POINTWISE BEST COAPPROXIMATION
IN THE SPACE OF BOCHNER INTEGRABLE FUNCTIONS

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Abstract. Let $X$ be a Banach space, $G$ be a closed subset of $X$, and $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space. In this paper we present some results on coproximinality (pointwise coproximinality) of $L^p(\mu, G)$, $1 \leq p \leq \infty$, in $L^p(\mu, X)$.

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1. INTRODUCTION

Let $G$ be a nonempty subset of a Banach space $X$ and let $x \in X$. An element $g_0$ in $G$ satisfying

$$\|x - g_0\| \leq \|x - g\|, \text{ for every } g \in G$$

is called a best approximation to $x$ from $G$. The set $G$ is called proximinal in $X$ if every element $x$ in $X$ has a best approximation from $G$.

Another kind of approximation, called best coapproximation was introduced by Franchetti and Furi [1], who considered those elements $g_0 \in G$, for which

$$\|g - g_0\| \leq \|x - g\|, \text{ for every } g \in G.$$ (1)

An element $g_0$ in $G$ satisfying (1) is called a best coapproximation to $x$ from $G$. $G$ is called coproximinal in $X$ if every element $x$ in $X$ has a best coapproximation from $G$.

Several papers have been devoted for studying when the space $L^p(\mu, G)$ is proximinal in $L^p(\mu, X)$ see for example [5], [9]–[12]. As a counter part of this problem is the problem of coproximinality of $L^p(\mu, G)$ in $L^p(\mu, X)$, which has been recently studied by some authors [2]–[4], [7], and it will be the object of this paper.

Throughout the whole paper, we always suppose that $(X, \|\cdot\|)$ is a Banach space and $(\Omega, \Sigma, \mu)$ is a given non trivial $\mu(\Omega) \neq \{0, \infty\}$ $\sigma$-finite measure space. We write $L(\mu, X)$ to denote the space of all $X$-valued strongly measurable functions, $L^p(\mu, X)$, $1 \leq p < \infty$, to denote the space of $p$-Bochner integrable functions defined on $\Omega$ with values in $X$ and, for $p = \infty$, $L^\infty(\mu, X)$.

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to denote the Banach space of all essentially bounded strongly measurable functions on $\Omega$ with values in $X$, endowed with the usual norm

$$\|f\|_{\infty} = \text{ess-sup} \|f(s)\|.$$ 

Finally, $\mathbb{N}$ stands for the set of natural numbers.

Finite measure spaces $(\Omega, \Sigma, \mu)$ in [3], [4], [7] played an important role in obtaining results on the coproximinality of $L^p(\mu, G)$ in $L^p(\mu, X)$. The purpose of the present paper is to further the topics using any $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$. The obtained results improve those in [3], [4], [7] and our methods are not only distinct but also seem to be simpler.

We start by recalling a few definitions.

Let $f : \Omega \to X$ be a function. Then

1. $f$ is called simple if its range contains only finitely many points $x_1, x_2, \ldots, x_n$ in $X$ and $f^{-1}(x_i)$ is measurable for $i = 1, 2, \ldots, n$. In this case we write $f = \sum_{i=1}^{n} x_i \chi_{E_i}$, where $\chi_{E_i}$ is the characteristic function of the set $E_i = f^{-1}(x_i)$.

2. $f$ is called strongly measurable if there exists a sequence $(f_n)$ of simple functions with $\lim \|f_n(s) - f(s)\| = 0$ for almost all $s \in \Omega$.

2. BEST COAPPROXIMATION IN $L^p(\mu, X), 1 \leq p \leq \infty$.

Definition 1. Let $f \in L(\mu, X)$ and $D \subset L(\mu, X)$. An element $h$ in $D$ is called a pointwise best coapproximation to $f$ from $D$ if for all $\varphi \in D$, we have

$$\|h(s) - \varphi(s)\| \leq \|f(s) - \varphi(s)\|, \text{ for almost all } s \in \Omega.$$ 

The notation of “pointwise coproximinal” is defined accordingly.

Proposition 2. Let $G$ be a closed subspace of $X$, $1 \leq p \leq \infty$, $f \in L^p(\mu, X)$, and $h \in L(\mu, G)$. If $h$ is a pointwise best coapproximation to $f$, then $h$ belongs to $L^p(\mu, G)$ and it is a best coapproximation to $f$ from $L^p(\mu, G)$.

Proof. Since $h(s)$ is a best coapproximation to $f(s)$ for almost all $s \in \Omega$, we have $\|h(s)\| \leq \|f(s)\|$ for almost all $s \in \Omega$. Then $\|h\|_p \leq \|f\|_p$ and so $h \in L^p(\mu, G)$. It is clear that $\|h - \varphi\|_p \leq \|f - \varphi\|_p$ for all $\varphi \in L^p(\mu, G)$.

Remark 3. Let $f$ be an element of $L^\infty(\mu, G)$. We recall that

$$\|f\|_{\infty} = \text{ess-sup} \|f(s)\|$$

where

$$\text{ess-sup} \|f(s)\| = \inf_{E \in \mathcal{E}} \sup_{s \in E^c} \|f(s)\|.$$ 

Here we put

$$\mathcal{E} = \{E \in \Sigma : \mu(E) = 0\}$$

and for $A \subset \Omega$ we let $A^c = \Omega \setminus A$. If $f$ is essentially bounded, one can show that there exists $E \in \mathcal{E}$ such that $\|f\|_{\infty} = \sup_{s \in E^c} \|f(s)\|$.
In [4, Th.2.2], It was shown that for a closed subspace \( G \) of \( X \), the coproximinality of \( L^\infty(\mu, G) \) in \( L^\infty(\mu, X) \) implies the coproximinality of \( G \) in \( X \). In fact there was a flaw in the proof given as follows:

For \( x \in X \) put \( f_x(s) = x, s \in \Omega \). Assume that for some \( h \in L^\infty(\mu, G) \) and \( x \in X \) we have \( \|h - f_g\|_\infty \leq \|f_x - f_g\|_\infty \) for all \( g \in G \). The author considered the existence of \( s_0 \in \Omega \) such that \( \|h(s_0) - g\| \leq \|x - g\| \) for all \( g \in G \). However, it does not follow automatically from the assumption. The author should have proved the existence of such an \( s_0 \). So the theorem may need more conditions to be correct as shown in the following Proposition.

**Proposition 4.** Let \( G \) be a separable subspace of \( X \). If \( L^\infty(\mu, G) \) is coproximinal in \( L^\infty(\mu, X) \) , then \( G \) is coproximinal in \( X \).

**Proof.** For \( x \in X \) put \( f_x(s) = x, s \in \Omega \). Then \( f_x \in L^\infty(\mu, X) \), so there exists \( h \in L^\infty(\mu, G) \) such that

\[
\|\varphi - h\|_\infty \leq \|f_x - \varphi\|_\infty
\]

for all \( \varphi \in L^\infty(\mu, G) \). Taking \( \varphi = f_g \), it follows

\[
\|h - f_g\|_\infty \leq \|f_x - f_g\|_\infty = \|x - g\|
\]

for all \( g \in G \). That is, for every \( g \in G \)

\[
\inf_{E \in \mathcal{K}} \sup_{s \in E} \|h(s) - g\| \leq \|x - g\|.
\]

Taking \( E_g \in \mathcal{K} \) such that

\[
\|h - f_g\|_\infty = \sup_{s \in E_g} \|h(s) - g\|
\]

it follows that

\[
\|h(s) - g\| \leq \|x - g\| \quad \text{for all} \quad s \in E_g^c.
\]

Thus for all \( g \in G \), there exists \( E_g \in \mathcal{K} \), \( \|h(s) - g\| \leq \|x - g\| \) for all \( s \in E_g^c \). Let \( G' \) be a countable dense subset of \( G \). Then for every \( g' \in G' \), there exists \( E_{g'} \in \mathcal{K} \) such that \( \|h(s) - g'\| \leq \|x - g'\| \) for all \( s \in E_{g'}^c \). If \( \cap \{ E_{g'} : g' \in G' \} \) is empty, then \( \cup \{ E_{g'} : g' \in G' \} = \Omega \) and so \( \mu(\Omega) = 0 \) which is false, since we have supposed the measure space is non-trivial. Thus there exists \( s_0 \in \cap \{ E_{g'} : g' \in G' \} \) such that \( \|h(s_0) - g'\| \leq \|x - g'\| \) for all \( g' \in G' \). Therefore \( \|h(s_0) - g\| \leq \|x - g\| \) for all \( g \in G \) as \( G' \) is dense in \( G \). \( \square \)

**Proposition 5.** Let \( (\Omega, \Sigma, \mu) \) be a measure space such that there exists \( A \in \Sigma \) with \( 0 < \mu(A) < \infty \), \( X \) be a Banach space, \( G \) be a closed subspace of \( X \) and \( 1 \leq p < \infty \). If \( L^p(\mu, G) \) is coproximinal in \( L^p(\mu, X) \), then \( G \) is coproximinal in \( X \).

**Proof.** Let \( x \in X \). Define \( f_x \in L^p(\mu, X) \) by

\[
f_x = \mu(A)^{\frac{1}{p} - 1} x \chi_A.
\]
By hypothesis, there exists \( h \in L^p(\mu, G) \) such that
\[
\|h - \varphi\|_p \leq \|f_x - \varphi\|_p \quad \text{for all } \varphi \in L^p(\mu, G).
\]
For \( g \in G \) let \( \varphi_g \in L^p(\mu, G) \) be given by
\[
\varphi_g = \mu(A)^{1/p - 1} g \chi_A.
\]
Then
\[
\|f_x - \varphi_g\|_p = \left( \int_{\Omega} \| f_x(s) - \varphi_g(s) \|^p \, d\mu(s) \right)^{1/p}
\]
(3)
\[
= \mu(A)^{1/p - 1} \left( \int_{\Omega} \| (x - g) \|^p \chi_A(s) \, d\mu(s) \right)^{1/p}
\]
\[
= \mu(A)^{1/p - 1} \| (x - g) \|.
\]
Now, let
\[
g_0 = \int_{\Omega} h(s) \chi_A(s) \, d\mu(s) \in G.
\]
If \( 1 < p < \infty \), then
\[
\|g - \mu(A)^{-1/p} g_0\| = 
\]
\[
= \left\| \mu(A_{k_0})^{-1} \int_{\Omega} g \chi_A(s) \, d\mu(s) - \mu(A)^{-1/p} \int_{\Omega} h(s) \, d\mu(s) \right\|
\]
\[
= \left\| \mu(A)^{-1/p} \int_{\Omega} (\varphi_g(s) - h(s)) \chi_A(s) \, d\mu(s) \right\|
\]
\[
\leq \mu(A)^{-1/p} \int_{\Omega} \| \varphi_g(s) - h(s) \| \chi_A(s) \, d\mu(s)
\]
\[
\leq \mu(A)^{-1/p} \left( \int_{\Omega} \| \varphi_g(s) - h(s) \|^p \, d\mu(s) \right)^{1/p} \mu(A)^{1-1/p} \quad \text{(Hölder Inequality)}
\]
\[
= \mu(A)^{1-2/p} \| \varphi_g - h \|
\]
\[
\leq \mu(A)^{1-2/p} \| \varphi_g - f_x \| \quad \text{(by (2))}
\]
\[
= \|x - g\|, \quad \text{(by (3))}
\]
which show that \( \mu(A)^{-1/p} g_0 \) is a best coapproximation element to \( x \) in \( G \).

If \( p = 1 \), then, instead of Hölder Inequality, the following inequality ca be used:
\[
\|g - \mu(A)^{-1} g_0\| \leq \mu(A)^{-1} \int_{\Omega} \| \varphi_g(s) - h(s) \| \chi_A(s) \, d\mu(s)
\]
\[
\leq \mu(A)^{-1} \int_{\Omega} \| \varphi_g(s) - h(s) \| \, d\mu(s)
\]
\[
= \mu(A)^{-1} \| \varphi_g - h \| \leq \mu(A)^{-1} \| \varphi_g - f_x \|
\]
\[
= \|x - g\|. \quad \square
\]
Remark 6. If by a trivial space one understands that \( \mu(\Sigma) \neq \{0, \infty\} \), i.e., \( \mu \) takes only the values 0 and \( \infty \), then in a non-trivial measure space \((\Omega, \Sigma, \mu)\) there exists always \( A \in \Sigma \) with \( 0 < \mu(A) < \infty \).

Corollary 7. Let \( G \) be a separable subspace of \( X \) and \( 1 \leq p \leq \infty \). If \( L^p(\mu, G) \) is coproximinal in \( L^p(\mu, X) \), then \( G \) is coproximinal in \( X \).

Proof. The proof follows from Proposition 4 and Proposition 5. \( \Box \)

REFERENCES


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