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INFINITELY HOMOCLINIC SOLUTIONS IN DISCRETE HAMILTONIAN SYSTEMS WITHOUT COERCIVE CONDITIONS

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Abstract. In this paper, we investigate the existence of infinitely many solutions for the second-order self-adjoint discrete Hamiltonian system

$$\Delta\left[p(n)\Delta u(n-1)\right] - L(n)u(n) + \nabla W(n, u(n)) = 0, \qquad (*)$$

where $n \in \mathbb{Z}, u \in \mathbb{R}^N, p, L : \mathbb{Z} \to \mathbb{R}^{N\times N}$ and $W : \mathbb{Z}\times\mathbb{R}^N \to \mathbb{R}$ are no periodic in n. The novelty of this paper is that $L(n)$ is bounded in the sense that there two constants $0 < \tau_1 < \tau_2 < \infty$ such that

$$\tau_1 \left|u\right|^2 < (L(n)u, u) < \tau_2 \left|u\right|^2, \ \forall n \in \mathbb{Z},\ u \in \mathbb{R}^N,$$

$W(t, u)$ satisfies Ambrosetti-Rabinowitz condition and some other reasonable hypotheses, we show that $(*)$ has infinitely many homoclinic solutions *via* the Symmetric Mountain Pass Theorem. Recent results in the literature are generalized and significantly improved.

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1. INTRODUCTION

Consider the second−order self-adjoint discrete Hamiltonian system

$$\Delta\left[p(n)\Delta u(n-1)\right] - L(n)u(n) + \nabla W(n, u(n)) = 0 \qquad (1)$$

where $n \in \mathbb{Z}, u \in \mathbb{R}^N, p, L : \mathbb{Z} \to \mathbb{R}^{N\times N}$ and $W : \mathbb{Z}\times\mathbb{R}^N \to \mathbb{R}$ is continuous differentiable in x, the forward difference operator Δ is defined by $\Delta u(n) = u(n+1) - u(n)$. As usual, we say a solution $u(n)$ of (1) is homoclinic (to 0) if $u(n) \to 0$ as $n \to \pm\infty$. In addition, if $u(n) \neq 0$ then $u(n)$ is called a nontrivial homoclinic solution. It is clear that (1) can be written as an equivalent first order nonlinear nonautonomous discrete Hamiltonian system

$$\Delta X(t) = J\nabla H_X(t, u(t+1), z(t)), \qquad (2)$$

where $X(t) = (u(t), z(t))^T$; $z(t)$ is a discrete momentum variable defined by $z(t) = p(t)\times\Delta u(t-1)$; $H(t, X(t)) = \frac{1}{2p(t)}z^2 + \frac{1}{2}q(t)u^2(t) - W(t, u(t))$ is called

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the Hamiltonian function, where J is the normal symplectic matrix. Moreover, (1) is a discretization of the following second order differential equation:

$$(p(t)u'(t))' - L(t)u(t) + \nabla W(t, u(t)) = 0, \tag{3}$$

which is also equivalent to a first order nonlinear Hamiltonian system.

Variational methods for difference equations, which allow one to achieve multiplicity results, were introduced by R.P. Agarwal, K. Perera, D. O'Regan see [3]. Recently, such methods received considerable attention. We mention here the works of M. Migda, J. Migda, M. Zdanowicz see [16], A. Pankov see [18], S. Stevic see [23]. Moreover, a lot of attention has been devoted in recent years to find periodic solutions of discrete dynamic models, for example, see [2, 3, 4, 5, 9, 10, 11, 19], other authors studied the existence of positive solutions of discrete fractional systems see [6, 12]. It is our purpose in the present work to find other types of solutions, namely the doubly asymptotic solutions, first discovered by Poincaré [20] in continuous Hamiltonian systems. In the past 40 years, system (3) has bee widely investigated, see [1, 13, 17, 22, 24] and references therein. System (3) is the special form of the Emden-Fowler equation, appearing in the study of astrophysics, gas dynamics, fluid mechanics, relativistic mechanics, nuclear physics and chemically reacting systems, and many well-known results concerning properties of solutions of (3) are collected in [25]. When $W(n, x)$ is an even function on x, there are few result on existence of infinitely many homoclinic orbits for discrete Hamiltonian systems, because it is often very difficult to verify the last condition of the Symmetric Mountain Pass Theorem, different from the Mountain Pass Theorem.

In this paper we show that the Palais-Smale condition is satisfied on the unbounded domain and we use the usual Mountain Pass Theorem to prove the existence of a homoclinic orbit of (1). Moreover, if $W(n, .)$ is an even function, we prove that (1) possesses an unbounded sequence of homoclinic orbits emanating from 0 by invoking the Symmetric Mountain Pass Theorem.

For the statement of our main result, the potential $W(t, x)$ is supposed to satisfy the following conditions:

- (P) $p(n)$ is symmetric and positive definite matrix for all $n \in \mathbb{Z}$.
- (L) $L(n)$ is symmetric and positive definite matrix for all $n \in \mathbb{Z}$ and there are two constants $0 < \tau_1 < \tau_2 < \infty$ such that

$$\tau_1 |x|^2 < (L(n)x, x) < \tau_2 |x|^2, \quad \forall n \in \mathbb{Z},\ x \in \mathbb{R}^N.$$

- (W_1) there exists a constant $\mu > 2$ such that

$$0 < \mu W(n, x) \leq (\nabla W(n, x), x), \quad \forall n \in \mathbb{Z},\ x \in \mathbb{R}^N \backslash \{0\}.$$

- (W_2) there exists some positive function $a : \mathbb{Z} \to \mathbb{R}$ with:

$$\lim_{|n| \to \infty} a(n) = 0, \tag{4}$$

such that

$$|\nabla W(n, x), x)| \leq a(n) |x|^{\mu - 1}, \quad \forall (n, x) \in \mathbb{Z} \times \mathbb{R}^n.$$

(W_3) $\varrho = \sup\{W(n,x) : n \in \mathbb{Z}, |x| = 1\} < \frac{1}{2C_2^2}$, where C_2 is defined in (5).

REMARK 1. (i) From (L), we assume that $L(n)$ is bounded. Therefore, the smallest eigenvalue of $L(n)$ does not tend to ∞ as $|n| \to \infty$, *i.e.*, $L(n)$ need not satisfy the various coercive conditions in the above mentioned papers.

(ii) As mentioned above, the coercive conditions are used to establish some compact embedding theorems to guarantee that (PS) condition holds, which is the essential step to obtain the existence of homoclinic solutions of (1) *via* Mountain Pass Theorem. In present paper, we assume that $L(n)$ is bounded and could not obtain some compact embedding theorem. Therefore, one difficulty is to adapt some new technique to overcome this difficulty and test that (PS) condition is verified. □

In this paper, we will prove the following Theorems.

THEOREM 2. *Suppose that (P), (L), (W_1), (W_2) and (W_3) hold. Then (1) possesses at least one nontrivial homoclinic solution.*

THEOREM 3. *Assume that (P),(L), (W_1), (W_2) and (W_3) are satisfied. Moreover, assume that $W(t,x)$ is even in x, that is,*

(W_4) *$W(n,-x) = W(n,x)$ for all $n \in \mathbb{Z}$, and $x \in \mathbb{R}^N$,*

then (1) has infinitely many nontrivial homoclinic solutions.

2. PRELIMINARIES

Let

$$S = \left\{\{u(n)\}_{n\in\mathbb{Z}}, u(n) \in \mathbb{R}, n \in \mathbb{Z}\right\},$$

$$E = \left\{u \in S; \sum_{n\in\mathbb{Z}} \left[(p(n+1)\Delta u(n), \Delta u(n)) + (L(n)u(n), u(n))\right] < \infty\right\},$$

for $u, v \in E$, let

$$\langle u, v\rangle = \sum_{n\in\mathbb{Z}} \left[(p(n+1)\Delta u(n), \Delta v(n)) + (L(n)u(n), v(n))\right].$$

Then E is a Hilbert space with the above inner product, and the corresponding norm is:

$$\|u\| = \sum_{n\in\mathbb{Z}} \left[(p(n+1)\Delta u(n), \Delta u(n)) + (L(n)u(n), u(n))\right], \quad u \in E.$$

As usual, for $1 \le p \le +\infty$, let

$$l^p(\mathbb{Z}, \mathbb{R}^N) = \left\{u \in S; \sum_{n\in\mathbb{Z}} |u(n)|^p < \infty\right\},$$

$$l^\infty(\mathbb{Z}, \mathbb{R}^N) = \left\{u \in S; \sup_{n\in\mathbb{Z}} |u(n)| < \infty\right\},$$

and their norms are defined by

$$\|u\|_q = \left(\sum_{n\in\mathbb{Z}} |u(n)|^q \right)^{\frac{1}{q}}, \quad \forall u \in l^q(\mathbb{Z}, \mathbb{R}^N);$$

$$\|u\|_\infty = \sup_{n\in\mathbb{Z}} |u(n)|, \quad \forall u \in l^\infty(\mathbb{Z}, \mathbb{R}^N);$$

respectively. Obviously, E is continuously embedded into $l^q(\mathbb{Z}, \mathbb{R}^N)$ for $2 \leq q \leq +\infty$, *i.e.*, there exists $C_q > 0$ such that

$$\|u\|_q \leq C_q \|u\|, \quad \forall u \in E. \tag{5}$$

For any $n_1, n_2 \in \mathbb{Z}$ with $n_1 < n_2$, we let $\mathbb{Z}(n_1, n_2) = [n_1, n_2] \cap \mathbb{Z}$, and for function $f : \mathbb{Z} \to \mathbb{R}$ and $a \in \mathbb{R}$, we set

$$\mathbb{Z}(f(n) \geq a) = \{n \in \mathbb{Z} : f(n) \geq a\}, \ \mathbb{Z}(f(n) \leq a) = \{n \in \mathbb{Z} : f(n) \leq a\}.$$

Define the functional $I : E \to \mathbb{R}$ by

$$I(u) = \tfrac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} W(n, u(n)). \tag{6}$$

Under the conditions of Theorem 2, we have

$$\begin{aligned} &I'(u)v = \\ &= \sum_{n\in\mathbb{Z}} [(p(n+1)\Delta u(n), \Delta v(n)) + (L(n)u(n), v(n)) - (\nabla W(n, u(n)), v(n))] \end{aligned} \tag{7}$$

for all $u, v \in E$. Moreover, I is a continuously Fréchet- differentiable functional defined on E, *i.e.*, $I \in C^1(E, \mathbb{R})$.

Observe that for all $u, v \in E$

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} [(p(n+1)\Delta u(n), \Delta v(n)) + (L(n)u(n), v(n)) - (\nabla W(n, u(n)), v(n))] = \\ &= \sum_{n\in\mathbb{Z}} [(-\Delta(p(n)\Delta u(n-1)) + L(n)u(n) + \nabla W(n, u(n)), v(n))]. \end{aligned} \tag{8}$$

It follows from (7) and (8) that $\langle I'(u), v\rangle = 0$ for all $v \in E$ if only if

$$\Delta [p(n)\Delta u(n-1)] - L(n)u(n) + \nabla W(n, u(n)) = 0, \quad \forall n \in \mathbb{Z}.$$

So, the critical points of I in E are the solutions of system (1) with $u(\pm\infty) = 0$.

We will obtain the critical points of I by the Mountain Pass Theorem and the Symmetric Mountain Pass Theorem. Therefore, we state the theorems precisely.

Lemma 4 ([21]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfying the Palais-Smale condition. If I satisfies the following conditions:*

(i) $I(0) = 0$,

(ii) *there exist constants $\rho, \beta > 0$ such that $I_{/\partial B_\rho(0)} \geq \beta$,*

(iii) *there exist* $e \in E \backslash \overline{B}_\rho(0)$ *such that* $I(e) \leq 0$.

Then I *possesses a critical value* $c \geq \beta$ *given by*

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0, g(1) = e\}.$$

Lemma 5 ([21]). *Let* E *be a real Banach space and* $I \in C^1(E, \mathbb{R})$ *satisfying the Palais-Smale condition. If* I *is even and satisfies the following conditions:*

(i) $I(0) = 0$,

(ii) *there exist constants* $\rho, \beta > 0$ *such that* $I_{/\partial B_\rho(0)} \geq \beta$,

(iii) *For each finite dimensional* $\tilde{E} \subset E$ *there is* $\gamma = \gamma(\tilde{E})$ *such that* $I(0) \leq 0$ *on* $\tilde{E} \backslash \rho_\gamma$.

Then I *possesses an unbounded sequence of critical values.*

3. PROOF OF THEOREMS

For the Proof of our Theorems we need some technical Lemmas.

Lemma 6. *Under the conditions of* Theorem 2, φ' *is compact, i.e.,* $\varphi'(u_k) \to \varphi'(u)$ *if* $u_k \rightharpoonup u$ *in* E, *where* $\varphi : E \to \mathbb{R}$ *is defined by*

$$\varphi(u) = \sum_{n\in\mathbb{Z}} W(n,u). \tag{9}$$

Proof. Assume that $u_k \rightharpoonup u$ *in* E. Then there exists a constant $M > 0$ such that

$$\|u_k\| \leq M \ \ \text{and} \ \ \|u\| \leq M$$

for $k \in \mathbb{N}$. In addition, from (W_2), for any $\epsilon > 0$, we can choose $T_0 > 0$ such that

$$|\nabla W(n,u)| \leq \epsilon\,|u|^{\mu-1}\,, \text{ and } |\nabla W(n,u_k)| \leq \epsilon\,|u_k|^{\mu-1}\,, \quad \forall\,|n| \geq T_0. \tag{10}$$

Consequently, for k large enough, and using Young inequality we have

$$\begin{aligned}
&|(\phi'(u_k) - \phi'(u))\,v| \leq \\
&\leq \sum_{n\in\mathbb{Z}} |\nabla W(n,u_k(n)) - \nabla W(n,u(n))|\,|v(n)| \\
&\leq \sum_{|n|\leq T_0} |\nabla W(n,u_k(n)) - \nabla W(n,u(n))|\,|v(n)| \\
&\quad + \sum_{|n|>T_0} |\nabla W(n,u_k(n))|\,|v(n)| + \sum_{|n|>T_0} |\nabla W(n,u(n))|\,|v(n)| \\
&\leq \epsilon\,\|v\|_\infty + \epsilon \sum_{|n|>T_0} |u_k(n)|^{\mu-1}\,|v| + \epsilon \sum_{|n|>T_0} |u(n)|^{\mu-1}\,|v| \\
&\leq \epsilon\gamma_\infty\,\|v\| + \epsilon \sum_{|n|>T_0} \left(\tfrac{\mu-1}{\mu}\,|u_k(n)|^\mu + \tfrac{1}{\mu}\,|v|^\mu\right) + \epsilon \sum_{|n|>T_0} \left(\tfrac{\mu-1}{\mu}\,|u(n)|^\mu + \tfrac{1}{\mu}\,|v|^\mu\right) \leq
\end{aligned}$$

$$\leq \epsilon\gamma_\infty \|v\| + \epsilon\frac{\mu-1}{\mu} \sum_{|n|>T_0} (|u_k|^\mu + |u|^\mu) + \epsilon\frac{2}{\mu}|v|^\mu . \tag{11}$$

Consequently, we obtain that

$$\begin{aligned} \|\varphi'(u_k) - \varphi'(u)\| &= \sup_{\|v\|=1} \left| \sum_{\mathbb{Z}} (\nabla W(n, u_k(n)) - \nabla W(n, u(n)), v(n)) \right| \\ &\leq \epsilon C_\infty + 2\epsilon (C_\mu M)^\mu \frac{\mu-1}{\mu} + \epsilon C_\mu^\mu \frac{2}{\mu}, \end{aligned} \tag{12}$$

which yields $\varphi'(u_k) \to \varphi'(u)$ as $u_k \rightharpoonup u$, that is, φ' is compact. □

Lemma 7. [14] *Under the assumption of* (W_1), *there exist two constants* $d_1 > 0,\ d_2 > 0$ *such that*

(i) $W(n,u) \leq d_1 |u|^\mu$ *for* $n \in \mathbb{Z}$ *and* $0 < |u| < 1$,
(ii) $W(n,u) \geq d_2 |u|^\mu$ *for* $n \in \mathbb{Z}$ *and* $|u| \geq 1$.

Lemma 8. *Under the condition of Theorem 2, I satisfies the Palais-Smale condition.*

Proof. Assume that $(u_k)_{k\in\mathbb{N}} \in E$ is a sequence such that $(I(u_k))$ is bounded and $I'(u_k) \to 0$ as $k \to \infty$. Then there exists a constant $C_1 > 0$ such that

$$|I(u_k)| \leq C_1 \ \ and \ \ \|I'(u_k)\| \leq C_1 \tag{13}$$

for every $k \in \mathbb{N}$. We first prove that (u_k) is bounded in E. By (6), (7), (W_1) and(W_2) , we obtain

$$\begin{aligned} & C_1 + \frac{C_1}{\mu}\|u_k\| \geq \\ & \geq I(u_k) - \frac{1}{\mu}\|I'(u_k)\| u_k \\ & = (\tfrac{1}{2} - \tfrac{1}{\mu}) \|u_k\|^2 + \sum_{n\in\mathbb{Z}} \left[W(n, u_k(n)) - \tfrac{1}{\mu}(\nabla W(n, u_k(n)), u_k(n)) \right] \\ & \geq (\tfrac{1}{2} - \tfrac{1}{\mu}) \|u_k\|^2 , \ k \in \mathbb{N}. \end{aligned} \tag{14}$$

Since $\mu > 2$, the inequality (14) shows that (u_k) is bounded in E. So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u$ in E, which yields that

$$(I'(u_k) - I'(u))\,(u_k - u) \to 0, \quad as\ k \to \infty. \tag{15}$$

Moreover, according to Lemma 6, we have

$$\phi'(u_k) \to \phi'(u), \quad as\ k \to \infty. \tag{16}$$

It follows from the definition of I that

$$\begin{aligned} & (I'(u_k) - I'(u))(u_k - u) = \\ & = \|u_k - u\|^2 - \sum_{n\in\mathbb{Z}} (\nabla W(n, u_k(n)) - \nabla W(n, u(n)), u_k - u) . \end{aligned} \tag{17}$$

combining (15), (16) with (17) we obtain that $u_k \to u$ in E. So the proof is complete. □

Proof of Theorem 2.

We will prove the existence of a nontrivial critical point of I.We have already shown that $I \in C^1(E,\mathbb{R})$, $I(0)=0$ and I satisfies the Palais-Smale condition. Hence it suffices to prove that I satisfies (ii) and (iii) of Lemma 4.

Let $\rho = \frac{1}{C_2}$, where C_2 is defined in (5). Assume that $u \in E$ with $\|u\| \leq \rho$, we have $\|u\|_\infty \leq \frac{1}{C_2}.C_2 = 1$. In consequence, combining this with (i) of Lemma 6, we obtain that

$$\begin{aligned} I(u) &= \tfrac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} W(n,u(n)) \\ &\geq \tfrac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} W(n, \tfrac{u(n)}{|u(n)|})\,|u(n)|^\mu \\ &\geq \tfrac{1}{2}\,\|u\|^2 - \varrho \sum_{\mathbb{Z}} |u(n)|^2 \\ &\geq \left(\tfrac{1}{2} - \varrho C_2^2\right)\|u\|^2, \quad \|u\| \leq \rho, \end{aligned} \tag{18}$$

where $\varrho = \sup\{W(n,u) : n \in \mathbb{Z}, |u| = 1\}$. Since $\varrho < \frac{1}{2C_2^2}$, then we get

$$I_{/\partial B_\rho(0)} \geq \frac{1}{C_2^2} - 2\varrho = \alpha > 0.$$

It remains to prove that there exists $e \in E$ such that $\|e\| > \rho$ and $I(e) \leq 0$, where ρ is defined above. Take some $u \in E$ such that $\|u\| = 1$. Then there exists a nonempty integer interval $I \subset \mathbb{Z}$ such that $u(n) \neq 0$ for $n \in I$. take $\sigma > 0$ such that $\sigma\,|u(n)| \geq 1$ for $n \in I$. Then, we obtain

$$\begin{aligned} I(\sigma u) &= \tfrac{\sigma^2}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} W(n,\sigma u(n)) \\ &\leq \tfrac{\sigma^2}{2}\|u\|^2 - \sigma^\mu \sum_{n\in I} W\left(n, \tfrac{u(n)}{|u(n)|}\right)|u(n)|^\mu \\ &\leq \tfrac{\sigma^2}{2}\|u\|^2 - m\sigma^\mu \sum_{n\in I} |u(n)|^\mu, \end{aligned} \tag{19}$$

where $m = \min\{W(n,u) : n \in I, |u| = 1\}$. Since $\mu > 2$, (19) implies that $I(\sigma u) < 0$ for some $\sigma > 0$ with $\sigma\,|u(n)| \geq 1$ for $n \in I$ and $\|\sigma u\| > \rho$. By Lemma 4, I possesses a critical value $c \geq \beta > 0$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1],E) : g(0) = 0, g(1) = e\}.$$

Hence there is $u \in E$ such that $I(u) = c, I'(u) = 0$. □

Proof of Theorem 3.

We have already known that $I \in C^1(E,\mathbb{R})$, $I(0) = 0$, I satisfies the Palais-Smale condition and we have I is even. To apply the Symmetric mountain Pass Theorem, it suffices to prove that I satisfies the conditions (iii) of Lemma 5. (ii) is identically the same as in Theorem 2, so it is already proved.

Let $\tilde{E}$ be a finite dimensional subspace of E. Since all norms of a finite dimensional normed space are equivalent, so there is a constant $c > 0$ such that

$$\|u\| < c\,\|u\|_\infty\,, \quad for\ u \in \tilde{E}. \tag{20}$$

Assume that $\dim \tilde{E} = m$ and $u_1, u_2, \ldots, u_m$ is the basis of $\tilde{E}$ such that

$$\langle u_i, u_j\rangle = \begin{cases} c^2 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \quad i, j = 1, 2, ..., m. \end{cases}$$

Since $u_i \in E$, we can choose an integer $\bar{T}$ such that

$$|u_i(n)| < \tfrac{1}{m}, \quad |n| > \bar{T}, \quad i = 1, 2, ..., m. \tag{21}$$

Set $\Sigma = \left\{u \in \tilde{E}, \|u\| = c\right\}$. Then for $u \in \Sigma$, there exist $\lambda_i \in \mathbb{R}, i = 1, 2, ..., m$ such that

$$u(n) = \sum_{i=1}^{m} \lambda_i u_i(n), \quad for\ n \in \mathbb{Z}, \tag{22}$$

it follows that

$$c^2 = \|u\|^2 = \sum_{i=1}^{m} {\lambda_i}^2 \langle u_i, u_i\rangle = c^2 \sum_{i=1}^{m} {\lambda_i}^2, \tag{23}$$

which implies that $|\lambda_i| \leq 1$ for $i = 1, 2, ..., m$. Hence, for $u \in \Sigma$, let $|u(n_0)| = \|u\|_\infty$, then by (20) and (22) we have

$$1 \leq \|u\|_\infty = |u(n_0)| \leq \sum_{i=1}^{m} |\lambda_i|\,|u_i(n_0)|\,, \quad u \in \Sigma. \tag{24}$$

This shows that there exists $i_0 \in \{1, 2, ..., m\}$ such that $|u_{i^0}(n_0)| \geq \frac{1}{m}$, which together with (21), implies that $|n_0| \leq \overline{T}$. For any $u \in E$, it follows from (21), (22), (24) and (W_2), we have for $u \in \sum$ and $\sigma > 1$

$$\begin{aligned} I(\sigma u) &= \tfrac{\sigma^2}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} W(n, \sigma u(n)) \\ &\leq \tfrac{\sigma^2}{2}\|u\|^2 - \sigma^\mu \sum_{n\in\mathbb{Z}} W(n, u(n)) \\ &= \tfrac{\sigma^2}{2}\|u\|^2 - \sigma^\mu \sum_{|n|>\overline{T}} W(n, u(n)) - \sigma^\mu \sum_{|n|\leq\overline{T}} W(n, u(n)) \\ &\leq \tfrac{\sigma^2}{2}\|u\|^2 - d_2\sigma^\mu\,|u(n_0)|\,. \end{aligned} \tag{25}$$

Since $\mu > 2$, we deduce that there is $\sigma_0 = \sigma(\widetilde{E}) > 1$ such that

$$I(\sigma u) < 0,\ for\ u \in \sum\ and\ \sigma \geq \sigma_0.$$

That is

$$I(u) < 0, \; for \; u \in \widetilde{E} \; and \; \|u\| \geq c\sigma_0.$$

This shows that (iii) of Lemma 5 holds. By Lemma 5, I possesses an unbounded sequence $\{d_k\}_{k\in\mathbb{N}}$ of critical values with $d_k = I(u_k)$, where u_k is such that $I'(u_k) = 0$ for $k = 1, 2, ...$

From (6), we have

$$\begin{aligned} \tfrac{1}{2}\|u_k\|^2 &= d_k + \sum_{n\in\mathbb{Z}} W(n, u_k(n)) \\ &\geq d_k, \end{aligned} \tag{26}$$

since $\{d_k\}_{k\in\mathbb{N}}$ is unbounded, then $\{\|u_k\|\}_{k\in\mathbb{N}}$ is unbounded. The proof is complete. □

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