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ITERATES OF A MODIFIED BERNSTEIN TYPE OPERATOR

TEODORA CĂTINAŞ*

Abstract. Using the weakly Picard operators technique and the contraction principle, we study the convergence of the iterates of some modified Bernstein type operators.

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1. INTRODUCTION

Let (X, d) be a metric space and $A : X \to X$ an operator. We denote by (see, *e.g.*, [21])

$F_A := \{x \in X \mid A(x) = x\}$-the fixed points set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y,\ Y \neq \emptyset\}$-the family of the nonempty invariant subsets of A;

$A^0 := 1_X,\ A^1 := A,\ \ldots,\ A^{n+1} := A \circ A^n,\quad n \in \mathbb{N}.$

DEFINITION 1. *The operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that:*
(i) $F_A = \{x^*\}$;
(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

DEFINITION 2. *The operator A is a weakly Picard operator if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.*

DEFINITION 3. *If A is a weakly Picard operator then we consider the operator A^∞, $A^\infty : X \to X$, defined by*

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

THEOREM 4. [21] *An operator A is a weakly Picard operator if and only if there exists a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, such that*

(a) $X_\lambda \in I(A),\ \forall \lambda \in \Lambda$;

*Babeş-Bolyai University, Faculty of Mathematics and Computer Science, Str. M. Kogălniceanu Nr. 1, RO-400084 Cluj-Napoca, Romania, E-mail: tcatinas@math.ubbcluj.ro.

(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator,* $\forall \lambda \in \Lambda$.

2. ITERATES OF A MODIFIED BERNSTEIN OPERATOR

We study the convergence of the iterates of the modified Bernstein type operators, given in (1), using the weakly Picard operators technique and the contraction principle. This approach for some other linear and positive operators lead to similar results, for example, in [1], [2], [6]-[9], [22]-[25].

The limit behavior for the iterates of some classes of positive linear operators were also studied, for example, in [10]-[20]. In the papers [10]-[12] there were introduced new methods (*e.g.*, Korovkin type technique) for the study of the asymptotic behavior of the iterates of positive linear operators, positive linear operators preserving the affine functions and defined on the space of bounded real-valued functions on $[0, 1]$. These techniques enlarge the class of operators for which the limit of the iterates can be computed. In [13], [14], [15] there were proposed methods to determine the degree of convergence for the iterates of certain positive linear operators towards the first Bernstein operator. In [16], [20] there were introduced new techniques (infinite products, rates of convergence), based on the results from [17], in order to prove that infinite products of certain positive linear operators weakly converge to the first Bernstein operator.

A modified Bernstein operator. Let f be an integrable function on $[0, 1]$. The modified Bernstein operator, introduced by J. L. Durrmeyer in [5], is defined by (see [3], [4])

$$(M_n f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt, \quad n \geq 1 \tag{1}$$

with

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{for } 0 \leq k \leq n. \tag{2}$$

For $f(x) = x^m$, with $m \leq n$, based on the properties of beta function, we get [3]

$$(M_n f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\tfrac{(k+m)!}{k!}\tfrac{n!}{(n+m+1)!},$$

or

$$(M_n f)(x) = \tfrac{(n+1)!}{(n+m+1)!}\sum_{k=0}^{n}\tbinom{m}{k}\tfrac{m!}{k!}\tfrac{n!}{(n-k)!}x^k.$$

Remark 5. The modified Bernstein polynomials (1) are obtained by the classical Bernstein polynomials

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\tfrac{k}{n}\right), \tag{3}$$

by replacing $f(\frac{k}{n})$ with $(n+1)\int_0^1 p_{n,k}(t)f(t)dt$. □

THEOREM 6. [4] *The operator M_n has the following properties:*

1) *it is linear and positive;*
2) *it preserves the constants;*
3) *it preserves the degree of polynomials when their degrees are $\leq n$;*
4) *it is a contraction on $L^p[0,1]$, $p \geq 1$.*

Next we give the main result of this note.

THEOREM 7. *The operator M_n is a weakly Picard operator and*

$$M_n^\infty(f) = \int_0^1 f(t)dt. \tag{4}$$

Proof. We define

$$X_\alpha = \left\{ f \in C[0,1] : \int_0^1 f(t)dt = \alpha \right\}, \quad \text{with } \alpha \in \mathbb{R},$$

and denote by

$$F_\alpha(x) := \alpha.$$

We have that X_α is a closed subsets of $C[0,1]$ and $C[0,1] = \underset{\alpha\in\mathbb{R}}{\cup} X_\alpha$ is a partition of $C[0,1]$.

We have that X_α is an invariant subset of M_n, which follows by linearity of the modified Bernstein operators and Theorem 6.

By Theorem 6 we know that M_n is a contraction.

On the other hand, as M_n reproduces constant functions it follows $F_\alpha \in X_\alpha$ is a fixed point of M_n.

From the contraction principle, it follows that α is the unique fixed point of M_n and $M_n|_{X_\alpha}$ is a Picard operator, with

$$M_n^\infty(f) = \int_0^1 f(t)dt.$$

By Theorem 4 it follows that the operators M_n, $n \geq 1$ are weakly Picard operators. □

REMARK 8. In [24] was shown that the classical Bernstein operator B_n given in (3) is a weakly Picard operator with

$$B_n^\infty(f)(x) = f(0) + [f(1) - f(0)]x. \qquad \square$$

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