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POINTWISE BEST COAPPROXIMATION IN THE SPACE OF BOCHNER INTEGRABLE FUNCTIONS

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Abstract. Let X be a Banach space, G be a closed subset of X, and (Ω, Σ, μ) be a σ-finite measure space. In this paper we present some results on coproximinality (pointwise coproximinality) of $L^p(\mu, G)$, $1 \leq p \leq \infty$, in $L^p(\mu, X)$.

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1. INTRODUCTION

Let G be a nonempty subset of a Banach space X and let $x \in X$. An element g_0 in G satisfying

$$\|x - g_0\| \leq \|x - g\|, \text{ for every } g \in G$$

is called a best approximation to x from G. The set G is called proximinal in X if every element x in X has a best approximation from G.

Another kind of approximation, called best coapproximation was introduced by Franchetti and Furi [1], who considered those elements $g_0 \in G$, for which

$$\|g - g_0\| \leq \|x - g\|, \quad \text{for every } g \in G. \tag{1}$$

An element g_0 in G satisfying (1) is called a best coapproximation to x from G. G is called coproximinal in X if every element x in X has a best coapproximation from G.

Several papers have been devoted for studying when the space $L^p(\mu, G)$ is proximinal in $L^p(\mu, X)$ see for example [5], [9]–[12]. As a counter part of this problem is the problem of coproximinality of $L^p(\mu, G)$ in $L^p(\mu, X)$,which has been recently studied by some authors [2]–[4], [7], and it will be the object of this paper.

Throughout the whole paper, we always suppose that $(X, \|\cdot\|)$ is a Banach space and (Ω, Σ, μ) is a given non trivial($\mu(\Omega) \neq \{0, \infty\}$) σ-finite measure space. We write $L(\mu, X)$ to denote the space of all X-valued strongly measurable functions, $L^p(\mu, X)$, $1 \leq p < \infty$, to denote the space of p-Bochner integrable functions defined on Ω with values in X and, for $p = \infty$, $L^{\infty}(\mu, X)$

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to denote the Banach space of all essentially bounded strongly measurable functions on Ω with values in X, endowed with the usual norm

$$\|f\|_{\infty} = \text{ess-sup}\, \|f(s)\| .$$

Finally, $\mathbb{N}$ stands for the set of natural numbers.

Finite measure spaces (Ω, Σ, μ) in [3], [4], [7] played an important role in obtaining results on the coproximinality of $L^p(\mu, G)$ in $L^p(\mu, X)$. The purpose of the present paper to further the topics using any σ-finite measure space (Ω, Σ, μ) . The obtained results improve those in [3], [4], [7] and our methods are not only distinct but also seem to be simpler.

We start by recalling a few definitions.

Let $f : \Omega \to X$ be a functions. Then

(1) f is called simple if its range contains only finitely many points $x_1, x_2, \ldots,$ x_n in X and $f^{-1}(x_i)$ is measurable for $i = 1, 2, \ldots, n$. In this case we write $f = \Sigma_{i=1}^{n} x_i \chi_{E_i}$,where χ_{E_i} is the characteristic function of the set $E_i = f^{-1}(x_i)$.

(2) f is called strongly measurable if there exists a sequence (f_n) of simple functions with $\lim \|f_n(s) - f(s)\| = 0$ for almost all $s \in \Omega$.

2. BEST COAPPROXIMATION IN $L^p(\mu, X)$, $1 \leq p \leq \infty$.

Definition 1. *Let $f \in L(\mu, X)$ and $D \subset L(\mu, X)$. An element h in D is called a pointwise best coapproximation to f from D if for all $\varphi \in D$, we have*

$$\|h(s) - \varphi(s)\| \leq ||f(s) - \varphi(s)||, \text{ for almost all } s \in \Omega.$$

The notation of "pointwise coproximinal" is defined accordingly.

Proposition 2. *Let G be a closed subspace of X , $1 \leq p \leq \infty$, $f \in L^p(\mu, X)$, and $h \in L(\mu, G)$. If h is a pointwise best coapproximation to f, then h belongs to $L^p(\mu, G)$ and it is a best coapproximation to f from $L^p(\mu, G)$.*

Proof. Since $h(s)$ is a best coapproximation to $f(s)$ for almost all $s \in \Omega$, we have $\|h(s)\| \leq ||f(s)||$ for almost all $s \in \Omega$. Then $\|h\|_p \leq ||f||_p$ and so $h \in L^p(\mu, G)$. It is clear that $\|h - \varphi\|_p \leq ||f - \varphi||_p$ for all $\varphi \in L^p(\mu, G)$. □

Remark 3. Let f be an element of $L^{\infty}(\mu, G)$. We recall that

$$\|f\|_{\infty} = \text{ess-sup}\, \|f(s)\|$$

where

$$\text{ess-sup}\, \|f(s)\| = \inf_{E \in \aleph} \sup_{s \in E^c} \|f(s)\| .$$

Here we put

$$\aleph = \{E \in \Sigma : \mu(E) = 0\}$$

and for $A \subset \Omega$ we let $A^c = \Omega \setminus A$. If f is essentially bounded, one can show that there exists $E \in \aleph$ such that $\|f\|_{\infty} = \sup_{s \in E^c} \|f(s)\|$.

In [4, Th.2.2], It was shown that for a closed subspace G of X, the coproximinality of $L^{\infty}(\mu, G)$ in $L^{\infty}(\mu, X)$ implies the coproximinality of G in X. In fact there was a flaw in the proof given as follows:

For $x \in X$ put $f_x(s) = x, s \in \Omega$. Assume that for some $h \in L^{\infty}(\mu, G)$ and $x \in X$ we have $\|h - f_g\|_{\infty} \leq \|f_x - f_g\|_{\infty}$ for all $g \in G$. The author considered the existence of $s_0 \in \Omega$ such that $\|h(s_0) - g\| \leq \|x - g\|$ for all $g \in G$. However, it does not follow automatically from the assumption. The author should have proved the existence of such an s_0. So the theorem may need more conditions to be correct as shown in the following Proposition.

Proposition 4. *Let G be a separable subspace of X. if $L^{\infty}(\mu, G)$ is coproximinal in $L^{\infty}(\mu, X)$, then G is coproximinal in X.*

Proof. For $x \in X$ put $f_x(s) = x, s \in \Omega$. Then $f_x \in L^{\infty}(\mu, X)$, so there exists $h \in L^{\infty}(\mu, G)$ such that

$$\|\varphi - h\|_{\infty} \leq \|f_x - \varphi\|_{\infty}$$

for all $\varphi \in L^{\infty}(\mu, G)$. Taking $\varphi = f_g$, it follows

$$\|h - f_g\|_{\infty} \leq \|f_x - f_g\|_{\infty} = \|x - g\|$$

for all $g \in G$. That is, for every $g \in G$

$$\inf_{E \in \aleph} \sup_{s \in E^c} \|h(s) - g\| \leq \|x - g\|.$$

Taking $E_g \in \aleph$ such that

$$\|h - f_g\|_{\infty} = \sup_{s \in E_g^c} \|h(s) - g\|,$$

it follows that

$$\|h(s) - g\| \leq \|x - g\| \text{ for all } s \in E_g^c.$$

Thus for all $g \in G$, there exists $E_g \in \aleph$, $\|h(s) - g\| \leq \|x - g\|$ for all $s \in E_g^c$. Let G' be a countable dense subset of G. Then for every $g' \in G'$, there exists $E_{g'} \in \aleph$ such that $\|h(s) - g'\| \leq \|x - g'\|$ for all $s \in E_{g'}^c$. If $\cap\{E_{g'}^c : g' \in G'\}$ is empty, then $\cup\{E_{g'} : g' \in G'\} = \Omega$ and so $\mu(\Omega) = 0$ which is false, since we have supposed the measure space is non-trivial. Thus there exists $s_0 \in \cap\{E_{g'}^c : g' \in G'\}$ such that $\|h(s_0) - g'\| \leq \|x - g'\|$ for all $g' \in G'$. Therefore $\|h(s_0) - g\| \leq \|x - g\|$ for all $g \in G$ as G' is dense in G. □

Proposition 5. *Let (Ω, Σ, μ) be a measure space such that there exists $A \in \Sigma$ with $0 < \mu(A) < \infty$, X be a Banach space, G be a closed subspace of X and $1 \leq p < \infty$. If $L^p(\mu, G)$ is coproximinal in $L^p(\mu, X)$, then G is coproximinal in X.*

Proof. Let $x \in X$. Define $f_x \in L^p(\mu, X)$ by

$$f_x = \mu(A)^{\frac{1}{p}-1} x \, \chi_A.$$

By hypothesis, there exists $h \in L^p(\mu, G)$ such that

$$\|h - \varphi\|_p \le \|f_x - \varphi\|_p \text{ for all } \varphi \in L^p(\mu, G). \tag{2}$$

For $g \in G$ let $\varphi_g \in L^p(\mu, G)$ be given by

$$\varphi_g = \mu(A)^{\frac{1}{p}-1} g\, \chi_A.$$

Then

$$\begin{aligned} \|f_x - \varphi_g\|_p &= \left(\int_\Omega \| f_x(s) - \varphi_g(s)\|^p \, d\mu(s)\right)^{1/p} \\ &= \mu(A)^{\frac{1}{p}-1} \left(\int_\Omega \| (x-g)\|^p \, \chi_A(s) d\mu(s)\right)^{1/p} \\ &= \mu(A)^{\frac{2}{p}-1} \| (x-g)\|. \end{aligned} \tag{3}$$

Now, let

$$g_0 = \int_\Omega h(s)\chi_A(s)d\mu(s) \in G.$$

If $1 < p < \infty$, then

$$\begin{aligned} &\left\|g - \mu(A)^{-1/p} g_0\right\| = \\ &= \left\|\mu(A_{k_0})^{-1}\int_\Omega g\ \chi_A(s)d\mu(s) - \mu(A)^{-1/p}\int_\Omega h(s)\ d\mu(s)\right\| \\ &= \left\|\mu(A)^{-1/p}\int_\Omega (\varphi_g(s) - h(s))\, \chi_A(s)d\mu(s)\right\| \\ &\le \mu(A)^{-1/p}\int_\Omega \|\varphi_g(s) - h(s)\ \| \,\chi_A(s)d\mu(s) \\ &\le \mu(A)^{-1/p}\left(\int_\Omega \|\varphi_g(s) - h(s)\|^p \, d\mu(s)\right)^{1/p} \mu(A)^{1-1/p} \qquad \text{(Hölder Inequality)} \\ &= \mu(A)^{1-2/p} \|\varphi_g - h\| \\ &\le \mu(A)^{1-2/p} \|\varphi_g - f_x\| \qquad \text{(by (2))} \\ &= \|x - g\|, \qquad \text{(by (3))} \end{aligned}$$

which show that $\mu(A)^{-1/p} g_0$ is a best coapproximation element to x in G.

If $p = 1$, then, instead of Hőlder Inequality, the following inequality ca be used:

$$\begin{aligned} \left\|g - \mu(A)^{-1} g_0\right\| &\le \mu(A)^{-1}\int_\Omega \|\varphi_g(s) - h(s)\ \| \,\chi_A(s)d\mu(s) \\ &\le \mu(A)^{-1}\int_\Omega \|\varphi_g(s) - h(s)\ \| \,d\mu(s) \\ &= \mu(A)^{-1}\|\varphi_g - h\| \le \mu(A)^{-1}\|\varphi_g - f_x\| \\ &= \|x - g\|. \end{aligned}$$

□

REMARK 6. If by a trivial space one understands that $\mu(\Sigma) \neq \{0, \infty\}$, *i.e.* μ takes only the values 0 and ∞, then in a non-trivial measure space (Ω, Σ, μ) there exists always $A \in \Sigma$ with $0 < \mu(A) < \infty$.

COROLLARY 7. *Let G be a separable subspace of X and $1 \leq p \leq \infty$. If $L^p(\mu, G)$ is coproximinal in $L^p(\mu, X)$, then G is coproximinal in X.*

Proof. The proof follows from Proposition 4 and Proposition 5. □

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