JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 49 (2020) no. 1, pp. 54–65
ictp.acad.ro/jnaat

QUANTITATIVE APPROXIMATION BY NONLINEAR ANGHELUŢĂ-CHOQUET SINGULAR INTEGRALS

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Abstract. By using the concept of nonlinear Choquet integral with respect to a capacity and as a generalization of the Poisson-Cauchy-Choquet operators, we introduce the nonlinear Angheluţă-Choquet singular integrals with respect to a family of submodular set functions. Quantitative approximation results in terms of the modulus of continuity are obtained with respect to some particular possibility measures and with respect to the Choquet measure $\mu(A) = \sqrt{M(A)}$, where M represents the Lebesgue measure. For some subclasses of functions we prove that these Choquet type operators can have essentially better approximation properties than their classical correspondents. The paper ends with the important, independent remark that for Choquet-type operators which are comonotone additive too, like Kantorovich-Choquet operators, Szász-Mirakjan-Kantorovich-Choquet operators and Baskakov-Kantorovich-Choquet operators studied in previous papers, the approximation results remain identically valid not only for non-negative functions, but also for all functions which take negative values too, if they are lower bounded.

MSC 2010. 41A36, 41A25, 28A10, 28A12, 28A25.

Keywords. Submodular set function, nonlinear Choquet integral, nonlinear Angheluţă-Choquet operators.

1. INTRODUCTION

In a very old paper [3], Th. Angheluţă has introduced the following singular integral by (with his notations)

$$A_r(f)(x) = \frac{2}{\pi}\int_{\mathbb{R}} f\left(x + u\log(\tfrac{1}{r})\right) \cdot \tfrac{du}{(1+u^2)^2}$$

and proves that for $r \searrow 1$, $A_r(f)(x)$ approximates $f(x)$ with the rate of the modulus of continuity $\omega_1(f; \log(1/r))$. Denoting $\log(1/r) = t$, $r \searrow 1$ with $t \searrow 0$ and changing the variable $x + u\log(1/r) := x + ut = v$ under the above integral, one obtains the following singular integral of Poisson-Cauchy-type

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given by

$$Q_t(f)(x) = \frac{2}{t\pi}\int_{\mathbb{R}} \frac{f(v)}{(1+(x-v)^2/t^2)^2}dv = \frac{2t^3}{\pi}\int_{\mathbb{R}} \frac{f(v)}{[t^2+(x-v)^2]^2}dv, \quad x \in \mathbb{R},$$

which approximates f with the rate $\omega_1(f;t)$, $t \searrow 0$.

It is worth noting that quantitative results in approximation by the classical Poisson-Cauchy singular integral can be found in, *e.g.*, the paper [5], the books [2], [4] and in the correspondent references therein.

On the other hand, the well-known Feller's probabilistic scheme in constructing linear and positive approximation operators (see, *e.g.*, [1], or [9, ch. 7], or, [2, §5.2, pp. 283–319]), was extended in [10] by replacing the classical linear integral with respect to a measure, with the nonlinear Choquet integral with respect to a monotone set-valued function (capacity). Also, in the papers [10]–[19], approximation results for various nonlinear approximation operators based on the Choquet integral with respect to a family of submodular set functions were obtained.

In this paper, these ideas will be applied to the above Angheluţă integral operators, namely replacing the usual integral with the nonlinear Choquet integral, the corresponding Angheluţă-Choquet nonlinear operators are introduced and quantitative approximation results in terms of the modulus of continuity $\omega_1(f;\cdot)$ are obtained with respect to some particular possibility measures and with respect to the Choquet measure $\mu(A) = \sqrt{M(A)}$ where M represents the Lebesgue measure. For some subclasses of functions we prove that these Choquet type operators have essentially better approximation properties than their classical correspondents. The paper ends with the important, independent remark that for classes of Choquet-type operators which are comonotone additive too, like the Kantorovich-Choquet operators, Szász-Mirakjan-Kantorovich-Choquet operators and Baskakov-Kantorovich-Choquet operators studied in previous papers, the approximation results remain identically valid not only for non-negative functions, but also for all functions which take negative values too, if they are lower bounded.

2. PRELIMINARIES

In order to give the reader a flavor on the topic, firstly in this section we present some concepts and results concerning the Choquet integral.

Definition 1. *Let $(\Omega, \mathcal{C})$ be a measurable space, i.e. Ω is a nonempty set and $\mathcal{C}$ be a σ-ring (or σ-algebra) of subsets in Ω with $\emptyset \in \mathcal{C}$.*

(i) (see, e.g., [20], p. 63) The set function $\mu : \mathcal{C} \to [0, +\infty]$ is called a monotone measure (or capacity) if $\mu(\emptyset) = 0$ and $A, B \in \mathcal{C}$, with $A \subset B$, implies $\mu(A) \le \mu(B)$. The monotone measure μ is called normalized if $\Omega \in \mathcal{C}$ and $\mu(\Omega) = 1$.

(ii) (see [6], or, e.g., [20, p. 179]) Let μ be a normalized monotone measure and consider $\mathcal{G} = \{X : \Omega \to \mathbb{R}_+; X$ is measurable on $(\Omega, \mathcal{C})\}$. Recall that

$X : \Omega \to \mathbb{R}$ is measurable (or more precisely $\mathcal{C}$-measurable), if for any B, Borelian subset in $\mathbb{R}$, we have $X^{-1}(B) \in \mathcal{C}$.

For $A \in \mathcal{C}$ and $X \in \mathcal{G}$, the Choquet integral of X on A with respect to a monotone measure μ is defined by

$$(C)\int_A X d\mu = \int_0^{\infty} \mu(F_\alpha(X)\bigcap A)d\alpha,$$

where $F_\alpha(X) = \{\omega \in \Omega; X(\omega) \geq \alpha\}$. If $(C)\int_A X d\mu < +\infty$ then X is called Choquet integrable on A.

If $X : \Omega \to \mathbb{R}$ is of arbitrary sign, then the Choquet integral is defined by (see [20, p. 233]*)*

$$(C)\int_A X d\mu = \int_0^{+\infty} \mu(F_\alpha(X)\bigcap A)d\alpha + \int_{-\infty}^{0} [\mu(F_\alpha(X)\bigcap A) - \mu(A)]d\alpha.$$

When μ is the Lebesgue measure, then the Choquet integral $(C)\int_A X d\mu$ reduces to the Lebesgue integral.

(iii) A possibility measure is a set function $P : \mathcal{P}(\Omega) \to [0, 1]$, satisfying the axioms $P(\emptyset) = 0$, $P(\Omega) = 1$ and $P(\bigcup_{i\in I} A_i) = \sup\{P(A_i); i \in I\}$ for all $A_i \subset \Omega$, and any I, an at most countable family of indices. Note that if $A, B \subset \Omega$, $A \subset B$, then the last property easily implies that $P(A) \leq P(B)$ and that $P(A\bigcup B) \leq P(A) + P(B)$.

A function $\lambda : \Omega \to [0, 1]$ is called possibility distribution if $\sup\{\lambda(\omega); \omega \in \Omega\} = 1$. Any possibility distribution λ on Ω, induces the possibility measure $P_\lambda : \mathcal{P}(\Omega) \to [0, 1]$, given by the formula $P_\lambda(A) = \sup\{\lambda(s); s \in A\}$, for all $A \subset \Omega$ (see, e.g., [8, ch. 1]*).*

Some known properties of the Choquet integral are expressed by the following.

Remark 2. Let us suppose that μ is a monotone measure. Then, the following properties hold :

(i) $(C)\int_A$ is non-additive (*i.e.* $(C)\int_A (f + g)d\mu \neq (C)\int_A f d\mu + (C)\int_A g d\mu$) but it is positive homogeneous, *i.e.* for all $a \geq 0$ we have $(C)\int_A af d\mu = a \cdot (C)\int_A f d\mu$ (for $f \geq 0$ see, *e.g.*, [20], Theorem 11.2, (5), p. 228 and for f of arbitrary sign, see, *e.g.*, [7], p. 64, Proposition 5.1, (ii)).

The Choquet integral is comonotone additive, that is, if f and g are comonotone on A (*i.e.*, $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for all $x, y \in A$), then $(C)\int_A (f + g)d\mu = (C)\int_A f d\mu + (C)\int_A g d\mu$.

If $f \leq g$ on A then the Choquet integral is monotone, that is $(C)\int_A f d\mu \leq (C)\int_A g d\mu$ (see, *e.g.*, [20], p. 228, Theorem 11.2, (3) for $f, g \geq 0$ and p. 232 for f, g of arbitrary sign).

If μ is submodular too (*i.e.* $\mu(A\bigcup B) + \mu(A\bigcap B) \leq \mu(A) + \mu(B)$ for all A, B) then the Choquet integral is subadditive, that is $(C)\int_A (f + g)d\mu \leq (C)\int_A f d\mu + (C)\int_A g d\mu$, for all f, g of arbitrary sign (see, *e.g.*, [7], p. 75, Theorem 6.3).

If $\overline{\mu}$ denotes the dual measure of μ (that is $\overline{\mu}(A) = \mu(\Omega) - \mu(\Omega \setminus A)$, for all $A \in \mathcal{C}$), then for all f of arbitrary sign we have $(C)\int_A(-f)d\mu = -(C)\int_A f d\overline{\mu}$ (see, *e.g.*, [20], Theorem 11.7, p. 233).

If $c \in \mathbb{R}$ and f is of arbitrary sign, then $(C)\int_A(f+c)d\mu = (C)\int_A f d\mu + c \cdot \mu(A)$ (see, *e.g.*, [20], pp. 232-233, or [7], p. 65).

By the definition of the Choquet integral, if $F \geq 0$ and μ is subadditive, then it is immediate that

$$(C)\int_{A\bigcup B} F d\mu \leq (C)\int_A F d\mu + (C)\int_B F d\mu.$$

Note that if μ is submodular then it is clear that it is subadditive too.

(ii) Simple concrete examples of monotone and submodular set functions μ, can be obtained from a probability measure M on $\mathcal{P}(\mathbb{X})$ (*i.e.* $M(\emptyset) = 0$, $M(\mathbb{X}) = 1$ and M is countable additive), by the formula $\mu(A) = \gamma(M(A))$, where $\gamma : [0,1] \to [0,1]$ is an increasing and concave function, with $\gamma(0) = 0$, $\gamma(1) = 1$ (see, *e.g.*, [7], pp. 16-17, Example 2.1).

Also, any possibility measure μ is monotone and submodular. While the monotonicity is immediate from the axiom $\mu(A\bigcup B) = \max\{\mu(A), \mu(B)\}$, the submodularity is immediate from the property $\mu(A\bigcap B) \leq \min\{\mu(A), \mu(B)\}$.

(iii) Many other properties of the Choquet integral can be found in, *e.g.*, Chapter 11 in [20], or in [7]. □

Now, we present the following general approximation result which will be used in the next sections.

Theorem 3. *([10], Theorem 3.3 and Remark 3.5) Denoting by $\mathcal{P}(\mathbb{R})$ the class of all subsets of $\mathbb{R}$, let $(\mathbb{R}, \mathcal{C})$ be a measurable space with $\mathcal{C} \subset \mathcal{P}(\mathbb{R})$ and $\mu_{n,x} : \mathcal{C} \to [0, +\infty)$, be a monotone and submodular family of set functions.*

For $\lambda_{n,x} : \mathbb{R} \to \mathbb{R}_+$, $n \in \mathbb{N}$, $x \in \mathbb{R}$, Choquet densities with respect to $\mu_{n,x}$, (that is, $(C)\int_{\mathbb{R}} \lambda_{n,x}(t)d\mu_{n,x}(t) = 1$), let us define by $UC(\mathbb{R})$, the class of all functions $f : \mathbb{R} \to \mathbb{R}_+$, uniformly continuous on $\mathbb{R}$, such that $f \cdot \lambda_{n,x}$ are $\mathcal{C}$-measurable and $T_n(f)(x) < +\infty$, for all $n \in \mathbb{N}$, $x \in \mathbb{R}$, where

$$T_n(f)(x) = (C)\int_{\mathbb{R}} f(t) \cdot \lambda_{n,x}(t)d\mu_{n,x}(t).$$

Then, denoting $\varphi_x(t) = |t - x|$, for all $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $\delta > 0$ we have

$$|T_n(f)(x) - f(x)| \leq \left[1 + \frac{T_n(\varphi_x)(x)}{\delta}\right] \cdot \omega_1(f;\delta)_{\mathbb{R}}.$$

Also, choosing above $\delta = T_n(\varphi_x)(x)$, it follows

$$|T_n(f)(x) - f(x)| \leq 2\omega_1(f; T_n(\varphi_x)(x))_{\mathbb{R}}.$$

Remark 4. An important problem in Theorem 3 is to determine the functions f with $T_n(f)(x) < +\infty$, for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. Since $T_n(e_0)(x) = 1$, from the positive homogeneity of T_n it easily follows that for $f_0(t) = c > 0$ for all $t \in \mathbb{R}$, we have $T_n(f_0)(x) = c$, for all $x \in \mathbb{R}$, $n \in \mathbb{R}$. Then, for any bounded

$f : \mathbb{R} \to \mathbb{R}_+$ and $\alpha \geq 0$, denoting $\|f\| = \sup\{f(x); x \in \mathbb{R}\} < +\infty$, we get $\{t \in \mathbb{R}; f(t) \cdot \lambda_{n,x}(t) \geq \alpha\} \subset \{t \in \mathbb{R}; \|f\| \cdot \lambda_{n,x}(t) \geq \alpha\}$, which implies

$$\mu(\{t \in \mathbb{R}; f(t) \cdot \lambda_{n,x}(t) \geq \alpha\}) \leq \mu(\{t \in \mathbb{R}; \|f\| \cdot \lambda_{n,x}(t) \geq \alpha\})$$

and therefore

$$\begin{aligned} T_n(f)(x) = & \int_0^{+\infty} \mu(\{t \in \mathbb{R}; f(t) \cdot \lambda_{n,x}(t) \geq \alpha\}) d\alpha \\ \leq & \int_0^{+\infty} \mu(\{t \in \mathbb{R}; \|f\| \cdot \lambda_{n,x}(t) \geq \alpha\}) d\alpha \\ = & \|f\| \cdot T_n(e_0)(x) = \|f\| < +\infty. \end{aligned}$$

Also, from these reasonings, it follows that if for an unbounded function $F_0 : \mathbb{R} \to \mathbb{R}_+$ we have $T_n(F_0)(x) < +\infty$, for all $x \in \mathbb{R}$, $n \in \mathbb{N}$, then for any unbounded function f satisfying $f(t) \leq F_0(t)$, for all $t \in \mathbb{R}$, we have $T_n(f)(x) < +\infty$, for all $x \in \mathbb{R}$, $n \in \mathbb{N}$. □

Remark 5. The above Theorem 3 remains valid for functions and operators defined on compact intervals too. Indeed, analysing the proof of Theorem 3.3 in [10], it is easily seen that it remains valid for $\lambda_{n,x} : I \to \mathbb{R}_+$, $f : I \to \mathbb{R}_+$ and $T_n(f)(x) = (C)\int_I f(t) \cdot \lambda_{n,x}(t) d\mu_{n,x}(t)$, where $I \subset \mathbb{R}$ is a compact subinterval. In fact, for $I = [0, 1]$, Theorem 3.3 in [10] was implicitly used in the case of Bernstein-Durrmeyer-Choquet operators (see the proof of Theorem 3.1, (i) in [18]) and in the case of Bernstein-Kantorovich-Choquet operators (see the proof of Theorem 3.3 in [11]). □

3. ANGHELUŢĂ-CHOQUET OPERATORS

The linear Angheluţă operators defined in Introduction can be generalized to the nonlinear Angheluţă-Choquet operators with respect to a family of monotone and submodular set functions $(\mu_{t,x})$, $t > 0$, $x \in \mathbb{R}$, by the formula

$$Q_{t,\mu_{t,x}}(f)(x) = \frac{1}{c(t,x,\mu_{t,x})} \cdot (C)\int_{\mathbb{R}} \frac{f(s)}{(|s-x|^2/t^2+1)^2} d\mu_{t,x}(s),$$

with $c(t, x, \mu_{t,x}) = (C)\int_{\mathbb{R}} \frac{1}{(|s-x|^2/t^2+1)^2} d\mu_{t,x}(s)$.

In this section we deal with the approximation properties of $Q_{t,\mu_{t,x}}(f)(x)$ for some particular choices of the set functions $\mu_{t,x}$, $t > 0$, $x \in \mathbb{R}$.

The first main result of this section is the following :

Theorem 6. *Let $\mu_{t,x}(A) := \mu(A) = \sqrt{M(A)}$ for all $t > 0$, $x \in \mathbb{R}$, where $M(A)$ denotes the Lebesgue measure of A. If $f : \mathbb{R} \to \mathbb{R}_+$ is uniformly continuous on $\mathbb{R}$, such that $Q_{t,\mu}(f)(x) < +\infty$, for all $t > 0$, $x \in \mathbb{R}$, then for all $t > 0$ and $x \in \mathbb{R}$ we have*

$$|Q_{t,\mu}(f)(x) - f(x)| \leq 4\omega_1(f; t)_{\mathbb{R}}.$$

Proof. By Theorem 3 (replacing there n by $\frac{1}{t}$), we get

$$|Q_{n,\mu}(f)(x) - f(x)| \leq 2\omega_1(f; Q_{t,\mu}(\varphi_x)(x))_{\mathbb{R}}, \tag{1}$$

where $\varphi_x(s) = |s-x|$. Therefore, the convergence of $Q_{t,\mu}(f)$ to f one relies on the convergence to zero, as $t \searrow 0$, of the quantity

$$Q_{t,\mu}(\varphi_x)(x) = \frac{1}{c(t,x,\mu)} \cdot (C)\int_{\mathbb{R}} \frac{|x-s|}{(|s-x|^2/t^2+1)^2} d\mu(s)$$
$$= \frac{1}{c(t,x,\mu)} \cdot \int_0^\infty \mu[\{s \in \mathbb{R}; \frac{|s-x|}{(|s-x|^2/t^2+1)^2} \geq \alpha\}] d\alpha.$$

Denoting

$$F_\alpha\left(\frac{1}{((x-\cdot)^2/t^2+1)^2}\right) = \{s \in \mathbb{R}; \frac{1}{((x-s)^2/t^2+1)^2} \geq \alpha\},$$

by simple calculation we get $F_\alpha\left(\frac{1}{((x-\cdot)^2/t^2+1)^2}\right) = \emptyset$ for $\alpha > 1$ and if $\alpha \leq 1$, then

$$F_\alpha\left(\frac{1}{((x-\cdot)^2/t^2+1)^2}\right) = \left[x - t\sqrt{(1-\sqrt{\alpha})/\sqrt{\alpha}}, x + t\sqrt{(1-\sqrt{\alpha})/\sqrt{\alpha}}\right],$$

which for $\mu(A) = \sqrt{M(A)}$, leads us to

$$c(t,x,\mu) = (C)\int_{\mathbb{R}} \frac{1}{((x-s)^2/t^2+1)^2} d\mu(s)$$
$$= \int_0^1 \mu\left(\left[x - t\sqrt{(1-\sqrt{\alpha})/\sqrt{\alpha}}, x + t\sqrt{(1-\sqrt{\alpha})/\sqrt{\alpha}}\right]\right) d\alpha$$
$$= \sqrt{2}\sqrt{t} \cdot \int_0^1 \left(\frac{1}{\sqrt{\alpha}} - 1\right)^{1/4} d\alpha.$$

By the substitution $\frac{1}{\sqrt{\alpha}} - 1 = s$, it easily follows $\alpha = \frac{1}{(s+1)^2}$, $d\alpha = -\frac{2}{(s+1)^3} ds$ and

$$c(t,x,\mu) = 2\sqrt{2}\sqrt{t} \cdot \int_0^{+\infty} \frac{s^{1/4}}{(s+1)^3} ds$$
$$= 8\sqrt{2}\sqrt{t} \cdot \int_0^{+\infty} \frac{v^4}{(1+v^4)^3} dv = 8\sqrt{2}\sqrt{t} \cdot \frac{3\pi\sqrt{2}}{128} = \frac{3\pi\sqrt{t}}{8}.$$

Therefore, the convergence of $Q_{t,\mu}(f)$ to f one relies on the convergence to zero of the quantity $Q_{t,\mu}(\varphi_x)(x) = \frac{1}{c(t,x,\mu)} \cdot (C)\int_{\mathbb{R}} \frac{|x-s|}{((x-s)^2/t^2+1)^2} d\mu(s)$.

Now, denoting $F(v) = \frac{v}{(v^2/t^2+1)^2}$, $v \geq 0$, we have $F'(v) = \frac{1-4v^2/t^2}{(v^2/t^2+1)^3}$, which immediately implies that $v = \frac{t}{2}$ is a maximum point for F on $[0,+\infty)$ and $F(t) = \frac{8t}{25}$ is the maximum value for F.

This implies that for $\alpha > \frac{8t}{25}$ we have $\{s \in \mathbb{R}; |s-x|/((|s-x|^2/t^2+1)^2) \geq \alpha\} = \emptyset$ and therefore

$$\begin{aligned} Q_{t,\mu}(\varphi_x)(x) &= \frac{1}{c(t,x,\mu)} \cdot \int_0^{\infty} \mu[\{s \in \mathbb{R}; \frac{|s-x|}{(|s-x|^2/t^2+1)^2} \geq \alpha\}]d\alpha \\ &= \frac{8}{3\pi\sqrt{t}} \cdot \int_0^{8t/25} \mu[\{s \in \mathbb{R}; \frac{|s-x|}{(|s-x|^2/t^2+1)^2} \geq \alpha\}]d\alpha \\ &\leq \frac{8}{3\pi\sqrt{t}} \cdot \int_0^{8t/25} \mu[\{s \in \mathbb{R}; \frac{|s-x|}{|s-x|^2/t^2+1} \geq \alpha\}]d\alpha \\ &= \frac{8}{3\pi\sqrt{t}} \cdot \int_0^{8t/25} \mu[\{s \in \mathbb{R}; \alpha(|s-x|)^2/t^2+1) - |s-x| \leq 0\}]d\alpha. \end{aligned}$$

But in the proof of Theorem 5.1 in [12], we have obtained

$$\mu[\{s \in \mathbb{R}; \alpha(|s-x|)^2/t^2+1) - |s-x| \leq 0\}] \leq 2\frac{(1-4\alpha^2/t^2)^{1/4}}{\sqrt{\alpha}/t},$$

which implies

$$\begin{aligned} Q_{t,\mu}(\varphi_x)(x) &\leq \frac{8}{3\pi\sqrt{t}} \cdot \int_0^{8t/25} \left[2\frac{(1-4\alpha^2/t^2)^{1/4}}{\sqrt{\alpha}/t}\right] d\alpha \\ &\leq \frac{8}{3\pi\sqrt{t}} \cdot \int_0^{t/2} \left[2\frac{(1-4\alpha^2/t^2)^{1/4}}{\sqrt{\alpha}/t}\right] d\alpha \\ &\leq \frac{16\sqrt{t}}{3\pi} \cdot \int_0^{t/2} \left(\frac{1}{\alpha^2} - 4/t^2\right)^{1/4} d\alpha, \end{aligned}$$

where by the calculations in the proof of Theorem 5.1 in [12], we easily get

$$\int_0^{t/2} \left(\frac{1}{\alpha^2} - 4/t^2\right)^{1/4} d\alpha \leq \frac{\sqrt{t}}{2\sqrt{2}} \cdot \mathrm{Beta}(5/4, 1/4),$$

with the Euler's Beta function satisfying $\mathrm{Beta}(5/4, 1/4) < 2$.

Therefore, it follows

$$Q_{t,\mu}(\varphi_x)(x) \leq \frac{16\sqrt{t}}{3\pi} \cdot \frac{\sqrt{t}}{2\sqrt{2}} \cdot \mathrm{Beta}(5/4, 1/4) \leq \frac{16}{3\pi\sqrt{2}} t < 2t.$$

Concluding, by (1), it follows

$$|Q_{t,\mu}(f)(x) - f(x)| \leq 4\omega_1(f;t)_{\mathbb{R}},$$

which ends the proof. □

Remark 7. For $\mu(A) = M(A)$ in Theorem 6 we recapture the Angheluţă operator and the order of approximation in [3] mentioned in Introduction. □

The second main result is for another choice for the family of set functions.

Theorem 8. *Let us consider the measures of possibility (depending on t and x too) defined by $\mu_{t,x}(A) = \sup\{\frac{1}{((s-x)^2/t^2+1)^2}; s \in A\}$ and define the Angheluţă-Choquet operators*

$$Q_{t,\mu_{t,x}}(f)(x) = \frac{1}{c(t,x,\mu_{t,x})} \cdot (C)\int_{\mathbb{R}} \frac{f(s)}{(|s-x|^2/t^2+1)^2} d\mu_{t,x}(s),$$

with $c(t, x, \mu_{t,x}) = (C)\int_{\mathbb{R}} \frac{1}{(|s-x|^2/t^2+1)^2} d\mu_{t,x}(s)$.

If $f : \mathbb{R} \to \mathbb{R}_+$ *is uniformly continuous on* $\mathbb{R}$*, such that* $Q_{t,\mu_{t,x}}(f)(x) < +\infty$*, for all* $t > 0$*,* $x \in \mathbb{R}$*, then*

$$|Q_{t,\mu_{t,x}}(f)(x) - f(x)| \le 2\omega_1 \left(f; \tfrac{t}{3}\right)_{\mathbb{R}},$$

for all $t \in [0,1]$ *and* $x \in \mathbb{R}$*. Here* $\omega_1 (f;\delta)_{\mathbb{R}} = \sup\{|f(x) - f(y)|; x, y \in \mathbb{R}, |x - y| \le \delta\}$.

Proof. Firstly, it is easy to see that any possibility measure $\mu_{n,x}$ is bounded, monotone and submodular, therefore we are under the hypothesis of Theorem 3.

Denoting $\varphi_s(t) = |s - x|$, by Theorem 3 (replacing there n by $\frac{1}{t}$), we need to estimate the quantity

$$Q_{t,\mu_{t,x}}(\varphi_x)(x) = \frac{1}{c(t,x,\mu_{t,x})} \cdot (C)\int_{\mathbb{R}} |s - x| \cdot \frac{1}{(|s-x|^2/t^2+1)^2} d\mu_{t,x}(s).$$

Firstly, we have

$$\begin{aligned} c(t, x, \mu_{t,x}) &= \int_0^{+\infty} \mu_{t,x}(\{s \in \mathbb{R}; \tfrac{1}{(|s-x|^2/t^2+1)^2} \ge \alpha\})d\alpha \\ &= \int_0^1 \mu_{t,x}(\{s \in \mathbb{R}; \tfrac{1}{(|s-x|^2/t^2+1)^2} \ge \alpha\})d\alpha \\ &= \int_0^1 \sup\{\tfrac{1}{(|s-x|^2/t^2+1)^2}; s \in \mathbb{R}, \tfrac{1}{(|s-x|^2/t^2+1)^2} \ge \alpha\}d\alpha \\ &= \int_0^1 1 d\alpha = 1, \end{aligned}$$

since $\frac{1}{(|s-x|^2/t^2+1)^2} \le 1$ for all $x, s \in \mathbb{R}$ and that its supremum is equal to 1 and is attained for $s = x$.

On the other hand, it follows

$$\begin{aligned} &\sup\{\tfrac{1}{(|s-x|^2/t^2+1)^2}; s \in \mathbb{R}, \tfrac{|s-x|}{(|s-x|^2/t^2+1)^2} \ge \alpha\} = \\ &= \sup\{\tfrac{1}{(v^2/t^2+1)^2}; v \in \mathbb{R}_+, \tfrac{v}{(v^2/t^2+1)^2} \ge \alpha\}. \end{aligned}$$

Also, denoting $F(v) = \frac{v}{(v^2/t^2+1)^2}$, $v \in \mathbb{R}_+$, the equation $F'(v) = 0$ implies that $v = \frac{t}{\sqrt{3}}$ is a maximum point for F, with the maximum value $F(t/\sqrt{3}) = \frac{9t}{16\sqrt{3}} \le \frac{t}{3} < 1$, for all $t \in [0,1]$.

Therefore, we get

$$\begin{aligned} Q_{t,\mu_{t,x}}(\varphi_x)(x) &= \int_0^{\infty} \sup\{\tfrac{1}{(|s-x|^2/t^2+1)^2}; s \in \mathbb{R}, \tfrac{|s-x|}{(|s-x|^2/t^2+1)^2} \ge \alpha\}d\alpha \\ &\le \int_0^{t/3} \sup\left\{\tfrac{1}{(v^2/t^2+1)^2}; v \in \mathbb{R}_+\right\} d\alpha \le \int_0^{t/3} 1 \cdot d\alpha = \tfrac{t}{3}. \end{aligned}$$

Concluding, by Theorem 3 we immediately get the approximation estimate

$$|Q_{t,\mu_{t,x}}(f)(x) - f(x)| \le 2\omega_1 \left(f; \tfrac{t}{3}\right)_{\mathbb{R}},$$

which proves the theorem. □

Remark 9. There are classes of functions for which $Q_{t,\mu_{t,x}}(f)(x)$ gives an essentially better estimate than in Theorem 6 and than that given by the Angheluţă operators.

In this sense, firstly we prove here that for any $f:\mathbb{R}\to\mathbb{R}_+$, with the properties $1\le f(x)\le 2$, for all $x\in\mathbb{R}$, f is nondecreasing on $\mathbb{R}$ and $\ln(f(x))$ is a Lipschitz function on $\mathbb{R}$ with the Lipschitz constant 2, we have $f(x)\le Q_{t,\mu_{t,x}}(f)(x)\le f(x+t^2)$, for all $x\in\mathbb{R}$. Then, for the order of approximation we get

$$0\le Q_{t,\mu_{t,x}}(f)(x)-f(x)\le f(x+t^2)-f(x)\le \omega_1(f;t^2)_{\mathbb{R}}, \tag{2}$$

which for $t\in(0,1)$ is essentially better than the order $\mathcal{O}(\omega_1(f;t)_{\mathbb{R}})$ given by Theorem 6.

Indeed, let $x\in\mathbb{R}$ be fixed. Since evidently that $c(t,x,\mu_{t,x})=1$, we easily obtain

$$Q_{t,\mu_{t,x}}(f)(x)=\int_0^{+\infty}\sup\{\tfrac{1}{(|t-x|^2/t^2+1)^2}; s\in\mathbb{R}, f(s)\cdot\tfrac{1}{(|s-x|^2/t^2+1)^2}\ge\alpha\}d\alpha.$$

We firstly show that for all $s\in\mathbb{R}$ and $t>0$, we have $f(s)\cdot\frac{1}{(|s-x|^2/t^2+1)^2}\le f(x+t^2)$.

If $s\le x$ then $f(s)\le f(x)\le f(x+t^2)$, which immediately implies $f(s)\cdot\frac{1}{(|s-x|^2/t^2+1)^2}\le f(x+t^2)$. Also, when $s>x$, let us denote $s=x+h$, with $h>0$. We have two cases : (i) $h\le t^2$; (ii) $h>t^2$.

In the case (i), since $f(x+h)\le f(x+t^2)$, we immediately get $f(s)\cdot\frac{1}{(|s-x|^2/t^2+1)^2}\le f(x+t^2)$.

Let us consider now the case (ii). The inequality required to be proved is evidently equivalent to

$$0\le\ln(f(x+h))-\ln(f(x+t^2))\le 2\ln(h^2/t^2+1),\ \text{for all } x\in\mathbb{R}, h>t^2, t>0.$$

But by the boundedness hypothesis we get

$$0\le\ln(f(x+h))-\ln(f(x+t^2))\le\ln(2)\le\ln(h^2/t^2+1)\le 2\ln(h^2/t^2+1),$$

for all $h\ge 1$, $x\in\mathbb{R}$, $t>0$. Therefore, it remains to settle the case when $t^2<h<1$.

By hypothesis, we also have $0\le\ln(f(x+h))-\ln(f(x+t^2))\le 2(h-t^2)$, which combined with the inequality $h-t^2\le\ln(h^2/t^2+1)$, valid for all $t^2\le h<1$, leads again to the above required inequality.

Therefore, it remains to prove that $h-t^2\le\ln\left(h^2/t^2+1\right)$, for all $t^2\le h<1$. For that purpose, denoting $G(h)=\ln\left(h^2/t^2+1\right)-(h-t^2)$, we get $G(t^2)=\ln(t^2+1)>0$ and

$$G'(h)=\tfrac{2h}{h^2+t^2}-1=\tfrac{(h-h^2)+(h-t^2)}{h^2+t^2}>0,$$

for all $t^2\le h<1$. This implies the required inequality.

In continuation, we easily get

$$Q_{t,\mu_{t,x}}(f)(x) = \int_0^{+\infty} \sup\{\tfrac{1}{(|t-x|^2/t^2+1)^2}; s \in \mathbb{R}, f(s) \cdot \tfrac{1}{(|s-x|^2/t^2+1)^2} \geq \alpha\} d\alpha$$
$$= \int_0^{f(x+t^2)} \sup\{\tfrac{1}{(|t-x|^2/t^2+1)^2}; s \in \mathbb{R}, f(s) \cdot \tfrac{1}{(|s-x|^2/t^2+1)^2} \geq \alpha\} d\alpha$$
$$+ \int_{f(x+t^2)}^{+\infty} \sup\{\tfrac{1}{(|t-x|^2/t^2+1)^2}; s\in \mathbb{R}, f(s) \cdot \tfrac{1}{(|s-x|^2/t^2+1)^2} \geq \alpha\} d\alpha$$
$$= \int_0^{f(x+t^2)} \sup\{\tfrac{1}{(|t-x|^2/t^2+1)^2}; s \in \mathbb{R}, f(s) \cdot \tfrac{1}{(|s-x|^2/t^2+1)^2} \geq \alpha\} d\alpha$$
$$\leq \int_0^{f(x+t^2)} 1 d\alpha = f(x+t^2).$$

Then,

$$\int_0^{f(x+t^2)} \sup\{\tfrac{1}{(|t-x|^2/t^2+1)^2}; s \in \mathbb{R}, f(s) \cdot \tfrac{1}{(|s-x|^2/t^2+1)^2} \geq \alpha\} d\alpha \geq$$
$$\geq \int_0^{f(x)} \sup\{\tfrac{1}{(|t-x|^2/t^2+1)^2}; s \in \mathbb{R}, f(s) \cdot \tfrac{1}{(|s-x|^2/t^2+1)^2} \geq \alpha\} d\alpha$$
$$= \int_0^{f(x)} 1 d\alpha = f(x).$$

Secondly, in order to compare the order of approximation given by $Q_{t,\mu_{t,x}}(f)(x)$ with that obtained by the classical Poisson-Cauchy singular integral, suppose that in addition to the hypothesis from the beginning of this remark, f is also a α-Lipschitz function on $\mathbb{R}$, with $0 < \alpha < 1$. From (2) it follows that the order of approximation of f by $Q_{t,\mu_{t,x}}(f)$ is $\mathcal{O}(t^{2\alpha})$.

Now, if the order $\mathcal{O}(h^{\alpha})$ of $\omega_1(f;h)_{\mathbb{R}}$ is the best possible, then combining this with Exercise 5, p. 150 in [4], it follows that the order of approximation of such f by the classical Poisson-Cauchy singular integral is only $\mathcal{O}(t^{\alpha})$, which for $t \in (0,1)$ is essentially worst than $\mathcal{O}(t^{2\alpha})$. □

Remark 10. Suppose that f may have negative values too on $\mathbb{R}$ but is lower bounded on $\mathbb{R}$, that is $f(x) \geq c$, for all $x \in \mathbb{R}$, with $c < 0$. We can use the trick in [17], [18], [11], [15] and [12], to construct slightly modified operators for approximation of functions which take negative values too and maintain the same order of approximation. Indeed, since $f(x) - c \geq 0$, where $c < 0$, *i.e.* $-c > 0$, it follows that for $R_{t,\mu}(f) = Q_{t,\mu}(f-c) + c$ we get (for example in the Theorem 6 case)

$$|R_{t,\mu}(f)(x) - f(x)| = |Q_{t,\mu}(f-c)(x) - (f(x)-c)|$$
$$\leq 4\omega_1(f-c;t)_{\mathbb{R}} = 4\omega_1(f;t)_{\mathbb{R}}.$$

It is worth mentioning here that we have applied the above construction to [17], [18] for the Bernstein-Durrmeyer-Choquet operators, to [11] for the

Kantorovich-Choquet operators, to [15] for the Stancu-Durrmeyer-Choquet-Šipoš operators, to [12] for the Picard-Choquet, Gauss-Weierstrass-Choquet and Poisson-Cauchy-Choquet singular integrals and to [11] for the Szász-Mirakjan-Kantorovich-Choquet and Baskakov-Kantorovich-Choquet opera tors.

But in the case of Kantorovich-Choquet operators, Szász-Mirakjan-Kantorovich-Choquet operators and Baskakov-Kantorovich-Choquet operators, since these operators obviously are comonotonic additive and a constant function is always comonotone additive with an arbitrary function, denoting any from these operators with L_n, we easily get that $L_n(f-c)+c = L_n(f)+L_n(-c)+c = L_n(f) - c + c = L_n(f)$, Therefore, in these cases, we have the convergence of $L_n(f)$ to f not only for positive functions too, but also for lower bounded functions which takes negative values too.

For example, in the case of Kantorovich-Choquet operator

$$K_{n,\mu}(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot \frac{(C)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)d\mu(t)}{\mu([k/(n+1),(k+1)/(n+1)])},$$

with $p_{n,k}(x) = \binom{n}{}x^k(1-x)^{n-k}$, $\mu(A) = \sqrt{m(A)}$, $m(A)$-the Lebesgue measure, by Corollary 3.6 in [11] we have

$$|K_{n,\mu}(f)(x) - f(x)| \leq 2\omega_1\left(f; \sqrt{x(1-x)}/\sqrt{n} + 1/n\right)_{[0,1]},$$

for all f continuous and nonnegative on $[0,1]$, $n \in \mathbb{N}$, $x \in [0,1]$.

Therefore, since each K_n, $n \in \mathbb{N}$, is comonotone additive too, by the above considerations, the previous quantitative estimate remain valid if f takes negative values too. □

REFERENCES

[1] O. Agratini, *Approximation by Linear Operators*, Cluj University Press, Cluj-Napoca, 2000, (in Romanian).

[2] F. Altomare, M. Campiti, *Korovkin-type Approximation Theory and its Applications*, de Gruyter Studies in Mathematics **17**, New York, Berlin, 1994.

[3] Th. Angheluţă, *Une remark sur l'integrale de Poisson*, Bull. Sci. Math. (Paris), **XLVIII** (1924), pp. 138–140.

[4] P.L. Butzer, R.J. Nessel, *Fourier Analysis and Approximation*, vol. 1, One-Dimensional Theory, Academic Press, New York, London, 1971.

[5] P.L. Butzer, W. Trebels, *Opérateurs de Gauss-Weiesrtrass et de Cauchy-Poisson et conditions lipschitzienne dans $L^1(E_n)$*, C. R. Acad. Sci. Paris sér. A-B., **268** (1969), pp. 700–703.

[6] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble), **5** (1953-1954), pp. 131–292.

[7] D. Denneberg, *Non-Additive Measure and Integral*, Kluwer Academic Publisher, Dordrecht, Boston, London, 2010.

[8] D. Dubois, H. Prade, *Possibility Theory*, Plenum Press, New York, 1988.

[9] W. Feller, *An Introduction to Probability Theory and Its Applications*, II, Wiley, New York, 1966.

[10] S.G. Gal, *Approximation by nonlinear Choquet integral operators*, Annali di Mat. Pura Appl., **195**(3) (2016), pp. 881–896.

[11] S.G. Gal, *Uniform and pointwise quantitative approximation by Kantorovich-Choquet type integral operators with respect to monotone and submodular set functions*, Mediterr. J. Math., **14**(5) (2017), pp. 205–216.

[12] S.G. Gal, *Quantitative approximation by nonlinear Picard-Choquet, Gauss-Weierstrass-Choquet and Poisson-Cauchy-Choquet singular integrals*, Results Math., **73**(3) (2018), Art. 92, 23 pp.

[13] S.G. Gal, *Correction to: Quantitative Approximation by Nonlinear Picard-Choquet, Gauss-Weierstrass-Choquet and Poisson-Cauchy-Choquet Singular Integrals*, Results Math., **75** (1) (2020), Art. 31, 3 pp.

[14] S.G. Gal, *Shape preserving properties and monotonicity properties of the sequences of Choquet type integral operators*, J. Numer. Anal. Approx. Theory, **47**(2) (2018), pp. 135–149.

[15] S.G. Gal, *Quantitative approximation by Stancu-Durrmeyer-Choquet-Šipoš operators*, Math. Slovaca, **69**(3) (2019), pp. 625–638.

[16] S.G. Gal, I.T. Iancu, *Quantitative approximation by nonlinear convolution operators of Landau-Choquet type*, Carpath. J. Math., **36** (2020) no. 3 (to appear).

[17] S.G. Gal, B.D. Opris, *Uniform and pointwise convergence of Bernstein-Durrmeyer operators with respect to monotone and submodular set functions*, J. Math. Anal. Appl., **424** (2015), pp. 1374–1379.

[18] S.G. Gal, S. Trifa, *Quantitative estimates in uniform and pointwise approximation by Bernstein-Durrmeyer-Choquet operators*, Carpath. J. Math., **33** (2017), pp. 49–58.

[19] S.G. Gal, S. Trifa, *Quantitative estimates in L^p-approximation by Bernstein-Durrmeyer-Choquet operators with respect to distorted Borel measures*, Results Math., **72**(3) (2017), pp. 1405–1315.

[20] S. Wang, G.J. Klir, *Generalized Measure Theory*, Springer, New York, 2009.

Received by the editors: March 8, 2020; accepted: May 17, 2020; published online: August 11, 2020.