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ASYMPTOTIC PROPERTIES AND BEHAVIOR OF SOME NONTRIVIAL SEQUENCES

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Abstract. Convergence properties and asymptotic behavior of several real sequences are investigated analytically. Some remarkable properties of these sequences are established including their limits.

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1. INTRODUCTION.

The investigation of the general properties and behavior of sequences is a subject that is always of interest and makes use of many tools and ideas from analysis. It is the objective here to study some related sequences none of which is easy to study analytically [1, 2]. These sequences are defined on the natural numbers by an analytic formula and lead to many interesting consequences. Asymptotic analysis is very effective, for example as a method of determining limiting behavior [2, 3].

Let us start by defining the two main sequences discussed here at the outset. The first sequence to be studied here is referred to as (z_n) and is defined for all $n \in \mathbb{N}$ by the equation

$$z_n = (n+1)^{\frac{1}{n+1}} - 1 - \tfrac{\log(n)}{n}. \tag{1}$$

There is another sequence somewhat more untractable than (1) and it is defined for $n \in \mathbb{N}$ by the formula

$$x_n = \sum_{k=1}^{n} k^{\frac{1}{k}} - n - \tfrac{1}{2}\log(n)^2. \tag{2}$$

There is also a version similar to (2) that is involves a definite integral to define it and it is considered last. This sequence is defined for $n \in \mathbb{N}$ as follows

$$\beta_n = \int_1^n \left(x^{\frac{1}{x}} - 1 - \tfrac{\log(x)}{x}\right) dx. \tag{3}$$

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These sequences will now be studied in turn. It will be seen that infinite series also play a major role here [4, 5, 6]. There has been interest recently in the study of various kinds of sequences [7, 8]. It is also worth mentioning that the asymptotic expansions that appear in Theorem 1, for example, can be generated by means of symbolic manipulation.

2. DISCRETE SEQUENCES.

THEOREM 1. *The sequence* (z_n) *consists of terms which are strictly positive and which decrease monotonically from above. The limit of sequence* (z_n) *as* $n \to \infty$ *is zero.*

Proof. The claim can be verified numerically to $n = 100$ with $z_{100} \doteq 0.00070262816\ldots$ quite easily. Using Maple, the accuracy of numerical calculations can be augmented by setting digits accordingly. For values of n larger than this, (z_n) admits the following asymptotic expansion,

$$\begin{aligned} z_n =& \tfrac{1}{n^2}\Big[(\tfrac{1}{2}\log(n) - 1)\log(n) + \tfrac{1}{n}(\tfrac{1}{6}(\log(n) - 6)\log(n)^2 + 3(4\log(n) - 3))\Big] \\ &+ \tfrac{1}{24n^4}\Big[(\log(n) - 12)\ \log(n)^3 + 48(\log(n) - \tfrac{7}{2})\log(n) + 56\Big] \\ &+ \mathcal{O}(\tfrac{\log(n)^5}{n^5}). \end{aligned} \tag{4}$$

For $n > 100$, the first term inside the brackets is positive and behaves like $\log(n)^2$. In comparison with the first term, the second and third terms approach zero as n grows. More over, the second term is positive for $\log(n) > 6$, and the third term as well once $\log(n) > 12$. Combining the numerical work with (4), since $(\log(n)/n)^5 < 2 \cdot 10^{-7}$ for all $n > 100$, it can be concluded that the sequence $z_n > 0$ for all $n \in \mathbb{N}$.

To show that (z_n) is a decreasing sequence, define the function

$$g(x) = (x+1)^{\frac{1}{x+1}} - 1 - \tfrac{\log(x)}{x}. \tag{5}$$

The sequence (z_n) is recovered by putting $x = n$ in (5). Since $g(x)$ has a derivative when $x \in (0, \infty)$, the derivative is found to be

$$g'(x) = -(x+1)^{\frac{1}{x+1}}(\tfrac{\log(x+1)-1}{(x+1)^2}) + \tfrac{\log(x)-1}{x}. \tag{6}$$

It can be verified that $g'(n) < 0$ for $1 \le n \le 100$. Moreover, $g'(n)$ admits the following asymptotic expansion,

$$\begin{aligned} g'(n) =& -\tfrac{1}{n^3}\big[(\log(n) - 3)\log(n) + 3 + \tfrac{1}{2n}((\log(n) - 7)\log(n)^2 + 16\log(n) - 13) \\ &+ \tfrac{1}{6n^2}((\log(n) - 13)\log(n)^3 + (5\log(n) - 108)\log(n)\big] + \mathcal{O}(\tfrac{\log(n)^5}{n^6}). \end{aligned} \tag{7}$$

The first terms inside the first bracket grows like $\log(n)^2$ whereas the second and third terms become positive too and continue to approach zero as n grows. This implies that the derivative of $g(x)$ is negative, and therefore z_n is a decreasing sequence. From the definition (1), it is clear that in (1) the first two terms and the last approach zero as n gets large and in fact

$$\lim_{n\to\infty} z_n = 0.$$

This proves the claim. □

The sequence (x_n) is somewhat more challenging and the next theorem begins the study of this sequence.

THEOREM 2. *The sequence (x_n) defined in (2) has positive values and is strictly increasing for all $n \in \mathbb{N}$.*

Proof. Consider the difference $x_{n+1} - x_n$. It is calculated from (2) to be exactly

$$x_{n+1} - x_n = (n+1)^{\frac{1}{n+1}} - 1 - \tfrac{1}{2}\big(\log(n+1)^2 - \log(n)^2\big) \tag{8}$$

By the mean value theorem, there exists a $\tau_n \in (n, n+1)$ such that

$$\log(n+1)^2 - \log(n)^2 = 2\tfrac{\log(\tau_n)}{\tau_n}. \tag{9}$$

Since the function $\log(x)/x$ is strictly decreasing for $x > 3$, there are the inequalities

$$\tfrac{\log(n+1)}{n+1} < \tfrac{\log(\tau_n)}{\tau_n} < \tfrac{\log(n)}{n}. \tag{10}$$

Using (9) and (10) in (8), it is concluded that

$$x_{n+1} - x_n > (n+1)^{\frac{1}{n+1}} - 1 - \tfrac{\log(n)}{n}. \tag{11}$$

The right-hand side of (11) is exactly (1), the sequence (z_n). By Theorem 1, it is known that $z_n > 0$ for all natural numbers. Applying Theorem 1 to (11), it follows that $x_{n+1} - x_n > 0$ or $x_{n+1} > x_n$. This is stating that sequence (x_n) is strictly increasing. Since $x_1 > 0$ and (x_n) is strictly increasing, it has to be that $x_n > 0$ for all $n \in \mathbb{N}$. □

From (8) it follows that

$$x_{n+1} < x_n + (n+1)^{\frac{1}{n+1}} - 1. \tag{12}$$

The last two terms on the right of (12) approach zero as n gets large. If all x_n can be bounded by a large constant for all n up to $n = N$, then (12) implies that x_{N+1} can be bounded by the same constant. Therefore, (x_n) is strictly increasing and bounded, so it has to converge by the monotone convergence theorem.

In order to obtain the limit, it is necessary to be able to place bounds on certain sums in what follows. To do this it is useful to recall the Euler summation formula.

THEOREM 3 (Euler Summation Formula). *For $m \le n$,*

$$\sum_{k=m}^{n} f(k) = \int_m^n f(x)\,dx + \tfrac{1}{2}[f(m)+f(n)] + \tfrac{1}{12}[f'(n)-f'(m)] + \rho(f;m,n), \tag{13}$$

with

$$|\rho(f;m,n)| \le \tfrac{1}{120} \int_m^n |f'''(t)|\,dt.$$

In order to calculate the limit, it is necessary to expand this sequence in the following way. To begin, for $N \ge 1$ we can write

$$\begin{aligned} x_N &= \sum_{k=1}^{N} k^{\frac{1}{k}} - N - \tfrac{1}{2}\log(N)^2 = \sum_{k=1}^{N}\sum_{m=0}^{\infty} \tfrac{1}{m!}\left(\tfrac{\log k}{k}\right)^m - N - \tfrac{1}{2}\log(N)^2 \\ &= \sum_{k=1}^{N}\sum_{m=1}^{\infty} \tfrac{1}{m!}\left(\tfrac{\log k}{k}\right)^m - \tfrac{1}{2}\log(N)^2 \\ &= \sum_{k=1}^{N} \tfrac{\log k}{k} - \tfrac{1}{2}\log(N) + \sum_{m=2}^{\infty} \tfrac{1}{m!} \sum_{k=1}^{N} \left(\tfrac{\log k}{k}\right)^m. \end{aligned} \tag{14}$$

To estimate the k-dependent series in (14), the Euler Summation formula is used. For the first series in (14), we have

$$\begin{aligned} \sum_{k=1}^{N} \tfrac{\log k}{k} &= \int_1^N \tfrac{\log x}{x}\,dx + \tfrac{\log(N)}{2N} + \mathcal{O}\left(\tfrac{\log(N)^2}{N^2}\right) = \tfrac{1}{2}\log(N)^2 + \tfrac{\log(N)}{2N} \\ &\quad + \mathcal{O}\left(\tfrac{\log(N)^2}{N^2}\right). \end{aligned} \tag{15}$$

For the last series in (14), the following integral for $m > 1$ is required,

$$\begin{aligned} \int_1^N \tfrac{(\log x)^m}{x^m}\,dx &= \int_0^{\log(N)} u^m e^{-(m+1)u}\,du \\ &= \int_0^{\infty} u^m e^{-(m-1)u}\,du - \int_{\log(N)}^{\infty} u^m e^{-(m-1)u}\,du \\ &= \tfrac{m!}{(m-1)^{m-1}} - \int_{\log(N)}^{\infty} u^m e^{-(m-1)u}\,du. \end{aligned} \tag{16}$$

For each $m \ge 2$ the remaining integral on the right side of (16) approaches zero as $N \to \infty$. Consequently, the final sum in x_N takes the following form,

$$\begin{aligned} &\sum_{m=2}^{\infty} \tfrac{1}{m!} \sum_{k=1}^{N} \left(\tfrac{\log k}{k}\right)^m = \\ &= \sum_{m=2}^{\infty} \tfrac{1}{(m-1)^{m+1}} - \sum_{m=2}^{\infty} \tfrac{1}{m!} \int_{\log(N)}^{\infty} u^m e^{-(m-1)u}\,du + \mathcal{O}\left(\left(\tfrac{\log(N)}{N}\right)^m\right). \end{aligned} \tag{17}$$

Substituting (15) and (17) into (14), we obtain that

$$x_N = \sum_{n=1}^{\infty} \tfrac{1}{n^{n+2}} - \sum_{m=2}^{\infty} \tfrac{1}{m!} \int_{\log(N)}^{\infty} u^m e^{-(m-1)u}\, du + \mathcal{O}(\tfrac{\log(N)}{N}). \tag{18}$$

The result in (18) is presented in the following Theorem.

THEOREM 4. *The sequence* (x_n) *for* $n \in \mathbb{N}$ *defined by* (2) *approaches a finite limit which is given by*

$$\lim_{n\to\infty} x_n = \sum_{n=1}^{\infty} \tfrac{1}{n^{n+2}}. \tag{19}$$

3. INTEGRAL VERSION RELATED TO THESE SEQUENCES.

There is an analogue of the sequence defined in (2) which is worth studying. This sequence employs a definite integral in its definition rather than a sum. To motivate the definition of this sequence, begin by defining the sequence (I_n) in the following form,

$$I_n = \int_0^n (x^{\frac{1}{x}} - 1)\, dx. \tag{20}$$

Writing the function $x^{1/x}$ in the form of an exponential, expanding this exponential and then using some substitutions, equation (20) can be put in the following form,

$$\begin{aligned} I_n &= \int_1^n (x^{\frac{1}{x}} - 1)\, dx = \int_1^n \Big(\sum_{k=0}^{\infty} \tfrac{1}{k!} (\tfrac{\log(x)}{x})^k - 1\Big)\, dx \\ &= \int_1^{\infty} \sum_{k=1}^{\infty} \tfrac{1}{k!} (\tfrac{\log(x)}{x})^k\, dx = \int_1^n \tfrac{\log(x)}{x}\, dx + \sum_{k=2}^{\infty} \tfrac{1}{k!} \int_1^n (\tfrac{\log(x)}{x})^k\, dx \\ &= \tfrac{1}{2}\log(n)^2 + \sum_{k=2}^{\infty} \tfrac{1}{k!} \int_1^{\log(n)} u^k e^{-(k-1)u}\, du \\ &= \tfrac{1}{2}\log(n)^2 + \sum_{k=2}^{\infty} \tfrac{1}{k!} \tfrac{1}{(k-1)^{k+1}} \int_0^{(k-1)\log(n)} t^k e^{-t}\, dt. \end{aligned} \tag{21}$$

Based on the final equation in (21), since the series on the right has to converge, define the following sequence for $n \in \mathbb{N}$ already introduced in (3),

$$\beta_n = \int_1^n (x^{\frac{1}{x}} - 1 - \tfrac{\log(x)}{x})\, dx.$$

It is desired to put β_n in a form in which its behavior for large n is clear and which implies the limit exists and permits its evaluation. Using the integral form of the gamma function, it is clear that

$$\begin{aligned}
\beta_n &= \sum_{k=2}^{\infty} \tfrac{1}{k!} \tfrac{1}{(k-1)^{k+1}} \Big(\int_0^{\infty} t^k e^{-t}\,dt - \int_{(k-1)\log(n)}^{\infty} t^k e^{-t}\,dt \Big) \\
&= \sum_{k=2}^{\infty} \tfrac{1}{(k-1)^{k+1}} - \sum_{k=2}^{\infty} \tfrac{1}{k!} \tfrac{1}{(k-1)^{k+1}} \int_{(k-1)\log(n)}^{\infty} t^k e^{-t}\,dt \\
&= \sum_{k=2}^{\infty} \tfrac{1}{(k-1)^{k+1}} - \tfrac{1}{2n} (\log(n)^2 + 2\log(n) + 2) \\
&\quad - \sum_{k=3}^{\infty} \tfrac{1}{k!} \tfrac{1}{(k-1)^{k+1}} \int_{(k-1)\log(n)}^{\infty} t^k e^{-t}\,dt. \qquad (22)
\end{aligned}$$

The second term on the right-hand side approaches zero as $n \to \infty$. The remaining series on the right side also converges for each large n and in fact approaches zero in the limit as well. It is not hard to give support to this claim by developing the following rough upper bound

$$\begin{aligned}
&\int_{(k-1)\log(n)}^{\infty} t^k e^{-t} e^{-(t-1)}\,dt \leq \\
&\leq ((k-1)\log(n))^k e^{-(k-1)\log(n)} \int_{(k-1)\log(n)}^{\infty} e^{-(t-1)}\,dt \\
&= \tfrac{(k-1)^k}{n^{k-1}} (\log(n))^k \, (e^{(\log(n))^{-(k-1)}+1}) = e\,(k-1)^k \tfrac{(\log(n))^k}{(n^{k-1})^2} \\
&\leq e\,(k-1)^k \, \tfrac{n^k}{n^{2k-2}}. \qquad (23)
\end{aligned}$$

Substituting this into the sum in (22), we have the upper bound

$$\begin{aligned}
\sum_{k=3}^{\infty} \tfrac{1}{k!} \tfrac{1}{(k-1)^{k+1}} \int_{(k-1)\log(n)}^{\infty} t^k e^{-t}\,dt &< e \sum_{k=3}^{\infty} \tfrac{1}{k!(k-1)} \tfrac{1}{n^{k-2}} \\
&< e \sum_{k=3}^{\infty} \tfrac{n^{k-2}}{(k-2)!} < 3 \cdot (e^{1/n} - 1). \qquad (24)
\end{aligned}$$

This implies that the series on the left of (24) is squeezed to zero as $n \to \infty$. With these results, it is possible to quickly prove the following theorem.

THEOREM 5. *The sequence β_n defined by (3) is increasing and has a finite limit which is given by*

$$\lim_{n \to \infty} \beta_n = \sum_{n=1}^{\infty} \tfrac{1}{n^{n+2}}. \qquad (25)$$

Proof. From the last term in (21), sequence β_n is given by

$$\beta_n = \sum_{k=2}^{\infty} \tfrac{1}{k!} \tfrac{1}{(k-1)^{k+1}} \int_0^{(k-1)\log(n)} t^k e^{-t}\,dt. \qquad (26)$$

Hence (26) implies that β_n is increasing because the integrand is strictly positive on $(0, \infty)$ and the upper limit on the integral grows like $\log(n)$. Since the right side of the inequality in (24) is finite the remainder term in (22) converges and since it converges to zero as noted already, this implies the limit of sequence β_n in (3) exists. Based on result (22), the limit must be given by (25). □

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