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ON THE CONVERGENCE RATES OF THE PAIRS OF ADJACENT SEQUENCES

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Abstract. In this paper we give a suitable definition for the pairs of adjacent (convergent) sequences of real numbers, we present some two-sided estimations which characterize the order of convergence to its limits of some of these sequences and we give certain general explanations for its similar orders of convergence.

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1. INTRODUCTION

In the mathematical literature (*e.g.*, [12, p. 112]) some of the most usual pairs of adjacent sequences are the following:

(a) $a_n = e_n \overset{def}{=} \left(1+\frac{1}{n}\right)^n \nearrow \mathrm{e};$[1) $\qquad b_n = f_n \overset{def}{=} \left(1+\frac{1}{n}\right)^{n+1} \searrow \mathrm{e};$

(b) $a_n = E_n \overset{def}{=} 1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!} \nearrow \mathrm{e}; \quad b_n = E_n + \frac{1}{n!n} \searrow \mathrm{e};$

(c) $a_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} - 2\sqrt{n+1} \nearrow l;$

$\quad b_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} - 2\sqrt{n} \searrow l$

(where we have denoted by l the common limit of these last two sequences);

(d) $a_n = \left(1+\frac{1}{n}\right)^n \left(1+\frac{x}{n}\right) \nearrow \mathrm{e}; \qquad b_n = \left(1+\frac{1}{n}\right)^n \left(1+\frac{y}{n}\right) \searrow \mathrm{e}$ (with $x < \frac{1}{2} \le y$) (also see [16, p. 38]).

To these we can also add some other pairs deduced from [16, pp. 181–185]:

(e) $a_n = \left(1+\frac{1}{n}\right)^{n+\alpha} \nearrow \mathrm{e}; \quad b_n = \left(1+\frac{1}{n}\right)^{n+\beta} \searrow \mathrm{e}$ (with $\alpha < \frac{1}{2} \le \beta$),

(f) $a_n = n!n^{-n-1/2}\mathrm{e}^n / \exp\left(\frac{1}{12n}\right) \nearrow \sqrt{2\pi};$

$\quad b_n = n!n^{-n-1/2}\mathrm{e}^n / \exp\left(\frac{1}{12n+1/4}\right) \searrow \sqrt{2\pi}$ (see [11]),

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1) The oblique arrows show that the sequences tend increasing respectively decreasing to its limits. (N.A.)

(g) $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \nearrow \ln 2$;
$b_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} + \frac{1}{2n+1} \searrow \ln 2$,

(h) $a_n = \left(1 - \frac{1}{n}\right)^{n+1} \nearrow 1/\mathrm{e}$; $\quad b_n = \left(1 - \frac{1}{n}\right)^{n} \searrow 1/\mathrm{e}$;

(i) $a_n = H_n - \ln(n+1) \nearrow \gamma$; $\quad b_n = \gamma_n \overset{def}{=} H_n - \ln n \searrow \gamma$

(where $H_n = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ is the harmonic sum of order n and γ is the constant of Euler), see also [18, pp. 31–32, item 2.10];

(j) $a_n(s) = \zeta_n(s) - \frac{(n+1)^{1-s}}{1-s} \nearrow a(s)$;
$b_n(s) = \zeta_n(s) - \frac{n^{1-s}}{1-s} \searrow a(s)$ (with $0 < s < 1$),

(where for $s > 0$ we note $\zeta_n(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots + \frac{1}{n^s}$, for $s \in (0,1)$, $\zeta_n(s) \to \infty$ and $a(s) = \lim\limits_{n\to\infty} a_n(s) = \lim\limits_{n\to\infty} b_n(s)$).

(k) $a_n(s) = \zeta_n(s) + \frac{1}{(s-1)(n+1)^{s-1}} \nearrow \zeta(s)$;
$b_n(s) = \zeta_n(s) + \frac{1}{(s-1)n^{s-1}} \searrow \zeta(s)$ (with $s > 1$).

For $s > 1$, $\zeta(s) = \lim\limits_{n\to\infty} \zeta_n(s)$ is the zeta-function of Riemann. For $0 < s < 1$ the sequence $(b_n(s))_n$ of (j) was considered by L. Euler (see [6], [7, pp. 112–113]). The sequences of (k) appear in the proof of [29] of the known inequality:

$$\frac{1}{(s-1)(n+1)^{s-1}} < \zeta_n - \zeta_n(s) < \frac{1}{(s-1)n^{s-1}}$$

(see [8, vol. II, pp. 262–263] and [29]).

We must add now a basic standard pair of adjacent sequences namely

(l) $a_n = l - \varepsilon_n \nearrow l$; $\quad b_n = l + \varepsilon_n \searrow l$,

(where l is a fixed real number and $(\varepsilon_n)_n$ is a given sequence of positive numbers which tends strictly decreasing to 0).

2. THE DEFINITION OF PAIRS OF ADJACENT SEQUENCES AND SOME EXPLANATIONS

All the pairs of convergent sequences previously mentioned satisfy, related to a given limit l, the conditions of Cantor Dedekind type in a strict form:

$$a_1 < a_2 < a_3 < \ldots < a_n < \ldots < b_n < \ldots < b_3 < b_2 < b_1 \quad \text{(C-D1)} \tag{1}$$

(which express simultaneously two monotonicities and two boundednesses) and also the condition:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l. \tag{2}$$

[Of course, if we have the hypothesis (1) satisfied, the second condition of Cantor-Dedekind:

$$\lim_{n\to\infty} (b_n - a_n) = 0 \qquad \text{(C-D2)} \tag{3}$$

implies that it exists an unique real number l, such that we have the equality (2).]

But these conditions are not sufficient to assure that is suitable to call that the sequences $(a_n)_n$ and $(b_n)_n$ constitute a pair of adjacent sequences. We see a necessity to impose a certain condition of analytic relationship between the two sequences. Moreover, we also must put a condition of equal „velocity" of tending to its common limit of the two sequences.

So we formulate the following

Definition 1. *Two sequences $(a_n)_n$ and $(b_n)_n$ are called to be adjacent related to a given limit l if its satisfy the conditions (1) and (2) and, moreover, it exists a nondegenerate interval $I \subset \mathbb{R}$ and a function $f : I \times \mathbb{N} \to \mathbb{R}$, such that:*

(i) For any $t \in I$, we have $\lim\limits_{n\to\infty} f(t,n) = l$;

(ii) It exists $\alpha, \beta \in I$, $\alpha < \beta$, such that $f(\alpha, n) = a_n$ and $f(\beta, n) = b_n$, for any $n \in \mathbb{N}$;

(iii) We have $\lim\limits_{n\to\infty} \frac{b_n - l}{l - a_n} = 1$.

In the case of the pairs of sequences which were previous mentioned, we can take $I = \mathbb{R}$ and the functions $f : I \times \mathbb{N} \to \mathbb{R}$ together with the values α and β can be obtained without difficulties. So, as example:

for (a): $f(t,n) = (1 + \frac{1}{n})^{n+t}$; $\alpha = 0$, $\beta = 1$;

for (b): $f(t,n) = E_n + \frac{t}{n!n}$; $\alpha = 0$, $\beta = 1$;

for (c); $f(t,n) = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} - 2\sqrt{n + (1-t)}$; $\alpha = 0$, $\beta = 1$;

for (d): $f(t,n) = e_n\left(1 + \frac{t}{n}\right)$; $\alpha = x < \frac{1}{2} \le y = \beta$;

for (e): $f(t,n) = \left(1 + \frac{1}{n}\right)^{n+t}$; $\alpha < \frac{1}{2} \le \beta$;

for (f): $f(t,n) = n!n^{-n-1/2}\exp\left(n - \frac{1}{12n+t}\right)$; $\alpha = 0$, $\beta = \frac{1}{4}$;

for (g): $f(t,n) = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} + \frac{t}{2n+1}$; $\alpha = 0$, $\beta = 1$;

for (h): $f(t,n) = \left(1 - \frac{1}{n}\right)^{n+(1-t)}$; $\alpha = 0$, $\beta = 1$;

for (i): $f(t,n) = H_n - \ln(n + (1-t))$; $\alpha = 0$, $\beta = 1$;

for (j): $f(t,n) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots + \frac{1}{n^s} - \frac{(n+(1-t))^{1-s}}{1-s}$; $\alpha = 0$, $\beta = 1$;

for (k): $f(t,n) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots + \frac{1}{n^s} - \frac{1}{(s-1)(n+(1-t))^{s-1}}$; $\alpha = 0$, $\beta = 1$;

for (l): $f(t,n) = l + \varepsilon_n \cdot t$; $\alpha = -1$, $\beta = 1$.

For $x = 0$ and $y = 1$ in (d) we obtain (a); For $\alpha = 0$ and $\beta = 1$ in (e) we obtain again (a). (So we see that, for a given pair of adjacent sequences, the function f can be not unique.) For $s = \frac{1}{2}$ in (j) we obtain (c).

Also a certain convergent and monotonic sequence admits no an unique adjacent pair; if $(a_n)_n$ and $(b_n)_n$ are as in the definition, then, for any $\beta_1 > \beta$ ($\beta_1 \in I$), the sequence $n \mapsto f(\beta_1, n)$ is a pair of $(a_n)_n$; also, for any $\alpha_1 < \alpha$ ($\alpha_1 \in I$), the sequence $n \mapsto f(\alpha_1, n)$ is a pair of $(b_n)_n$. To illustrate the necessity of the condition (iii) of our definition, consider the sequences of general term:

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{3n} - \ln n;$$

$$y_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{2n} - \ln n \qquad \text{(see [30])}.$$

Both these sequences converge to the Euler's constant γ; they are obtained modifying not the logarithm (as in [4], [13]), but the last term of the harmonic sum H_n. The sequence $(x_n)_n$ is strictly increasing and the sequence $(y_n)_n$ is strictly decreasing. Also, we can choose the function $f : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$, $f(t, n) = H_{n-1} + \frac{t}{n} - \ln n$ and the values $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{2}$. But the sequences $(x_n)n$ and $(y_n)_n$ are not adjacent because the condition (iii) from definition is not satisfyied; we have: $\lim\limits_{n\to\infty} \frac{y_n-\gamma}{\gamma-x_n} = 0$. [More precisely, we observe that $\lim\limits_{n\to\infty} n^2(\gamma - y_n) = \frac{1}{12}$ and $\lim\limits_{n\to\infty} n(\gamma - x_n) = \frac{1}{6}$. All the three last results can be obtained using the so called lemma of Stolz-Cesàro for the case $\frac{0}{0}$ (see [9, p. 54] and [17]). In [30] the two-estimate $\frac{1}{12(n+1)^2} < \gamma - y_n < \frac{1}{12n^2}$ is proved. The explanation consists in the asymptotic expansion of H_n, namely:

$$H_n = \ln n + \gamma + \frac{1}{2n} + \frac{1}{12n^2} + \ldots]$$

3. A SPECIAL CASE

For some strictly monotonic, convergent sequences the finding of an adjacent pair may appear sometime unexpected. So, the sequence $(b_n)_n$ of general term $b_n = \sqrt[n]{n}$, which tends to 1 and is strictly decreasing (for $n \geq 3$) can not have as adjacent pair the sequence of general term $a_n = \sqrt[n+1]{n}$, because this, although tends to 1, is also strictly decreasing (for $n \geq 3$). A possible adjacent pair of $(b_n)_n$ is the sequence $a_n = \frac{1}{\sqrt[n]{n}}$, $(n \geq 3)$ which tends strictly increasing to 1; in this case $I = \mathbb{R}$ and $f(t, n) = \left(\sqrt[n]{n}\right)^t$ with the values $\alpha = -1$, $\beta = 1$; the conditiion (iii) is also satisfyied.

4. SOME TWO-SIDED ESTIMATIONS WHICH DESCRIBE THE FIRST ORDER OF CONVERGENCE

The order of convergence of the majority of the sequences of the Section 1 can be described by certain pairs of two-sided estimations. So, we have:

- for (a):

$$\frac{\mathrm{e}}{2n+2} < \mathrm{e} - \left(1 + \frac{1}{n}\right)^n < \frac{\mathrm{e}}{2n+1}; \qquad \text{(see [16, p. 38], [27])} \tag{4}$$

$$\frac{\mathrm{e}}{2n+1} < \left(1 + \frac{1}{n}\right)^{n+1} - \mathrm{e} < \frac{\mathrm{e}}{2n}; \qquad \text{(see [25])} \tag{4'}$$

- for (c):

$$\frac{1}{2\sqrt{n+1}} < a - a_n < \frac{1}{2\sqrt{n}}; \tag{5}$$

$$\frac{1}{2\sqrt{n+1}} < b_n - a < \frac{1}{2\sqrt{n}}, \qquad \text{(see [28])} \tag{5'}$$

- for (g):

$$\frac{1}{4n+\alpha} < \ln 2 - a_n < \frac{1}{4n+1} \quad (\alpha > 1); \qquad \text{(see [31])} \tag{6}$$

$$\frac{1}{4n+\beta} < b_n - \ln 2 < \frac{1}{4n+3} \quad (\beta > 3); \tag{6'}$$

- for (h):

$$\frac{1}{2n\mathrm{e}} < \frac{1}{\mathrm{e}} - \left(1 - \frac{1}{n}\right)^{n+1} < \frac{1}{(2n-1)\mathrm{e}}; \qquad \text{(see [14], [15])} \tag{7}$$

$$\frac{1}{(2n-1)\mathrm{e}} < \left(1 - \frac{1}{n}\right)^{n} - \frac{1}{\mathrm{e}} < \frac{1}{(2n-2)\mathrm{e}} \; (n \geq 2) \quad \text{(see [14], [15])}, \tag{7'}$$

- for (i):

$$\frac{1}{2n+1} < (H_n - \ln n) - \gamma < \frac{1}{2n}; \qquad \text{(see [26])} \tag{8}$$

$$\frac{1}{2n+2} < \gamma - (H_n - \ln(n+1)) < \frac{1}{2n+1}. \tag{8'}$$

So, for the examples (5) and (5′), we have $a - a_n = \mathcal{O}(\frac{1}{\sqrt{n}})$, $b_n - a = \mathcal{O}(\frac{1}{\sqrt{n}})$, and for all the other the median term of the two sided estimations is $\mathcal{O}(\frac{1}{n})$ (recall that $\alpha_n = \mathcal{O}(\beta_n)$ if it exists a constant $C > 0$ and $n_0 \in \mathbb{N}$ such that $|\alpha_n| \leq C|\beta_n|$, for all $n \geq n_0$).

5. THE FIRST ITERATED LIMIT

We recall now some basic facts of the asymptotic analysis; we present its directly for the sequences, *i.e.* for the functions of natural variable, in a neighborhood of the unique accumulation point of the domain of definition $\mathbb{N}$, of the sequences, namely ∞.

If $(x_n)_n$ and $(y_n)_n$ are two sequences we call that these are asymptotic equivalent (and we write $x_n \sim y_n$) if $\lim\limits_{n\to\infty} x_n/y_n$ exists and is a finite and different of 0 number.

Consider now a convergent sequence $(x_n)_n = (x(n))_n$ with the limit l. A sequence $(u_k)_k$ of functions of natural variable n, $u_k = u_k(n)$, $k = 0, 1, 2, \ldots$, with $n \in \mathbb{N}$ is called to be an asymptotic scale for the given sequence $(x_n)_n$ if it exists a sequence of real numbers $(l_k)_k$, where $l_0 = l$, such that, for any $k \in \mathbb{N}$, we have:

$$x_n \sim l_0 + l_1 u_1(n) + l_2 u_2(n) + \ldots + l_k u_k(n),$$

where, for every $j = 0, 1, \ldots, k-1$, we have $u_{j+1} = o(u_j)$, *i.e.* $\lim\limits_{n\to\infty} \frac{u_{j+1}(n)}{u_j(n)} = 0$.

The coefficients l_1, l_2, l_3,..., are also called the iterated limits of sequence $(x_n)_n$ (with respect to the asymptotic scale $(u_k)_k$). (See the expository books of Copson [2], De Bruijn [3], Erdely [5], van der Corput [19]–[24].)

All our inequalities of the section 3 implies that, for two adjacent sequences, the first iterated limit (related to the same function $n \mapsto u_1(n)$ of a given asymptotic scale of functions of natural variable $u_k = u_k(n)$, $k = 0, 1, 2, \ldots$) is the same. So, we have:

- for (a):

$$\lim_{n\to\infty} n\left(\mathrm{e} - (1+\tfrac{1}{n})^n\right) = \tfrac{\mathrm{e}}{2}; \tag{9}$$

$$\lim_{n\to\infty} n\left((1+\tfrac{1}{n})^{n+1} - \mathrm{e}\right) = \tfrac{\mathrm{e}}{2}; \tag{9'}$$

- for (c):

$$\lim_{n\to\infty} \sqrt{n}(l - a_n) = \tfrac{1}{2}; \tag{10}$$
$$\lim_{n\to\infty} \sqrt{n}(b_n - l) = \tfrac{1}{2}; \tag{10'}$$

- for (g):

$$\lim_{n\to\infty} n(\ln 2 - a_n) = \tfrac{1}{4}; \tag{11}$$
$$\lim_{n\to\infty} n(b_n - \ln 2) = \tfrac{1}{4}; \tag{11'}$$

- for (h):

$$\lim_{n\to\infty} n\left(\tfrac{1}{\mathrm{e}} - \left(1-\tfrac{1}{n}\right)^{n+1}\right) = \tfrac{1}{2\mathrm{e}}; \tag{12}$$

$$\lim_{n\to\infty} n\left(\left(1-\tfrac{1}{n}\right)^{n} - \tfrac{1}{\mathrm{e}}\right) = \tfrac{1}{2\mathrm{e}}; \tag{12'}$$

- for (i):

$$\lim_{n\to\infty} n\left(\gamma - (H_n - \ln(n+1))\right) = \tfrac{1}{2}; \tag{13}$$

$$\lim_{n\to\infty} n\left((H_n - \ln n) - \gamma\right) = \tfrac{1}{2}. \tag{13'}$$

We can present an explanation of this fact.

Proposition 2. *If $(a_n)_n$ and $(b_n)_n$ are two adjacent sequences, then, for any asymptotic scale $(u_k)_k$ (the same for both the sequences), its first iterated limits are equal.*

Proof. Let l_1 and λ_1 be the first iterated limits of $(a_n)_n$ and $(b_n)_n$. Suppose, *ad absurdum*, that $l_1 \neq \lambda_1$. Then we have:

$$\lim_{n\to\infty} \frac{b_n - l}{l - a_n} = \frac{\lim\limits_{n\to\infty} u_1(n)(b_n - l)}{\lim\limits_{n\to\infty} u_1(n)(l - a_n)} = \frac{\lambda_1}{l_1} \neq 1,$$

a contradiction! Therefore $l_1 = \lambda_1$.

This shows for what, in all the previous two-sided estimations, the principal coefficient must be the same. □

6. SOME RESULTS

We can give now some general results concerning the order of convergence of the sequences of adjacent pairs.

PROPOSITION 3. *Let* $(a_n)_n$ *and* $(b_n)_n$ *be two adjacent sequences. If* f *admits a partial derivative respecting the variable* t *and, for any* $n \in \mathbb{N}$, *it exists* $m_n, M_n \in \mathbb{R}$, $m_n < M_n$ *such that:*

$$m_n \le \frac{\partial f}{\partial t}(t, n) \le M_n \qquad (\textit{for any}\, t \in I).$$

Then:

$$(\beta - \alpha) m_n \le b_n - a_n \le (\beta - \alpha) M_n \qquad (\textit{for any}\, n \in \mathbb{N}).$$

Proof. Let $n \in \mathbb{N}$ be fixed. So we can consider the function $t \mapsto f(t, n)$ as a function of one variable t and $f'(t, n) = \frac{\partial f}{\partial t}(t, n)$. In view of the mean value theorem of Lagrange, there exists $\tau \in (\alpha, \beta)$ such that:

$$b_n - a_n = f(\beta, n) - f(\alpha, n) = \frac{\partial f}{\partial t}(\tau, n)(\beta - \alpha),$$

which gives our conclusion. □

COROLLARY 4. *In the same hypotesis we have:*

$$l - a_n < (\beta - \alpha) M_n, \qquad b_n - l < (\beta - \alpha) M_n$$

(*because* $l - a_n < b_n - a_n$ *and* $b_n - l < b_n - a_n$).

7. THE CASE OF CONTINUATION

For many sequences of real numbers $(a_n)_{n \ge 1}$ it is possible to find a function $g : [1, \infty) \to \mathbb{R}$ such that $g(n) = a_n$, for any $n \in \mathbb{N}$, called a continuation of the sequence on the positive real axis.

Two of nontrivial examples are the factorial and the harmonic sum.

So, for $a_n = n!$, the continuation to $[0, \infty)$ is made by the Γ function of Euler,

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} \mathrm{d}t, \quad x \ge 1.$$

As $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = 1$, we have $\Gamma(n+1) = n!$. Therefore the function $g : [0, \infty) \to \mathbb{R}$, $g(x) = \int_0^\infty t^x \mathrm{e}^{-t} \mathrm{d}t$ is the continuation of the factorial on the positive axis $[0, \infty)$.

The functiom Γ is also logarithmically-convex. A beautiful theorem of Bohr and Mollerup characterizes completely the function: if $f : [0, \infty) \to [0, \infty)$ satisfies the functional equation $f(x+1) = xf(x)$, is logarithmically-convex and $f(1) = 1$, then $f = \Gamma$.

For the sequence $(H_n)_{n\geq 1}$ of the harmonic sums, the continuation function is $H:(0,\infty)\to\mathbb{R}$, defined by the equality $H(x)=\psi(x+1)-\gamma$, where ψ is the logarithmic derivative of Γ, $\psi(x)=\Gamma'(x)/\Gamma(x)$, and γ is the constant of Euler.

Suppose now that the adjacent sequences $(a_n)_n$ and $(b_n)_n$ admits continuations on $[1,\infty)$. In this case the function $f: I\times\mathbb{N}\to\mathbb{R}$ admits a natural continuation on $I\times[1,\infty)$.

Proposition 5. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences which admit continuations to $[1,\infty)$ and for which it exists a function f which satisfies the conditions (i) and (ii) of the definition. If f admits a partial derivative and it exists in $\mathbb{R}$ $\lim\limits_{x\to x_0}\frac{\partial f}{\partial x}(\alpha,x)=\lim\limits_{x\to x_0}\frac{\partial f}{\partial x}(\beta,x)\neq 0$, then the sequences $(a_n)_n$ and $(b_n)_n$ are adjacent.*

Proof. The sequences of general term b_n-l, respectively $l-a_n$ has strictly positive terms and tend decreasing to zero. We obtain:

$$\lim_{x\to x_0}\frac{(b_{n+1}-l)-(b_n-l)}{(a_{n+1}-l)-(a_n-l)}=\lim_{x\to x_0}\frac{f(\beta,n+1)-f(\beta,n)}{f(\alpha,n+1)-f(\alpha,n)}=\lim_{x\to x_0}\frac{\frac{\partial f}{\partial x}(\beta,\nu_1)}{\frac{\partial f}{\partial x}(\alpha,\nu_2)}$$

where ν_1 and ν_2 are contained in the interval $(n,n+1)$ and tend to ∞ when $n\to\infty$. From the hypotesis, the last limit is equal to 1. According to the lemma of Stolz-Cesàro for the case $\frac{0}{0}$ (see [9, p. 56], [17]) we have the conclusion. $\square$

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