is called divided difference of order two of Q by points x,y,z if it satisfies the condition*

$$Q(x,y,z)(y-z)=Q(x,y)-Q(x,z).$$

Theorem 3. *Let F and Q are nonlinear operator, which are defined on open convex subset D of a Banach space E_1 with values in a Banach space E_2. Suppose, that equation (1) has a solution $x_*\in D$ and the inverse Fréchet derivative $[H'(x_*)]^{-1}$ exists. Let $Q(\cdot,\cdot)$ and $Q(\cdot,\cdot,\cdot)$ are the divided differences of order one and two of operator Q, which are defined on the set D, and the Lipschitz conditions are satisfied for each $x,y,z\in D$*

$$\|H'(x_*)^{-1}(F'(x)-F'(x_*))\|\le 2l_*^0\|x-x_*\|, \tag{3}$$

$$\|H'(x_*)^{-1}(Q(x_*,x_*)-Q(x,x_*))\|\le a\|x_*-x\|, \tag{4}$$

$$\|H'(x_*)^{-1}(Q(z,x_*)-Q(z,x))\|\le b\|x_*-x\|, \tag{5}$$

$$\|H'(x_*)^{-1}(Q(x,x_*,y)-Q(z,x_*,y))\|\le q_*^0\|x-z\|, \tag{6}$$

for each $x,y,z\in D_0=D\cap U(x_*,r_0)$,

$$r_0=\frac{1}{l_*^0+p_*^0+\sqrt{(l_*^0+p_*^0)^2+2q_*^0}}, \tag{7}$$

$$\|H'(x_*)^{-1}(F'(x)-F'(y))\|\le 2l_*\|x-y\|, \tag{8}$$

$$\|H'(x_*)^{-1}(Q(x,x_*)-Q(x,x))\|\le c\|x_*-x\|, \tag{9}$$

$$\|H'(x_*)^{-1}(Q(x,x)-Q(x,y))\|\le d\|x-y\|, \tag{10}$$

$$\|H'(x_*)^{-1}(Q(x,x,y)-Q(z,x,y))\|\le q_*\|x-z\|, \tag{11}$$

where $2p_*^0=a+b$, $p_*=c+d$,

$$r_*=\frac{2}{2(l_*^0+p_*^0)+l_*+p_*+\sqrt{(2(l_*^0+p_*^0)+l_*+p_*)^2+8(q_*+q_*^0)}} \tag{12}$$

Then for each $x_{-2},x_{-1},x_0\in U$ iterative process (2) is well defined, and generates a sequence $\{x_n\}_{n\ge 0}\in U$, which converges to x_* and satisfies the estimation

$$\begin{aligned}&\|x_{n+1}-x_*\|\le\\ &\le\frac{l_*+p_*}{C_n}\|x_n-x_*\|^2+\frac{q_*}{C_n}(\|x_n-x_*\|+\|x_{n-2}-x_*\|)\|x_{n-1}-x_*\|\|x_n-x_*\|,\end{aligned} \tag{13}$$

where $C_n=1-2(l_*^0+p_*^0)\|x_n-x_*\|-q_*^0(\|x_n-x_*\|+\|x_{n-2}-x_*\|)\|x_{n-1}-x_*\|$.

Proof. Let $x, y, z \in U$. Denote $A = F'(x) + Q(x,y) + Q(z,x) - Q(z,y)$. Then in view of conditions (3)–(6), we obtain

$$\begin{aligned}
&\|I - H'(x_*)^{-1}A\| = \\
&= \|H(x_*)^{-1}[F'(x_*) - F'(x) + Q(x_*,x_*) - Q(x,y) - Q(z,x) + Q(z,y)]\| \\
&\le \|H(x_*)^{-1}\,[F'(x_*) - F'(x) + Q(x_*,x_*) - Q(x,x_*) \\
&\quad + Q(z,x_*) - Q(z,x) + Q(x,x_*) - Q(x,y) + Q(z,y) - Q(z,x_*)]\,\| \\
&\le \|H(x_*)^{-1}\,[F'(x_*) - F'(x) + Q(x_*,x_*) - Q(x,x_*) \\
&\quad + Q(z,x_*) - Q(z,x) + [Q(x,x_*,y) - Q(z,x_*,y)](x_* - y)]\,\| \\
&\le 2l_*^0\|x - x_*\| + 2p_*^0\|x - x_*\| + q_*^0\|x - z\|\|y - x_*\| \\
&\le 2(l_*^0 + p_*^0)\|x - x_*\| + q_*^0(\|x - x_*\| + \|z - x_*\|)\|y - x_*\|.
\end{aligned}$$

By the definition of r_* (12), we get

$$2(l_*^0 + p_*^0)r_0 + 2q_*^0 r_0^2 < 1. \tag{14}$$

Then, by the Banach Lemma on invertible operators [2], A is invertible and

$$\begin{aligned}
&\|(I - (I - H'(x_*)^{-1}A))^{-1}\| = \|A^{-1}H'(x_*)\| \\
&\le [1 - 2(l_*^0 + p_*^0)\|x - x_*\| - q_*^0(\|x - x_*\| + \|z - x_*\|)\|y - x_*\|]^{-1}.
\end{aligned} \tag{15}$$

Suppose, that $x_{n-2}, x_{n-1}, x_n \in U$. Then the operator

$$A_n = F'(x_n) + Q(x_n, x_{n-1}) + Q(x_{n-2}, x_n) - Q(x_{n-2}, x_{n-1})$$

is invertible. Next, we can write

$$\begin{aligned}
\|x_{n+1} - x_*\| &= \|x_n - x_* - A_n^{-1}(H(x_n) - H(x_*))\| \\
&\le \|A_n^{-1}H'(x_*)\|\|H'(x_*)^{-1}[H(x_n) - H(x_*) - A_n(x_n - x_*)]\|.
\end{aligned} \tag{16}$$

In view of (8)–(11), we get

$$\begin{aligned}
&\|H'(x_*)^{-1}[H(x_n) - H(x_*) - A_n(x_n - x_*)]\| = \\
&= \|H'(x_*)^{-1}[F(x_n) - F(x_*) + Q(x_n) - Q(x_*) - A_n(x_n - x_*)]\| \\
&\le \|H'(x_*)^{-1}\int_0^1 (F'(x_* + t(x_n - x_*)) - F'(x_n))dt\|\|x_n - x_*\| \\
&\quad + \|H'(x_*)^{-1}\,[Q(x_n,x_*) - Q(x_n,x_n) + Q(x_n,x_n) - Q(x_n,x_{n-1}) - Q(x_{n-2},x_n) \\
&\quad + Q(x_{n-2},x_{n-1})]\,\|\|x_n - x_*\| \\
&\le (l_* + p_*)\|x_n - x_*\|^2 \\
&\quad + \|H'(x_*)^{-1}[Q(x_n,x_n,x_{n-1}) - Q(x_{n-2},x_n,x_{n-1})]\|\|x_n - x_{n-1}\|\|x_n - x_*\| \\
&\le (l_* + p_*)\|x_n - x_*\|^2 + q_*\|x_n - x_{n-2}\|\|x_n - x_{n-1}\|\|x_n - x_*\|.
\end{aligned}$$

Then, from (15) and (16), we obtain the estimate (13). Moreover, from inequalities (13), (14) we have in a turn

$$\|x_{n+1} - x_*\| < \|x_n - x_*\| < r_*, \quad n \geq 0.$$

Hence, iterative process (2) is well defined, generated sequence $\{x_n\}_{n\geq 0}$ is in U, and converges to the solution x_*. Finally, by the last inequality, and estimate (13) we get, that $\lim_{n\to\infty} \|x_n - x_*\| = 0$. □

Corollary 4. *The R-convergence order of the combined iterative solver* (2) *is* 1.839...

Proof. By estimate (13), we have that there exist a constant C, and a natural number N, such that

$$\|x_{n+1} - x_*\| \leq C\|x_n - x_*\|\|x_{n-1} - x_*\|\|x_{n-2} - x_*\|, \quad n \geq N.$$

Hence, the R-convergence order of solver (2) is the unique positive root of nonlinear equation $t^3 - t^2 - t - 1 = 0$ [22], which is 1.839... □

Remark 5. The conditions used in [25] instead of (3)–(11) are:
for each $x, y, u, v \in D$

$$\|H'(x_*)^{-1}(F'(x) - F'(y))\| \leq 2l_*^1\|x - y\|, \tag{17}$$

$$\|H'(x_*)^{-1}(Q(x, y) - Q(u, v))\| \leq p_*^1(\|x - u\| + \|y - v\|), \tag{18}$$

$$\|H'(x_*)^{-1}(Q(u, x, y) - Q(v, x, y))\| \leq q_*^1\|u - v\|, \tag{19}$$

$$r_*^1 = \frac{2}{3(l_*^1+p_*^1)+\sqrt{9(l_*^1+p_*^1)^2+16q_*^1}}, \tag{20}$$

$$C_n^1 = 1 - 2(l_*^1 + p_*^1)\|x_n - x_*\| - q_*^1(\|x_n - x_*\| + \|x_{n-2} - x_*\|)\|x_{n-1} - x_*\|. \tag{21}$$

But

$$D_0 \subseteq D,$$

so

$$\begin{aligned} l_*^0 &\leq l_*^1, \qquad p_*^0 \leq p_*^1, \qquad q_*^0 \leq q_*^1, \\ l_* &\leq l_*^1, \qquad p_* \leq p_*^1, \qquad q_* \leq q_*^1 \end{aligned} \tag{22}$$

and

$$(C_n)^{-1} \leq (C_n^1)^{-1}. \tag{23}$$

In view of (22)–(23), the new results give compared to the ones in [25].

At least as many initial points, and fewer iterations to achieve a predetermined accuracy. The improvements are obtained under the same or less computational cost as in [25], since the new constants are special cases of ones in [25]. Examples where (22)–(23) hold as strict inequalities can be found in [27]. This technique is used to expand applicability of some solvers [7] and can be used to do the same on other solvers.

3. SEMI-LOCAL CONVERGENCE OF SOLVER (2)

Set $U_0(x_0,r) = \{x : \|x - x_0\| \le r\}$. The semi-local convergence of the combined Newton-Potra solver (2) is presented in what follows.

THEOREM 6. *Let F and Q are nonlinear operators, which are defined in open convex subset D of a Banach space E_1, with values in a Banach space E_2. $Q(\cdot,\cdot)$ and $Q(\cdot,\cdot,\cdot)$ are the divided differences of order one and two of function Q, which are defined on set D.*

Suppose, that the linear operator $A_0 = F'(x_0) + Q(x_0,x_{-1}) + Q(x_{-2},x_0) - Q(x_{-2},x_{-1})$, where $x_{-2},x_{-1},x_0 \in D$, is invertible fore each $x,y,u,v \in D$ satisfies the Lipschitz conditions

$$\|A_0^{-1}(F'(x) - F'(x_0))\| \le 2l_0^0\|x - x_0\|, \tag{24}$$

$$\|A_0^{-1}(Q(x_0,x_0) - Q(x,x_0))\| \le \lambda\|x_0 - x\|, \tag{25}$$

$$\|A_0^{-1}(Q(x,x_0) - Q(x,y))\| \le \mu\|x_0 - y\|, \tag{26}$$

$$\|A_0^{-1}(Q(z,u) - Q(z,x))\| \le \xi\|u - x\|, \tag{27}$$

$$\|A_0^{-1}(Q(x_0,x_{-1},x_0) - Q(x_{-2},x_{-1},x_0))\| \le \overline{q}_0\|x_0 - x_{-2}\|, \tag{28}$$

Set

$$D_1 = D \cap U(x_0,r_0), \qquad r_0 = \tfrac{1-\overline{q}_0 a(a+b)}{2(l_0^0+p_0^0)} \;\; for \;\; \overline{q}_0 a(a+b) < 1 \tag{29}$$

and $p_0^0 = \max\{\lambda,\mu,\xi\}$.

For each $x,y,u,v \in D_1$

$$\|A_0^{-1}(F'(x) - F'(y))\| \le 2l_0\|x - y\|, \tag{30}$$

$$\|A_0^{-1}(Q(x,y) - Q(u,v))\| \le p_0(\|x - u\| + \|y - v\|), \tag{31}$$

$$\|A_0^{-1}(Q(u,x,y) - Q(v,x,y))\| \le q_0\|u - v\|. \tag{32}$$

Let a,b and c are a nonnegative numbers, such that

$$\|x_0 - x_{-1}\| \le a, \quad \|x_{-1} - x_{-2}\| \le b, \quad \|A_0^{-1}(F(x_0) + Q(x_0))\| \le c. \tag{33}$$

Let r_1 is a nonnegative number, such that

$$r_1 > \tfrac{c}{1-\gamma},$$

$$\gamma = \tfrac{(l_0+p_0)c + q_0 a(a+b)}{1-\overline{q}_0 a(a+b) - 2(l_0^0+p_0^0)r_1}, \quad 0 < \gamma < 1,$$

and the closed ball $U_0(x_0,r_1)$ is included in D. Then, real sequence $\{t_k\}_{k\ge -2}$ defined as

$$\begin{aligned} &t_{-2} = r_1 + a + b, \quad t_{-1} = r_1 + a, \quad t_0 = r_1, \quad t_1 = r_1 - c \\ &t_{n+1} - t_{n+2} = \tfrac{(l_0+p_0)(t_n - t_{n+1}) + q_0(t_{n-1} - t_n)(t_{n-2} - t_n)}{1-(2l_0^0+\lambda+\xi)(t_0 - t_{n+1}) - (\mu+\xi)(t_0 - t_n) - \overline{q}_0 a(a+b)}(t_n - t_{n+1}). \end{aligned} \tag{34}$$

is nonnegative and decreasing converging to some $t_* \in \mathbb{R}$, *such that*

$$r_1 - \tfrac{c}{1-\gamma} \leq t_* \leq t_{-1}.$$

Then the iterative process (2) *is well defined, remains in* $U_0(x_0, r_1)$ *and converges to a solution* $x \in U_0(x_0, r_1)$ *of equation* $F(x) + Q(x) = 0$, *Moreover, the following estimates are true*

$$\|x_n - x_*\| \leq t_n - t_*. \tag{35}$$

Proof. Using mathematical induction, we show that the iterative process (34) is well defined

$$t_{k+1} - t_{k+2} \leq \gamma(t_k - t_{k+1}), \tag{36}$$

$$t_{k+1} \geq t_{k+2} \geq r_1 - \tfrac{c}{1-\gamma}. \tag{37}$$

Using (34) and $k = 0$, we obtain

$$\begin{aligned} t_1 - t_2 =& \tfrac{(l_0+p_0)(t_0-t_1)+q_0(t_{-1}-t_0)(t_{-2}-t_0)}{1-(2l_0^0+\lambda+\xi)(t_0-t_1)-\overline{q}_0 a(a+b)}(t_0 - t_1) \\ \leq& \tfrac{(l_0+p_0)c+q_0 a(a+b)}{1-\overline{q}_0 a(a+b)-2(l_0^0+p_0^0)r_1}(t_0 - t_1), \\ t_0 \geq& t_1,\ t_1 \geq t_2 \geq t_1 - \gamma(t_0 - t_1) = r_1 - (1-\gamma)c = r_1 - \tfrac{(1-\gamma^2)c}{1-\gamma} \\ \geq& r_1 - \tfrac{c}{1-\gamma} \geq 0, \end{aligned}$$

so (36)–(37) are true for $k = 0$.

Suppose, that estimates (36)–(37) are satisfied for each $k \leq n$. Then, for $k = n$ we have the following

$$\begin{aligned} t_{n+1} - t_{n+2} =& \tfrac{(l_0+p_0)(t_n-t_{n+1})+q_0(t_{n-1}-t_n)(t_{n-2}-t_n)}{1-(2l_0^0+\lambda+\xi)(t_0-t_{n+1})-(\mu+\xi)(t_0-t_n)-\overline{q}_0 a(a+b)}(t_n - t_{n+1}) \\ \leq& \tfrac{(l_0+p_0)c+q_0 a(a+b)}{1-\overline{q}_0 a(a+b)-2(l_0^0+p_0^0)r_1}(t_n - t_{n+1}) = \gamma(t_n - t_{n+1}), \\ t_{n+1} \geq& t_{n+2} \geq t_{n+1} - \gamma(t_n - t_{n+1}) \geq r_1 - \tfrac{(1-\gamma^2)c}{1-\gamma} \geq r_1 - \tfrac{c}{1-\gamma} \geq 0. \end{aligned}$$

Hence, that $\{t_n\}_{n \geq -2}$ is decreasing, nonnegative sequence which converges to some $t_* \geq 0$. Next, we show, that iterative process (2) is well defined, and following estimate is true for each $n \geq -2$

$$\|x_n - x_{n+1}\| \leq t_n - t_{n+1},\ n \geq -2. \tag{38}$$

In view of Lipschitz conditions (24)–(28), for $k = n+1$, we obtain

$$\begin{aligned}
&\|I - A_0^{-1}A_{n+1}\| = \\
&= \|A_0^1(A_0 - A_{n+1})\| \\
&\leq \|A_0^{-1}[F'(x_0) - F'(x_{n+1})]\| \\
&\quad + \|A_0^{-1}\,[Q(x_0, x_{-1}) - Q(x_0, x_0) + Q(x_{-2}, x_0) - Q(x_{-2}, x_{-1}) + Q(x_0, x_0) \\
&\quad - Q(x_{n+1}, x_0) + Q(x_{n+1}, x_0) - Q(x_{n+1}, x_n) + Q(x_{n-1}, x_n) - Q(x_{n-1}, x_{n+1})]\,\| \\
&= \|A_0^{-1}[F'(x_0) - F'(x_{n+1})]\| + \|A_0^{-1}[(Q(x_0, x_{-1}, x_0) - Q(x_{-2}, x_{-1}, x_0)) \\
&\quad \times (x_{-1} - x_0) + Q(x_0, x_0) - Q(x_{n+1}, x_0) + Q(x_{n+1}, x_0) - Q(x_{n+1}, x_n) \\
&\quad + Q(x_{n-1}, x_n) - Q(x_{n-1}, x_{n+1})]\| \\
&\leq 2l_0^0\|x_0 - x_{n+1}\| + \overline{q}_0 a(a+b) + \lambda\|x_0 - x_{n+1}\| + \mu\|x_0 - x_n\| + \xi\|x_n - x_{n+1}\| \\
&\leq 2(l_0^0 + p_0^0)(t_0 - t_{n+1}) + \overline{q}_0 a(a+b) \leq 2(l_0^0 + p_0^0)t_0 + \overline{q}_0 a(a+b) \\
&\leq 2(l_0^0 + p_0^0)r_1 + \overline{q}_0 a(a+b) < 1.
\end{aligned}$$

Hence, A_{n+1} is invertible and

$$\begin{aligned}
&\|A_{n+1}^{-1}A_0\| \leq \\
&[1 - \overline{q}_0 a(a+b) - 2l_0^0\|x_0 - x_{n+1}\| - \lambda\|x_0 - x_{n+1}\| - \mu\|x_0 - x_n\| - \xi\|x_n - x_{n+1}\|]^{-1}.
\end{aligned}$$

Taking into account the definition of the divided difference and conditions (30)–(32) we get in a turn

$$\begin{aligned}
&\|A_0^{-1}[F(x_{n+1}) + Q(x_{n+1})]\| = \\
&= \|A_0^{-1}[F(x_{n+1}) + Q(x_{n+1}) - F(x_n) - Q(x_n) - A_n(x_{n+1} - x_n)]\| \\
&\leq \left\|A_0^{-1}\int_0^1 (F'(x_n + t(x_{n+1} - x_n)) - F'(x_n))\,dt\right\|\|x_n - x_{n+1}\| \\
&\qquad + \|A_0^{-1}[Q(x_n, x_{n+1}) - Q(x_n, x_n) + (Q(x_n, x_n, x_{n-1}) - Q(x_{n-2}, x_n, x_{n-1})) \\
&\qquad \times (x_n - x_{n-1})]\|\|x_n - x_{n+1}\| \\
&\leq (l_0 + p_0)\|x_n - x_{n+1}\|^2 + q_0\|x_{n-2} - x_n\|\|x_{n-1} - x_n\|\|x_n - x_{n+1}\|.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|x_{n+1} - x_{n+2}\| = \|A_0^{-1}H(x_{n+1})\| \leq \|A_0^{-1}A_0\|\|A_0^{-1}[F(x_{n+1}) + Q(x_{n+1})]\| \\
&\leq \tfrac{(l_0+p_0)\|x_n - x_{n+1}\|^2 + q_0\|x_{n-1} - x_n\|\|x_{n-2} - x_n\|\|x_n - x_{n+1}\|}{1 - (2l_0^0 + \lambda)\|x_0 - x_{n+1}\| - \mu\|x_0 - x_n\| - \xi\|x_n - x_{n+1}\| - \overline{q}_0 a(a+b)} \\
&\leq \tfrac{(l_0+p_0)(t_n - t_{n+1}) + q_0(t_{n-1} - t_n)(t_{n-2} - t_n)}{1 - (2l_0^0 + \lambda + \xi)(t_0 - t_{k+1}) - (\mu + \xi)(t_0 - t_n) - \overline{q}_0 a(a+b)}(t_n - t_{n+1}) \\
&= t_{n+1} - t_{n+2}.
\end{aligned}$$

That is, iterative process (2) is well defined for each n. Moreover

(39) $$\|x_n - x_k\| \leq t_n - t_k, \quad -2 \leq n \leq k,$$

so the sequence $\{x_n\}_{n\geq 0}$ is fundamental, and as such convergent in the Banach space E_1. By letting $k\to\infty$ in (39), we get (35).

Let us show, that x_* is a root of equation $F(x)+Q(x)=0$.

$$\begin{aligned}&\|A_0^{-1}H(x_{n+1})\|\leq\\&\leq(l_0+p_0)\|x_n-x_{n+1}\|^2+q_0\|x_n-x_{n-2}\|\|x_n-x_{n-1}\|\|x_n-x_{n+1}\|\to 0\end{aligned}$$

for $n\to\infty$. Hence, $F(x_*)+Q(x_*)=0$.

Next we will show the uniqueness of solution x_*. Suppose, that $x_{**}\in U_0(x_0,r_1)$, exists $x_{**}\neq x_*$ and $H(x_{**})=0$. Denote

$$P=\int_0^1 F'(x_*+t(x_{**}-x_*))dt+Q(x_{**},x_*).$$

Then $P(x_{**}-x_*)=H(x_{**})-H(x_*)$. In case operator P^{-1} is invertible, we obtain, that $x_{**}=x_*$.

$$\begin{aligned}&\|I-A_0^{-1}P\|=\|A_0^{-1}(A_0-P)\|\leq\\&\leq\|A_0^{-1}\int_0^1(F'(x_0)-F'(x_*+t(x_{**}-x_*)))dt\|\\&\quad+\|A_0^{-1}[Q(x_0,x_{-1})+Q(x_{-2},x_0)-Q(x_{-2},x_{-1})-Q(x_{**},x_*)]\|\\&\leq(l_0^0+p_0^0)(\|x_0-x_*\|+\|x_0-x_{**}\|)+\overline{q}_0a(a+b)\\&\leq 2(l_0^0+p_0^0)r_1+\overline{q}_0a(a+b)<1.\end{aligned}$$

Hence, P^{-1} exists. □

Theorem 7. *Let conditions of Theorem 6 are true. Then for each $n\geq 1$ the following estimate is true*

$$\|x_n-x_*\|\leq\frac{(l_0+p_0)(t_{n-1}-t_n)+q_0(t_{n-3}-t_{n-1})(t_{n-2}-t_{n-1})}{1-\overline{q}_0a(a+b)-(l_0^0+p_0^0)(2t_0-t_n)}(t_{n-1}-t_n). \tag{40}$$

Proof. Taking into account estimates (24)–(27), we get

$$\begin{aligned}&\Big\|I-A_0^{-1}\Big(\int_0^1 F'(x_*+t(x_n-x_*))dt+Q(x_n,x_*)\Big)\Big\|\leq\\&\leq\Big\|A_0^{-1}\int_0^1(F'(x_0)-F'(x_*+t(x_n-x_*)))dt\Big\|+\|A_0^{-1}[Q(x_0,x_{-1})\\&\quad-Q(x_0,x_0)+Q(x_{-2},x_0)-Q(x_{-2},x-1)+Q(x_0,x_0)-Q(x_n,x_*)]\|\\&\leq(l_0+p_0)(\|x_0-x_n\|+\|x_0-x_*\|)+q_0a(a+b)\\&\leq(l_0^0+p_0^0)(2t_0-t_n)+\overline{q}_0a(a+b)<1.\end{aligned}$$

Hence, $\int_0^1 F'(x_*+t(x_n-x_*))dt+Q(x_n,x_*)$ is invertible and

$$\begin{aligned}&\Big\|\Big(\int_0^1 F'(x_*+t(x_n-x_*)\Big)dt+Q(x_n,x_*))^{-1}A_0\Big\|\leq\\&\leq(1-\overline{q}_0a(a+b)-(l_0^0+p_0^0)(\|x_0-x_n\|+\|x_0-x_*|))^{-1}.\end{aligned}$$

Using the estimation

$$\begin{aligned}\|x_n - x_*\| =& \Big\| \Big(\int_0^1 F'(x_* + t(x_n - x_*))dt + Q(x_n, x_*) \Big)^{-1} (H(x_n) - H(x_*)) \Big\| \\ \leq& \|(\int_0^1 F'(x_* + t(x_n - x_*))dt + Q(x_n, x_*))^{-1} A_0\| \|A_0^{-1} H(x_n)\|,\end{aligned}$$

we obtain estimate (40). □

REMARK 8. The corresponding conditions in [25] are given for each $x, y, u, v \in D$ by

$$\begin{aligned}\|A_0^{-1}(F'(x) - F'(y))\| \leq& 2l_0^1 \|x - y\|, \\ \|A_0^{-1}(Q(x, y) - Q(u, v))\| \leq& p_0^1 (\|x - u\| + \|y - v\|), \\ \|A_0^{-1}(Q(u, x, y) - Q(v, x, y))\| \leq& q_0^1 \|u - v\|, \\ q_0 a(a + b) <& 1, \\ \overline{r}_1 >& \tfrac{c}{1 - \overline{\gamma}},\end{aligned}$$

where

$$\overline{\gamma} = \tfrac{(l_0^1 + p_0^1)c + q_0 a(a+b)}{1 - q_0^1 a(a+b) - 2(l_0^1 + p_0^1) r_1}, \quad 0 < \overline{\gamma} < 1.$$

We have that $D_1 \subseteq D$, so as in the local convergence case

$$\begin{aligned}l_0^0 \leq l_0^1, &\qquad l_0 \leq l_0^1, \\ p_0^0 \leq p_0^1, &\qquad p_0 \leq p_0^1, \\ q_0^0 \leq q_0^1, &\qquad q_0 \leq q_0^1\end{aligned}$$

and the old majorizing sequence call it $\{s_n\}$ (using l_0^1, p_0^1, q_0^1) is less tight than t_n [25]. Hence, the applicability of solver (2) has been extended in the semilocal convergence too. □

4. NUMERICAL EXPERIMENTS

In order to demonstrate the results of iterative solver (2), we carried out numerical experiments on test cases with nondifferentiable operator. The calculations are performed for different initial approximations with accuracy $\varepsilon = 10^{-10}$. The iterative process was performed until following conditions are satisfied:

$$\|x_{n+1} - x_n\|_\infty \leq \varepsilon, \quad \|H(x_{n+1})\|_\infty \leq \varepsilon.$$

Additional initial approximations were chosen by the following formula:

$$x_{-1} = x_0 - 10^{-4}, \; x_{-2} = x_0 - 2 \cdot 10^{-4}.$$

To compare the convergence speed of the combined Newton-Potra solver with a basic solvers, the number of iterations, required to obtain a solution of systems of nonlinear equations, are presented in a table. The Newton-type solver for equation (1) has the form [3]:

$$x_{n+1} = x_n - [F'(x_n)]^{-1} H(x_n), \; n \geq 0, \tag{41}$$

and the Potra solver [22]:

$$x_{n+1} = x_n - [H(x_n, x_{n-1}) + H(x_{n-2}, x_n) - H(x_{n-2}, x_{n-1})]^{-1} H(x_n), \; n \geq 0. \tag{42}$$

Consider the system of two equations

Example 9.

$$\begin{cases} 4xy^2 - x^3 + y^3 - 1 + |x| = 0, \\ 2y^2 - x^2y^2 + 1 + |x+y| = 0. \end{cases}$$

The solution of this system is $(x_*, y_*) = (2, -1)$. The numerical results are presented in Table 1.

x_0	Newton type solver (41)	Potra solver (42)	Newton-Potra solver (2)
(1.1,0.1)	23	19	14
(5,−5)	24	25	18
(1.85, −0.85)	13	11	7

Table 1. Number of iteration made to solve the problem, for initial approximation x_0.

Consider the system of three equations.

Example 10.

$$\begin{cases} z^2(1-y) - xy + |y - z^2| = 0, \\ z^2(x^3 - x) - y^2 + |3y^2 - z^2 + 1| = 0, \\ 6xy^3 + y^2z^2 - xy^2z + |x + z - y| = 0 \end{cases}$$

It is known, that one of solutions of the system is $(x_*, y_*, z_*)^T = (-1, 2, 3)^T$. The results of solvers (2), (41), (42) are presented in Table 2.

x_0	Newton type solver (41)	Potra solver (42)	Newton-Potra solver (2)
(−0.5, 2.3, 3.5)	142	11	10
(−1.5, 2.5, 3.5)	131	10	8
(−10, 20, 30)	128	23	17

Table 2. Number of iteration made to solve the problem.

5. CONCLUSIONS

Based on the obtained results we showed the advantages of combined solver (2) over basic solvers, in particular, over Potra solver (42), even the theoretical convergence order of both solvers are the same. Moreover the convergence region of iterative solvers in general is small, which limits the choice of initial points. So by using the new Lipschitz constants we get at least as many initial points and fewer iterations to achieve predetermined accuracy, without any additional cost. This technique can be applied to extend the applicability of other iterative solvers. Therefore, the proposed combined solver (2) is an effective alternative for solving nonlinear equations with nondifferentiable operator.

REFERENCES

[1] W.C. Rheinboldt, *Methods for Solving Systems of Nonlinear Equations*, SIAM, Philadelphia, 1998.

[2] L.V. Kantorovich, G.P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, UK, 1982.

[3] P.P. Zabrejko, D.F. Nguen, *The majorant method in the theory of Newton-Kantorovich approximations and the Pták error estimates*, Numer. Funct. Anal. Optim., **9** (1987), pp. 671–686.

[4] I.K. Argyros, *A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space*, J. Math. Anal. Appl., **298** (2004), pp. 374–397.

[5] I.K. Argyros, *Convergence and Applications of Newton-Type Iterations*, Springer, New York, NY, USA, 2008.

[6] I.K. Argyros, A.A. Magréñan, *A Contemporary Study of Iterative Methods*, Elsevier (Academic Press), New York, NY, USA, 2018.

[7] I.K. Argyros, S.M. Shakhno, *Extended local convergence for the combined Newton-Kurchatov method under the generalized Lipschitz conditions*, Mathematics, **7** (2019) 207.

[8] E. Catinas, *On some iterative methods for solving nonlinear equations*, Rev. Anal. Numér. Théor. Approx., **23** (1994), pp. 47–53.

[9] M.A. Hernandez, M.J. Rubio, *The secant method for nondifferentiable operators*, J. Math. Anal. Appl., (2004), pp. 374–397.

[10] R. Iakymchuk, S. Shakhno, H. Yarmola, *Combined Newton-Kurchatov method for solving nonlinear operator equations*, Proc. Appl. Math. Mech., **16** (2016), pp. 719–720.

[11] H. Ren, I.K. Argyros, *A new semilocal convergence theorem for a fast iterative method with nondifferentiable operators*, J. Appl. Math. Comp., **34** (2010) nos. 1–2, pp. 39–46.

[12] S.M. Shakhno, *Convergence of combined Newton-Secant method and uniqueness of the solution of nonlinear equations*, Visnyk Ternopil Nat. Tech. Univ., **69** (2013), pp. 242–252 (In Ukrainian).

[13] S.M. Shakhno, *Convergence of the two-step combined method and uniqueness of the solution of nonlinear operator equations*, J. Comp. Appl. Math., **261** (2014), pp. 378–386.

[14] S.M. Shakhno, O.P. Gnatyshyn, *On an iterative algorithm of order 1.839... for solving the nonlinear least squares problems*, Appl. Math. Comp., **161** (2005) no. 1, pp. 253–264.

[15] S.M. Shakhno, I.V. Melnyk, H.P. Yarmola, *Analysis of convergence of a combined method for the solution of nonlinear equations*, Journal of Mathematical Sciences, **201** (2014) no. 1, pp. 32–43.

[16] S.M. Shakhno, H.P. Yarmola, *On the two-step method for solving nonlinear equations with nondifferentiable operator*, Proc. Appl. Math. Mech., **12** (2012) no. 1, pp. 617–618.

[17] S.M. Shakhno, H.P. Yarmola, *On the convergence of Newton-Kurchatov method under the classical Lipschitz conditions*, Journal of Computational and Applied Mathematics, Kyiv, **1** (2016), pp. 89–97.

[18] S.M. Shakhno, H.P. Yarmola, *Two-point method for solving nonlinear equation with nondifferentiable operator*, Mat. Stud., **36** (2011) no. 2, pp. 213–220 (in Ukrainian).

[19] J.E. Dennis, R.B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, SIAM, Philadelphia, 1996.

[20] X. Wang, *Convergence of Newton's method and uniqueness of the solution of equations in Banach space*, IMA Journal of Numerical Analysis., **20** (2000), pp. 123–134.

[21] J.F. Traub, *Iterative methods for the solution of equations*, Prentice Hall, Englewood Cliffs, 1964.

[22] F.A. Potra, *On an iterative algorithm of order* 1.839 . . . *for solving nonlinear operator equations*, Numer. Funct. Anal. Optim., **7** (1984-1985) no. 1, pp. 75–106.

[23] S.M. Shakhno, *Iterative algorithm with convergence order* 1.839 . . . *under the generalized Lipschitz conditions for the divided differences*, Visnyk National University *Lviv Politechnic*, Ser. Phys.-Mat., **740** (2012), pp. 62–65 (in Ukrainian).

[24] S.M. Shakhno, O.M. Makukh, *About iterative methods in conditions of Hölder continuity of the divided differences of the second order*, Matematychni Metody ta Fizyko-Mekhanichni Polya, **49** (2006) no. 2, pp. 90–98 (in Ukrainian).

[25] S.M. Shakhno, A.-V.I. Babjak, H.P. Yarmola *Combined Newton-Potra method for solving nonlinear operator equations*, Journal of Computational and Applied Mathematics, Kyiv, **3** (2015) 120, pp. 170–178 (in Ukrainian).

[26] S.M. Shakhno, H.P. Yarmola, *On convergence of Newton-Potra method under weak conditions*, Visnyk Lviv Univ. Ser. Appl. Math. Inform., **25** (2017), pp. 49–55 (in Ukrainian).

[27] I.K. Argyros, A.A. Magréñan, *Iterative Methods and Their Dynamics with Applications*, CRC Press, New York, NY, USA, 2017.

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