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GENERALIZED OSTROWSKI INEQUALITIES AND COMPUTATIONAL INTEGRATION

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Abstract. We state and prove three generalized results related to Ostrowski inequality by using differentiable functions which are bounded, bounded below only and bounded above only, respectively. From our proposed results we get number of established results as our special cases. Some applications in numerical integration are also given which gives us some standard and nonstandard quadrature rules.

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1. INTRODUCTION.

Ostrowski inequality has gained supreme position among many types of integral inequalities. This interesting and useful inequality [19] was first presented by the Ukrainian mathematician Alexander Markowich Ostrowski in 1938, which is stated as follows:

THEOREM 1.1. *Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mappings on I^o, the interior of the interval I, such that g is differentiable and belongs to $L[a_0, a_1]$, where $a_0, a_1 \in I$ with $a_0 < a_1$. If $|g'(\eta)| \leq M$, valid for all $\eta \in [a_0, a_1]$ and M is positive real constant, then we have the following inequality:*

$$\left|g(\eta) - \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds\right| \leq M(a_1-a_0)\left[\tfrac{1}{4} + \tfrac{\left(\eta-\frac{a_0+a_1}{2}\right)^2}{(a_1-a_0)^2}\right],$$

where $\frac{1}{4}$ is the best possible constant that it cannot be replaced by any smaller one value.

Ostrowski inequality can be used to determine the absolute deviation of functional value from its integral mean. It also approximates area under the curve of a function by a rectangle. It has great importance because of its number of applications in statistics, probability theory, integral operator theory,

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numerical quadrature rules and special means. Due to its high importance most of the researchers are continuously working on its generalization by using various techniques. Even we can find research work on Ostrowski in 70's as can be seen in [15, 16]. For some of its recent generalizations and different variants we refer the reader to the following articles [1, 7–14, 20, 22, 23].

In this paper, we would use our main results to introduce standard quadratures and nonstandard quadratures rules:

$$A_1(g) : \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(\eta)d\eta \cong \tfrac{g(a_0)+g(a_1)}{2},$$
$$A_2(g) : \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(\eta)d\eta \cong g\left(\tfrac{a_0+a_1}{2}\right),$$
$$A_3(g) : \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(\eta)d\eta \cong \tfrac{1}{2}\left[g\left(\tfrac{a_0+a_1}{2}\right)+\tfrac{g(a_0)+g(a_1)}{2}\right],$$
$$A_4(g) : \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(\eta)d\eta \cong \tfrac{1}{2}\left[-g(a_0)+2g\left(\tfrac{a_0+a_1}{2}\right)+g(a_1)\right],$$
$$A_5(g) : \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(\eta)d\eta \cong \tfrac{1}{2}\left[g(a_0)+2g\left(\tfrac{a_0+a_1}{2}\right)-g(a_1)\right],$$
$$A_6(g) : \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(\eta)d\eta \cong g(a_1),$$
$$A_7(g) : \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(\eta)d\eta \cong g(a_0).$$

We need some lemmas that would be helpful in our main result.

LEMMA 1.2. *Let g be as in Theorem 1.1. Consider the following kernel on $[a_0, a_1]$ for all $\delta \in [0,1]$ and*

$$\eta + \delta\tfrac{a_1-a_0}{2} \leq \eta \leq \tfrac{a_0+a_1}{2}$$

$$K(\eta, s) = \begin{cases} s-\eta+\delta\frac{a_1-a_0}{2}, & \text{if} \quad s\in[a_0,\eta], \\ s-\eta-\delta\frac{a_1-a_0}{2}, & \text{if} \quad s\in(\eta,a_1]. \end{cases} \tag{1.1}$$

Then the following identity holds

$$\int_{a_0}^{a_1} K(\eta,s)g'(\eta)ds = \delta(a_1-a_0)\left[g(\eta)-\tfrac{g(a_0)+g(a_1)}{2}\right] + (a_1-\eta)g(a_1)+(\eta-a_0)g(a_0)-\int_{a_0}^{a_1} g(s)ds. \tag{1.2}$$

Proof. Applying integration-by-parts for the Riemann-Stieltjes integral in kernel (1.1), we get

$$\int_{a_0}^{\eta}\left(s-\eta+\delta\tfrac{a_1-a_0}{2}\right)dg(s) = \delta\tfrac{a_1-a_0}{2}g(\eta)-\left(\eta-a_0+\delta\tfrac{a_1-a_0}{2}\right)g(\eta)-\int_{a_0}^{\eta} g(s)ds, \tag{1.3}$$

and

(1.4)
$$\int_{\eta}^{a_1}\left(s-\eta-\delta\tfrac{a_1-a_0}{2}\right)dg(s)=\left(a_1-\eta-\delta\tfrac{a_1-a_0}{2}\right)g(a_1)+\delta\tfrac{a_1-a_0}{2}g(\eta)-\int_{a_1}^{\eta}g(s)ds.$$

By adding the equalities (1.3) and (1.4), we get (1.2). □

We use the following lemma from [13] to proceed further.

LEMMA 1.3. *If $\gamma(\eta)\le g'(\eta)\le\tau(\eta)$ for any $\gamma,\tau\in C[a_0,a_1]$ and $\eta\in[a_0,a_1]$, then we have*

$$\left|g'(\eta)-\tfrac{\gamma(s)+\tau(s)}{2}\right|\le\tfrac{\tau(s)-\gamma(s)}{2}. \tag{1.5}$$

With the help of kernel (1.1), our concentration is to derive different bounds of Ostrowski type inequality. Also explicit error bounds for numerical quadrature and nonstandard quadrature formulas will be discussed. Also we recaptured many established results from articles [12], [13], [14] and [21].

2. MAIN RESULTS

THEOREM 2.1. *Let $g\colon I\to\mathbb{R}$, be a differentiable function in the interior I^0 of interval I and let $[a_0,a_1]\subset I^0$. If $\gamma(\eta)\le g'(\eta)\le\tau(\eta)$ for any $\gamma,\tau\in C[a_0,a_1]$, $\eta\in[a_0,a_1]$, then for all $\delta\in[0,1]$ the following inequality holds*

$$\begin{aligned}m(\eta,\delta)\le&\delta\left[g(\eta)-\tfrac{g(a_0)+g(a_1)}{2}\right]+\tfrac{a_1-\eta}{a_1-a_0}g(a_1)\\&+\tfrac{\eta-a_0}{a_1-a_0}g(a_0)-\tfrac{1}{a_1-a_0}\int_{a_0}^{a_1}g(s)ds\le M(\eta,\delta),\end{aligned}\tag{2.1}$$

or

$$\begin{aligned}(a_1-a_0)m(\eta,\delta)\le&\delta(a_1-a_0)\left[g(\eta)-\tfrac{g(a_0)+g(a_1)}{2}\right]+(a_1-\eta)g(a_1)\\&+(\eta-a_0)g(a_0)-\int_{a_0}^{a_1}g(s)ds\le(a_1-a_0)M(\eta,\delta),\end{aligned}\tag{2.2}$$

where

$$m(\eta,\delta)=$$
$$=\tfrac{1}{a_1-a_0}\Bigg[\int_{-\delta\frac{a_1-a_0}{2}}^{a_1-\eta-\delta\frac{a_1-a_0}{2}}\left(\tfrac{\zeta+|\zeta|}{2}\gamma\left(\zeta+\eta+\delta\tfrac{a_1-a_0}{2}\right)+\tfrac{\zeta-|\zeta|}{2}\tau\left(\zeta+\eta+\delta\tfrac{a_1-a_0}{2}\right)\right)d\zeta$$
$$+\int_{\eta-a_0+\delta\frac{a_1-a_0}{2}}^{\delta\frac{a_1-a_0}{2}}\left(\tfrac{\zeta+|\zeta|}{2}\gamma\left(\zeta+\eta-\delta\tfrac{a_1-a_0}{2}\right)+\tfrac{\zeta-|\zeta|}{2}\tau\left(\zeta+\eta-\delta\tfrac{a_1-a_0}{2}\right)\right)d\zeta\Bigg]$$

and

$$M(\eta,\delta)=$$
$$=\tfrac{1}{a_1-a_0}\Bigg[\int_{-\delta\frac{a_1-a_0}{2}}^{a_1-\eta-\delta\frac{a_1-a_0}{2}}\left(\tfrac{\zeta-|\zeta|}{2}\gamma\left(\zeta+\eta+\delta\tfrac{a_1-a_0}{2}\right)+\tfrac{\zeta+|\zeta|}{2}\tau\left(\zeta+\eta+\delta\tfrac{a_1-a_0}{2}\right)\right)d\zeta+$$

$$+\int_{\eta-a_0+\delta\frac{a_1-a_0}{2}}^{\delta\frac{a_1-a_0}{2}}\left(\tfrac{\zeta-|\zeta|}{2}\gamma\left(\zeta+\eta-\delta\tfrac{a_1-a_0}{2}\right)+\tfrac{\zeta+|\zeta|}{2}\tau\left(\zeta+\eta-\delta\tfrac{a_1-a_0}{2}\right)\right)d\zeta\Bigg].$$

Proof. By referring to kernel (1.1) and identity (1.2), we first have

$$\begin{aligned}
&\int_{a_0}^{a_1}K(\eta,s)\left(g'(\eta)-\tfrac{\gamma(s)+\tau(s)}{2}\right)ds=\\
&=\int_{a_0}^{a_1}K(\eta,s)g'(\eta)ds-\tfrac{1}{2}\left(\int_{a_0}^{a_1}K(\eta,s)\left(\gamma(s)+\tau(s)\right)ds\right)\\
&=(a_1-a_0)\left[g(\eta)-\tfrac{g(a_0)+g(a_1)}{2}\right]+(a_1-\eta)g(a_1)+(\eta-a_0)g(a_0)\\
&\quad-\int_{a_0}^{a_1}g(s)ds-\tfrac{1}{2}\left[\int_{a_0}^{\eta}\left(s-\eta+\delta\tfrac{a_1-a_0}{2}\right)(\gamma(s)+\tau(s))ds\right.\\
&\quad\left.+\int_{\eta}^{a_1}\left(s-\eta-\delta\tfrac{a_1-a_0}{2}\right)(\gamma(s)+\tau(s))ds\right].
\end{aligned}\tag{2.3}$$

Therefore, we can conclude from (1.5) and (2.3) that

$$\begin{aligned}
&\Bigg|(a_1-a_0)\left[g(\eta)-\tfrac{g(a_0)+g(a_1)}{2}\right]+(a_1-\eta)g(a_1)+(\eta-a_0)g(a_0)\\
&\quad-\int_{a_0}^{a_1}g(s)ds-\tfrac{1}{2}\left[\int_{a_0}^{\eta}\left(s-\eta+\delta\tfrac{a_1-a_0}{2}\right)(\gamma(s)+\tau(s))ds\right.\\
&\quad\left.+\int_{\eta}^{a_1}\left(s-\eta-\delta\tfrac{a_1-a_0}{2}\right)(\gamma(s)+\tau(s))ds\right]\Bigg|=\\
&=\left|\int_{a_0}^{a_1}K(\eta,s)\left(g'(\eta)-\tfrac{\gamma(s)+\tau(s)}{2}\right)ds\right|\\
&\le\int_{a_0}^{a_1}|K(\eta,s)|\left|\left(g'(\eta)-\tfrac{\gamma(s)+\tau(s)}{2}\right)ds\right|\\
&\le\int_{a_0}^{a_1}|K(\eta,s)|\left(\tfrac{\tau(s)-\gamma(s)}{2}\right)ds\\
&=\tfrac{1}{2}\left[\int_{a_0}^{\eta}\left|s-\eta+\delta\tfrac{a_1-a_0}{2}\right|(\tau(s)-\gamma(s))ds\right.\\
&\quad\left.+\int_{\eta}^{a_1}\left|s-\eta-\delta\tfrac{a_1-a_0}{2}\right|(\tau(s)-\gamma(s))ds\right].
\end{aligned}\tag{2.4}$$

After re-arranging (2.4), we obtain

$$\begin{aligned}
m(\eta,\delta)=&\tfrac{1}{a_1-a_0}\left[\int_{a_0}^{\eta}\left(\left(s-\eta+\delta\tfrac{a_1-a_0}{2}-\left|s-\eta+\delta\tfrac{a_1-a_0}{2}\right|\right)\tfrac{\tau(s)}{2}\right.\right.\\
&\left.+\left(s-\eta+\delta\tfrac{a_1-a_0}{2}+\left|s-\eta+\delta\tfrac{a_1-a_0}{2}\right|\right)\tfrac{\gamma(s)}{2}\right)ds\\
&+\int_{\eta}^{a_1}\left(\left(s-\eta-\delta\tfrac{a_1-a_0}{2}-\left|s-\eta-\delta\tfrac{a_1-a_0}{2}\right|\right)\tfrac{\tau(s)}{2}\right.\\
&\left.+\left(s-\eta-\delta\tfrac{a_1-a_0}{2}+\left|s-\eta-\delta\tfrac{a_1-a_0}{2}\right|\right)\tfrac{\gamma(s)}{2}\right)ds=
\end{aligned}$$

$$=\frac{1}{a_1-a_0}\left[\int_{-\delta\frac{a_1-a_0}{2}}^{a_1-\eta-\delta\frac{a_1-a_0}{2}}\left(\frac{\zeta+|\zeta|}{2}\gamma\left(\zeta+\eta+\delta\frac{a_1-a_0}{2}\right)\right.\right.$$
$$\left.+\frac{\zeta-|\zeta|}{2}\tau\left(\zeta+\eta+\delta\frac{a_1-a_0}{2}\right)\right)d\zeta$$
$$+\int_{\eta-a_0+\delta\frac{a_1-a_0}{2}}^{\delta\frac{a_1-a_0}{2}}\left(\frac{\zeta+|\zeta|}{2}\gamma\left(\zeta+\eta-\delta\frac{a_1-a_0}{2}\right)\right.$$
$$\left.+\frac{\zeta-|\zeta|}{2}\tau\left(\zeta+\eta-\delta\frac{a_1-a_0}{2}\right)\right)d\zeta$$

and

$$M(\eta,\delta)=\frac{1}{a_1-a_0}\left[\int_{a_0}^{\eta}\left(\left(s-\eta+\delta\frac{a_1-a_0}{2}+\left|s-\eta+\delta\frac{a_1-a_0}{2}\right|\right)\frac{\tau(s)}{2}\right.\right.$$
$$\left.+\left(s-\eta+\delta\frac{a_1-a_0}{2}-\left|s-\eta+\delta\frac{a_1-a_0}{2}\right|\right)\frac{\gamma(s)}{2}\right)ds$$
$$+\int_{\eta}^{a_1}\left(\left(s-\eta-\delta\frac{a_1-a_0}{2}+\left|s-\eta-\delta\frac{a_1-a_0}{2}\right|\right)\frac{\tau(s)}{2}\right.$$
$$\left.+\left(s-\eta-\delta\frac{a_1-a_0}{2}-\left|s-\eta-\delta\frac{a_1-a_0}{2}\right|\right)\frac{\gamma(s)}{2}\right)ds$$
$$=\frac{1}{a_1-a_0}\left[\int_{-\delta\frac{a_1-a_0}{2}}^{a_1-\eta-\delta\frac{a_1-a_0}{2}}\left(\frac{\zeta-|\zeta|}{2}\gamma\left(\zeta+\eta+\delta\frac{a_1-a_0}{2}\right)\right.\right.$$
$$\left.+\frac{\zeta+|\zeta|}{2}\tau\left(\zeta+\eta+\delta\frac{a_1-a_0}{2}\right)\right)d\zeta$$
$$+\int_{\eta-a_0+\delta\frac{a_1-a_0}{2}}^{\delta\frac{a_1-a_0}{2}}\left(\frac{\zeta-|\zeta|}{2}\gamma\left(\zeta+\eta-\delta\frac{a_1-a_0}{2}\right)\right.$$
$$\left.+\frac{\zeta+|\zeta|}{2}\tau\left(\zeta+\eta-\delta\frac{a_1-a_0}{2}\right)\right)d\zeta.$$

This completes the proof of Theorem 2.1. □

COROLLARY 2.2. *Suppose that all the assumptions of Theorem 2.1 hold, also if we substitute* $\eta=\frac{a_0+a_1}{2}$ *in* (2.1), *we get the following bounds*

(2.5)
$$m_1(\delta)\le\delta\left[g\left(\frac{a_0+a_1}{2}\right)-\frac{g(a_0)+g(a_1)}{2}\right]+\frac{g(a_0)+g(a_1)}{2}-\frac{1}{a_1-a_0}\int_{a_0}^{a_1}g(s)ds\le M_1(\delta)$$

where

$$m_1(\delta)=$$
$$\frac{1}{a_1-a_0}\left[\int_{-\delta\frac{a_1-a_0}{2}}^{(1-\delta)\frac{a_1-a_0}{2}}\left(\frac{\zeta+|\zeta|}{2}\gamma\left(\zeta+\frac{a_0+a_1}{2}+\delta\frac{a_1-a_0}{2}\right)+\frac{\zeta-|\zeta|}{2}\tau\left(\zeta+\frac{a_0+a_1}{2}+\delta\frac{a_1-a_0}{2}\right)\right)d\zeta\right.$$
$$\left.+\int_{(\delta-1)\frac{a_1-a_0}{2}}^{\delta\frac{a_1-a_0}{2}}\left(\frac{\zeta+|\zeta|}{2}\gamma\left(\zeta+\frac{a_0+a_1}{2}-\delta\frac{a_1-a_0}{2}\right)+\frac{\zeta-|\zeta|}{2}\tau\left(\zeta+\frac{a_0+a_1}{2}-\delta\frac{a_1-a_0}{2}\right)\right)d\zeta\right]$$

and

$M_1(\delta) =$

$$\frac{1}{a_1-a_0}\left[\int_{-\delta\frac{a_1-a_0}{2}}^{(1-\delta)\frac{a_1-a_0}{2}}\left(\frac{\zeta-|\zeta|}{2}\gamma\left(\zeta+\frac{a_0+a_1}{2}+\delta\frac{a_1-a_0}{2}\right)+\frac{\zeta+|\zeta|}{2}\tau\left(\zeta+\frac{a_0+a_1}{2}+\delta\frac{a_1-a_0}{2}\right)\right)d\zeta\right.$$

$$\left.+\int_{(\delta-1)\frac{a_1-a_0}{2}}^{\delta\frac{a_1-a_0}{2}}\left(\frac{\zeta-|\zeta|}{2}\gamma\left(\zeta+\frac{a_0+a_1}{2}-\delta\frac{a_1-a_0}{2}\right)+\frac{\zeta+|\zeta|}{2}\tau\left(\zeta+\frac{a_0+a_1}{2}-\delta\frac{a_1-a_0}{2}\right)\right)d\zeta\right].$$

Corollary 3.2 can be more useful to get different quadrature bounds, which we see it as follows. Throughout the section $\gamma_0, \gamma_1, \tau_0$ and τ_1 are real constants.

Remark 2.3. If we choose $\delta = 0$ in (2.5) then we get the bounds for trapezoidal rule

$$m_2 \leq \frac{g(a_0)+g(a_1)}{2} - \frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq M_2, \tag{2.6}$$

where

$$m_2 = \frac{1}{a_1-a_0}\left[\int_{-\frac{a_1-a_0}{2}}^{\frac{a_1-a_0}{2}}\left(\frac{\zeta+|\zeta|}{2}\gamma\left(\zeta+\frac{a_0+a_1}{2}\right)+\frac{\zeta-|\zeta|}{2}\tau\left(\zeta+\frac{a_0+a_1}{2}\right)\right)d\zeta\right]$$

and

$$M_2 = \frac{1}{a_1-a_0}\left[\int_{-\frac{a_1-a_0}{2}}^{\frac{a_1-a_0}{2}}\left(\frac{\zeta-|\zeta|}{2}\gamma\left(\zeta+\frac{a_0+a_1}{2}\right)+\frac{\zeta+|\zeta|}{2}\tau\left(\zeta+\frac{a_0+a_1}{2}\right)\right)d\zeta\right].$$

The above result is obtained in [12] and [14]. □

Special Case 1. If we choose, $\gamma(\eta) = \gamma_0 \neq 0$ and $\tau(\eta) = \tau_0 \neq 0$ in (2.6), then

$$\frac{(a_1-a_0)}{8}(\gamma_0-\tau_0) \leq \frac{g(a_0)+g(a_1)}{2} - \frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq \frac{(a_1-a_0)}{8}(\tau_0-\gamma_0).$$

The above result is obtained in [12] and [21].

Special Case 2. If we choose, $\gamma(\eta) = \gamma_1\eta + \gamma_0 \neq 0$ and $\tau(\eta) = \tau_1\eta + \tau_0 \neq 0$ in (2.6), then

$$m_3 \leq \frac{g(a_0)+g(a_1)}{2} - \frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq M_3,$$

where

$$m_3 = \frac{a_1-a_0}{8}\left(\frac{a_1-a_0}{3}(\gamma_1+\tau_1)+\frac{a_0+a_1}{2}(\gamma_1-\tau_1)+\gamma_0-\tau_0\right)$$

and

$$M_3 = \frac{a_1-a_0}{8}\left(\frac{a_1-a_0}{3}(\gamma_1+\tau_1)+\frac{a_0+a_1}{2}(\tau_1-\gamma_1)+\tau_0-\gamma_0\right).$$

The above result is obtained in [12].

Remark 2.4. If we put $\delta = 1$ in (2.5), then we get the bounds for midpoint rule

$$m_4 \leq g\left(\tfrac{a_0+a_1}{2}\right) - \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq M_4, \tag{2.7}$$

where

$$\begin{aligned} m_4 =& \tfrac{1}{a_1-a_0}\Bigg[\int_{-\frac{a_1-a_0}{2}}^{0}\Big(\tfrac{\zeta+|\zeta|}{2}\gamma(\zeta+a_1)+\tfrac{\zeta-|\zeta|}{2}\tau(\zeta+a_1)\Big)d\zeta \\ &+\int_0^{\frac{a_1-a_0}{2}}\Big(\tfrac{\zeta+|\zeta|}{2}\gamma(\zeta+a_0)+\tfrac{\zeta-|\zeta|}{2}\tau(\zeta+a_0)\Big)d\zeta\Bigg] \end{aligned}$$

and

$$\begin{aligned} M_4 =& \tfrac{1}{a_1-a_0}\Bigg[\int_{-\frac{a_1-a_0}{2}}^{0}\Big(\tfrac{\zeta-|\zeta|}{2}\gamma(\zeta+a_1)+\tfrac{\zeta+|\zeta|}{2}\tau(\zeta+a_1)\Big)d\zeta \\ &+\int_0^{\frac{a_1-a_0}{2}}\Big(\tfrac{\zeta-|\zeta|}{2}\gamma(\zeta+a_0)+\tfrac{\zeta+|\zeta|}{2}\tau(\zeta+a_0)\Big)d\zeta\Bigg]. \end{aligned}$$

The above result is obtained in [12] and [13]. □

Special Case 3. If we choose, $\gamma(\eta) = \gamma_0 \neq 0$ and $\tau(\eta) = \tau_0 \neq 0$ in (2.7), then

$$\tfrac{(a_1-a_0)}{8}(\gamma_0-\tau_0) \leq g\left(\tfrac{a_0+a_1}{2}\right) - \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq \tfrac{(a_1-a_0)}{8}(\tau_0-\gamma_0).$$

The above result is obtained in [12], [13] and [21].

Special Case 4. If we choose, $\gamma(\eta) = \gamma_1\eta + \gamma_0 \neq 0$ and $\tau(\eta) = \tau_1\eta + \tau_0 \neq 0$ in (2.7), then

$$m_5 \leq g\left(\tfrac{a_0+a_1}{2}\right) - \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq M_3,$$

where

$$m_5 = \tfrac{(a_1-a_0)}{8}\Big(\tfrac{a_1-a_0}{3}(\gamma_1+\tau_1) + a_0\gamma_1 - a_1\tau_1 + \gamma_0 - \tau_0\Big)$$

and

$$M_5 = \tfrac{(a_1-a_0)}{8}\Big(\tfrac{a_1-a_0}{3}(\gamma_1+\tau_1) + a_0\tau_1 - a_1\gamma_1 + \tau_0 - \gamma_0\Big).$$

The above result is obtained in [12] and [13].

Remark 2.5. If we choose $\delta = \frac{1}{2}$ in (2.5) we get the bounds for the average of midpoint and trapezoidal rule

$$m_6 \leq \tfrac{1}{2}\left[\tfrac{g(a_0)+g(a_1)}{2} + g\left(\tfrac{a_0+a_1}{2}\right)\right] - \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq M_6, \tag{2.8}$$

where

$$m_6 = \frac{1}{a_1-a_0}\left[\int_{-\frac{a_1-a_0}{4}}^{\frac{a_1-a_0}{4}} \left(\frac{\zeta+|\zeta|}{2}\gamma\left(\zeta+\frac{3a_0+a_1}{4}\right)+\frac{\zeta-|\zeta|}{2}\tau\left(\zeta+\frac{3a_0+a_1}{4}\right)\right)d\zeta \right.$$
$$\left. + \int_{-\frac{a_1-a_0}{4}}^{\frac{a_1-a_0}{4}} \left(\frac{\zeta+|\zeta|}{2}\gamma\left(\zeta+\frac{a_0+3a_1}{4}\right)+\frac{\zeta-|\zeta|}{2}\tau\left(\zeta+\frac{a_0+3a_1}{4}\right)\right)d\zeta\right]$$

and

$$M_6 = \frac{1}{a_1-a_0}\left[\int_{-\frac{a_1-a_0}{4}}^{\frac{a_1-a_0}{4}} \left(\frac{\zeta-|\zeta|}{2}\gamma\left(\zeta+\frac{3a_0+a_1}{4}\right)+\frac{\zeta+|\zeta|}{2}\tau\left(\zeta+\frac{3a_0+a_1}{4}\right)\right)d\zeta \right.$$
$$\left. + \int_{-\frac{a_1-a_0}{4}}^{\frac{a_1-a_0}{4}} \left(\frac{\zeta-|\zeta|}{2}\gamma\left(\zeta+\frac{a_0+3a_1}{4}\right)+\frac{\zeta+|\zeta|}{2}\tau\left(\zeta+\frac{a_0+3a_1}{4}\right)\right)d\zeta\right].$$

The above result is obtained in [12]. □

Special Case 5. If we choose, $\gamma(\eta)=\gamma_0\neq 0$ and $\tau(\eta)=\tau_0\neq 0$ in (2.8) then

$$\frac{(a_1-a_0)}{16}(\gamma_0-\tau_0)\leq$$
$$\leq \frac{1}{2}\left[\frac{g(a_0)+g(a_1)}{2}+g\left(\frac{a_0+a_1}{2}\right)\right]-\frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq \frac{(a_1-a_0)}{16}(\tau_0-\gamma_0).$$

The above result is obtained in [12] and [21].

Special Case 6. If we choose, $\gamma(\eta)=\gamma_1\eta+\gamma_0\neq 0$ and $\tau(\eta)=\tau_1\eta+\tau_0\neq 0$ in (2.8), then

$$m_7(\eta)\leq \frac{1}{2}\left[\frac{g(a_0)+g(a_1)}{2}+g\left(\frac{a_0+a_1}{2}\right)\right]-\frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq M_7,$$

where

$$m_7=\frac{(a_1-a_0)}{16}\left[\frac{(a_1-a_0)}{6}(\gamma_1+\tau_1)+\frac{a_0}{2}(\gamma_1-\tau_1)+\frac{a_1}{2}(\gamma_1-\tau_1)+\gamma_0-\tau_0\right]$$

and

$$M_7=\frac{(a_1-a_0)}{16}\left[\frac{(a_1-a_0)}{6}(\gamma_1+\tau_1)+\frac{a_0}{2}(\tau_1-\gamma_1)+\frac{a_1}{2}(\tau_1-\gamma_1)+\tau_0-\gamma_0\right].$$

The above result is obtained in [12].

Corollary 2.6. *Suppose that all the assumptions of the Theorem 2.1 hold, also if we substitute $\eta=a_0$ in (2.1), then for any value of $\delta\in[0,1]$, we get the following bounds*

$$m_8\leq \delta\left[\frac{g(a_0)-g(a_1)}{2}\right]+g(a_1)-\frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds\leq M_8, \tag{2.9}$$

where

$m_8 =$

$$\frac{1}{a_1-a_0}\left[\int_{-\delta\frac{a_1-a_0}{2}}^{\left(1-\frac{\delta}{2}\right)(a_1-a_0)}\left(\frac{\zeta+|\zeta|}{2}\gamma\left(\zeta+a_0+\delta\frac{a_1-a_0}{2}\right)+\frac{\zeta-|\zeta|}{2}\tau\left(\zeta+a_0+\delta\frac{a_1-a_0}{2}\right)\right)d\zeta\right]$$

and

$M_8 =$

$$\frac{1}{a_1-a_0}\left[\int_{-\delta\frac{a_1-a_0}{2}}^{\left(1-\frac{\delta}{2}\right)(a_1-a_0)}\left(\frac{\zeta-|\zeta|}{2}\gamma\left(\zeta+a_0+\delta\frac{a_1-a_0}{2}\right)+\frac{\zeta+|\zeta|}{2}\tau\left(\zeta+a_0+\delta\frac{a_1-a_0}{2}\right)\right)d\zeta\right].$$

Remark 2.7. Suppose that all the assumptions of the Theorem 2.1 hold, also if we substitute $\delta = 0$ in (2.9), then we get the bounds for nonstandard quadrature rule as follows

$$m_9 \leq g(a_1) - \frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq M_9, \tag{2.10}$$

where

$$m_9 = \frac{1}{a_1-a_0}\left[\int_0^{a_1-a_0}\left(\frac{\zeta+|\zeta|}{2}\gamma(\zeta+a_0)+\frac{\zeta-|\zeta|}{2}\tau(\zeta+a_0)\right)d\zeta\right]$$

and

$$M_9 = \frac{1}{a_1-a_0}\left[\int_0^{a_1-a_0}\left(\frac{\zeta-|\zeta|}{2}\gamma(\zeta+a_0)+\frac{\zeta+|\zeta|}{2}\tau(\zeta+a_0)\right)d\zeta\right].\ \square$$

Special Case 7. If we choose, $\gamma(\eta) = \gamma_0 \neq 0$ and $\tau(\eta) = \tau_0 \neq 0$ in (2.10) then

$$\frac{a_1-a_0}{2}\gamma_0 \leq g(a_1) - \frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq \frac{a_1-a_0}{2}\tau_0.$$

Special Case 8. If we choose, $\gamma(\eta) = \gamma_1\eta + \gamma_0 \neq 0$ and $\tau(\eta) = \tau_1\eta + \tau_0 \neq 0$ in (2.10), then

$$m_{10} \leq g(a_1) - \frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq M_{10},$$

where

$$m_{10} = \frac{a_1-a_0}{6}\left[\gamma_1(2a_1+a_0)+3\gamma_0\right]$$

and

$$M_{10} = \frac{a_1-a_0}{6}\left[\tau_1(2a_1+a_0)+3\tau_0\right].$$

Corollary 2.8. *Suppose that all the assumptions of the Theorem 2.1 hold, also if we substitute $\eta = a_1$ in (2.1), then for any value of $\delta \in [0,1]$, we get the following bound*

$$m_{11} \leq \delta\left[\frac{g(a_1)-g(a_0)}{2}\right] + g(a_0) - \frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq M_{11}, \tag{2.11}$$

where

$m_{11} =$

$$\frac{1}{a_1-a_0}\left[\int_{\left(\frac{\delta}{2}-1\right)(a_1-a_0)}^{\delta\frac{a_1-a_0}{2}}\left(\frac{\zeta+|\zeta|}{2}\gamma\left(\zeta+a_1-\delta\frac{a_1-a_0}{2}\right)+\frac{\zeta-|\zeta|}{2}\tau\left(\zeta+a_1-\delta\frac{a_1-a_0}{2}\right)\right)d\zeta\right]$$

and

$$M_{11}=\frac{1}{a_1-a_0}\left[\int_{\left(\frac{\delta}{2}-1\right)(a_1-a_0)}^{\delta\frac{a_1-a_0}{2}}\left(\frac{\zeta-|\zeta|}{2}\gamma\left(\zeta+a_1-\delta\frac{a_1-a_0}{2}\right)+\frac{\zeta+|\zeta|}{2}\tau\left(\zeta+a_1-\delta\frac{a_1-a_0}{2}\right)\right)d\zeta\right].$$

Remark 2.9. Suppose that all the assumptions of the Theorem 2.1 hold, also if we substitute $\delta=0$ in (2.11), then we get the following bound for nonstandard quadrature

$$m_{12}\leq g(a_0)-\frac{1}{a_1-a_0}\int_{a_0}^{a_1}g(s)ds\leq M_{12}, \tag{2.12}$$

where

$$m_{12}=\frac{1}{a_1-a_0}\left[\int_{-(a_1-a_0)}^{0}\left(\frac{\zeta+|\zeta|}{2}\gamma(\zeta+a_0)+\frac{\zeta-|\zeta|}{2}\tau(\zeta+a_0)\right)d\zeta\right]$$

and

$$M_{12}=\frac{1}{a_1-a_0}\left[\int_{-(a_1-a_0)}^{0}\left(\frac{\zeta-|\zeta|}{2}\gamma(\zeta+a_0)+\frac{\zeta+|\zeta|}{2}\tau(\zeta+a_0)\right)d\zeta\right].\ \square$$

Special Case 9. If we choose $\gamma(\eta)=\gamma_0\neq0$ and $\tau(\eta)=\tau_0\neq0$ in (2.12), gives the following inequality as

$$-\frac{a_1-a_0}{2}\tau_0\leq g(a_0)-\frac{1}{a_1-a_0}\int_{a_0}^{a_1}g(s)ds\leq-\frac{a_1-a_0}{2}\gamma_0.$$

Special Case 10. If we choose $\gamma(\eta)=\gamma_1\eta+\gamma_0\neq0$ and $\tau(\eta)=\tau_1\eta+\tau_0\neq0$ in (2.12), gives

$$m_{13}\leq g(a_0)-\frac{1}{a_1-a_0}\int_{a_0}^{a_1}g(s)ds\leq M_{13},$$

where

$$m_{13}=\frac{a_1-a_0}{6}\left[\tau_1(2a_1-5a_0)-3\tau_0\right]$$

and

$$M_{13}=\frac{a_1-a_0}{6}\left[\gamma_1(2a_1-5a_0)-3\gamma_1\right].$$

Remark 2.10. Suppose that all the assumptions of the Theorem 2.1 hold, also if we substitute $\delta=1$ in both (2.9) and (2.11), gives the bound for trapezoidal rule

$$m_{14}\leq\frac{g(a_0)+g(a_1)}{2}-\frac{1}{a_1-a_0}\int_{a_0}^{a_1}g(s)ds\leq M_{14}, \tag{2.13}$$

where

$$m_{14}=\frac{1}{a_1-a_0}\left[\int_{-\frac{a_1-a_0}{2}}^{\frac{a_1-a_0}{2}}\left(\frac{\zeta+|\zeta|}{2}\gamma\left(\zeta+\frac{a_0+a_1}{2}\right)+\frac{\zeta-|\zeta|}{2}\tau\left(\zeta+\frac{a_0+a_1}{2}\right)\right)d\zeta\right]$$

and

$$M_{14} = \frac{1}{a_1-a_0}\left[\int_{-\frac{a_1-a_0}{2}}^{\frac{a_1-a_0}{2}} \left(\frac{\zeta-|\zeta|}{2}\gamma\left(\zeta+\frac{a_0+a_1}{2}\right)+\frac{\zeta+|\zeta|}{2}\tau\left(\zeta+\frac{a_0+a_1}{2}\right)\right)d\zeta\right]. \qquad \square$$

Special Case 11. If we choose, $\gamma(\eta) = \gamma_0 \neq 0$ and $\tau(\eta) = \tau_0 \neq 0$ in (2.13) then

$$\frac{a_1-a_0}{8}(\gamma_0-\tau_0) \leq \frac{g(a_0)+g(a_1)}{2} - \frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq -\frac{a_1-a_0}{8}(\tau_0-\gamma_0).$$

which is Corollary 2 of [21]

Special Case 12. If we choose, $\gamma(\eta) = \gamma_1\eta+\gamma_0 \neq 0$ and $\tau(\eta) = \tau_1\eta+\tau_0 \neq 0$ in (2.13), then

$$m_{15} \leq \frac{g(a_0)+g(a_1)}{2} - \frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq M_{15},$$

where

$$m_{15} = \frac{(a_1-a_0)}{8}\left[\frac{a_1-a_0}{3}(\gamma_1+\tau_1)+\frac{a_0+a_1}{2}(\gamma_1-\tau_1)+\gamma_0-\tau_0\right]$$

and

$$M_{15} = \frac{(a_1-a_0)}{8}\left[\frac{a_1-a_0}{3}(\gamma_1+\tau_1)+\frac{a_0+a_1}{2}(\tau_1-\gamma_1)+\tau_0-\gamma_0\right].$$

Although we have discussed bounded condition in Theorem 2.1, however sometimes we are not able to find both the bounds. In order to be more effective, we need to obtain the theorems only for the cases of bounded below and bounded above. So the first theorem would be useful when $a_0(\eta) \leq g'(\eta)$ and the second one would be useful when $g'(\eta) \leq a_1(\eta)$.

THEOREM 2.11. *Let $g : I \to \mathbb{R}$, where I is an interval, be differentiable function in the interior I^0 of I, and let $[a_0, a_1] \subset I^0$. If g' is unbounded from above then $\gamma(\eta) \leq g'(\eta)$ for any $\gamma \in C[a_0, a_1]$, $\eta \in [a_0, a_1]$, then for all $\delta \in [0, 1]$ the following inequality holds*

$$\begin{aligned} m_{16}(\eta,\delta) \leq &\delta\left[g(\eta)-\frac{g(a_0)+g(a_1)}{2}\right]+\frac{a_1-\eta}{a_1-a_0}g(a_1)\\ &+\frac{\eta-a_0}{a_1-a_0}g(a_0)-\frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq M_{16}(\eta,\delta), \end{aligned} \tag{2.14}$$

or

$$\begin{aligned} (a_1-a_0)m_{16}(\eta,\delta) &\leq (a_1-a_0)\delta\left[g(\eta)-\frac{g(a_0)+g(a_1)}{2}\right]+(a_1-\eta)g(a_1)\\ &+(\eta-a_0)g(a_0)-\int_{a_0}^{a_1} g(s)ds \leq (a_1-a_0)M_{16}(\eta,\delta), \end{aligned} \tag{2.15}$$

where

$$
\begin{aligned}
&m_{16}(\eta,\delta) = \\
&= \tfrac{1}{a_1-a_0}\left[\int_{a_0}^{a_1}(s-\eta)\gamma(s)ds + \delta\tfrac{a_1-a_0}{2}\left(\int_{a_0}^{\eta}\gamma(s)ds - \int_{\eta}^{a_1}\gamma(s)ds\right)\right.\\
&\quad - \max\left\{\delta\tfrac{a_1-a_0}{2}, \left|(\eta-a_0+\delta\tfrac{a_1-a_0}{2})\right|, \left|(a_1-\eta-\delta\tfrac{a_1-a_0}{2})\right|\right\}\\
&\quad \left.\times\left(g(a_1)-g(a_0)-\int_{a_0}^{a_1}\gamma(s)ds\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&M_{16}(\eta,\delta) = \\
&= \tfrac{1}{a_1-a_0}\left[\int_{a_0}^{a_1}(s-\eta)\gamma(s)ds + \delta\tfrac{a_1-a_0}{2}\left(\int_{a_0}^{\eta}\gamma(s)ds - \int_{\eta}^{a_1}\gamma(s)ds\right)\right.\\
&\quad + \max\left\{\delta\tfrac{a_1-a_0}{2}, \left|(\eta-a_0+\delta\tfrac{a_1-a_0}{2})\right|, \left|(a_1-\eta-\delta\tfrac{a_1-a_0}{2})\right|\right\}\\
&\quad \left.\times\left(g(a_1)-g(a_0)-\int_{a_0}^{a_1}\gamma(s)ds\right)\right].
\end{aligned}
$$

Proof. Since

$$
\begin{aligned}
&\int_{a_0}^{a_1}K(\eta,s)\left(g'(s)-\gamma(s)\right)ds = \\
&= \delta(a_1-a_0)\left[g(\eta)-\tfrac{g(a_0)+g(a_1)}{2}\right] + (a_1-\eta)g(a_1) + (\eta-a_0)g(a_0)\\
&\quad - \int_{a_0}^{a_1}g(s)ds - \int_{a_0}^{a_1}K(\eta,s)\gamma(s)ds.\\
&= \delta(a_1-a_0)\left[g(\eta)-\tfrac{g(a_0)+g(a_1)}{2}\right] + (a_1-\eta)g(a_1) + (\eta-a_0)g(a_0)\\
&\quad - \int_{a_0}^{a_1}g(s)ds - \left[\int_{a_0}^{\eta}\left(s-\eta+\delta\tfrac{a_1-a_0}{2}\right)\gamma(s)ds + \int_{\eta}^{a_1}\left(s-\eta-\delta\tfrac{a_1-a_0}{2}\right)\gamma(s)ds\right],
\end{aligned}
$$

using modulus property, we have

$$
\begin{aligned}
&\left|\delta(a_1-a_0)\left[g(\eta)-\tfrac{g(a_0)+g(a_1)}{2}\right] + (a_1-\eta)g(a_1) + (\eta-a_0)g(a_0)\right.\\
&\quad \left.- \int_{a_0}^{a_1}g(s)ds - \left[\int_{a_0}^{\eta}\left(s-\eta+\delta\tfrac{a_1-a_0}{2}\right)\gamma(s)ds + \int_{\eta}^{a_1}\left(s-\eta-\delta\tfrac{a_1-a_0}{2}\right)\gamma(s)ds\right]\right| = \\
&= \left|\int_{a_0}^{a_1}K(\eta,s)\left(g'(\eta)-\gamma(s)\right)ds\right|\\
&\leq \int_{a_0}^{a_1}|K(\eta,s)|\left(g'(s)-\gamma(s)\right)ds\\
&\leq \max_{s\in[a_0,a_1]}|K(\eta,s)|\int_{a_0}^{a_1}\left(g'(s)-\gamma(s)\right)ds.
\end{aligned}
$$

$$= \max\left\{\delta\tfrac{a_1-a_0}{2}, \left(\eta - a_0 + \delta\tfrac{a_1-a_0}{2}\right), \left(a_1 - \eta - \delta\tfrac{a_1-a_0}{2}\right)\right\}$$

$$\times \left(g(a_1) - g(a_0) - \int_{a_0}^{a_1} \gamma(s)ds\right). \tag{2.16}$$

Arrangement of (2.16) gives inequality (2.14). □

Remark 2.12. The inequality (2.14) represents the generalized case of Theorem 2 presented in [13] and [14]. □

Corollary 2.13. *Suppose that all the assumptions of Theorem 2.11 holds and if we substitute* $\eta = \frac{a_0+a_1}{2}$ *in* (2.14)*, then we obtain*

(2.17)

$$m_{17}(\delta) \leq \delta\left[g\left(\tfrac{a_0+a_1}{2}\right) - \tfrac{g(a_0)+g(a_1)}{2}\right] + \tfrac{g(a_0)+g(a_1)}{2} - \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq M_{17}(\delta),$$

where

$$m_{17}(\delta) =$$
$$= \tfrac{1}{a_1-a_0}\left[\int_{a_0}^{a_1}\left(s - \tfrac{a_0+a_1}{2}\right)\gamma(s)ds + \delta\tfrac{a_1-a_0}{2}\left(\int_{a_0}^{\eta}\gamma(s)ds - \int_{\eta}^{a_1}\gamma(s)ds\right)\right.$$
$$- \max\left\{\delta\tfrac{a_1-a_0}{2}, (1-\delta)\tfrac{a_1-a_0}{2}, (\delta-1)\tfrac{a_1-a_0}{2}\right\}$$
$$\left.\times\left(g(a_1) - g(a_0) - \int_{a_0}^{a_1}\gamma(s)ds\right)\right]$$

and

$$M_{17}(\delta) =$$
$$= \tfrac{1}{a_1-a_0}\left[\int_{a_0}^{a_1}\left(s - \tfrac{a_0+a_1}{2}\right)\gamma(s)ds + \delta\tfrac{a_1-a_0}{2}\left(\int_{a_0}^{\eta}\gamma(s)ds - \int_{\eta}^{a_1}\gamma(s)ds\right)\right.$$
$$+ \max\left\{\delta\tfrac{a_1-a_0}{2}, (1-\delta)\tfrac{a_1-a_0}{2}, (\delta-1)\tfrac{a_1-a_0}{2}\right\}$$
$$\left.\times\left(g(a_1) - g(a_0) - \int_{a_0}^{a_1}\gamma(s)ds\right)\right].$$

Theorem 2.14. *Let* $g : I \to \mathbb{R}$*, where* I *is an interval, be a function differentiable in the interior* I^0 *of* I*, and let* $[a_0, a_1] \subset I^0$*. If* g' *is unbounded from below then* $g'(\eta) \leq \tau(\eta)$ *for any* $\tau \in C[a_0, a_1]$*,* $\eta \in [a_0, a_1]$*, then for all* $\delta \in [0, 1]$ *the following inequality holds*

$$m_{18}(\eta,\delta) \leq \delta\left[g(\eta) - \tfrac{g(a_0)+g(a_1)}{2}\right] + \tfrac{a_1-\eta}{a_1-a_0}g(a_1)$$
$$+ \tfrac{\eta-a_0}{a_1-a_0}g(a_0) - \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \leq M_{18}(\eta,\delta), \tag{2.18}$$

or

$$(a_1 - a_0)m_{18}(\eta, \delta) \leq (a_1 - a_0)\delta \left[g(\eta) - \tfrac{g(a_0)+g(a_1)}{2}\right] + (a_1 - \eta)g(a_1)$$

$$+ (\eta - a_0)g(a_0) - \int_{a_0}^{a_1} g(s)ds \leq M_{18}(\eta, \delta), \tag{2.19}$$

where

$$\begin{aligned} &m_{18}(\eta, \delta) = \\ &= \tfrac{1}{a_1-a_0} \Bigg[\int_{a_0}^{a_1}(s-\eta)\tau(s)ds + \delta\tfrac{a_1-a_0}{2}\left(\int_{a_0}^{\eta}\tau(s)ds - \int_{\eta}^{a_1}\tau(s)ds\right) \\ &\quad - \max\left\{\delta\tfrac{a_1-a_0}{2}, \left|(\eta - a_0 + \delta\tfrac{a_1-a_0}{2})\right|, \left|(a_1 - \eta - \delta\tfrac{a_1-a_0}{2})\right|\right\} \\ &\quad \times \left(\int_{a_0}^{a_1}\tau(s) - g(a_1) - g(a_0)ds\right)\Bigg] \end{aligned}$$

and

$$\begin{aligned} M_{18}(\eta, \delta) = &\tfrac{1}{a_1-a_0} \Bigg[\int_{a_0}^{a_1}(s-\eta)\tau(s)ds + \delta\tfrac{a_1-a_0}{2}\left(\int_{a_0}^{\eta}\tau(s)ds - \int_{\eta}^{a_1}\tau(s)ds\right) \\ &+ \max\left\{\delta\tfrac{a_1-a_0}{2}, \left|(\eta - a_0 + \delta\tfrac{a_1-a_0}{2})\right|, \left|(a_1 - \eta - \delta\tfrac{a_1-a_0}{2})\right|\right\} \\ &\times \left(\int_{a_0}^{a_1}\tau(s) - g(a_1) - g(a_0)ds\right)\Bigg]. \end{aligned}$$

Proof. Since

$$\begin{aligned} &\int_{a_0}^{a_1} K(\eta, s)\left(g'(s) - \tau(s)\right)ds = \\ &= \delta(a_1 - a_0)\left[g(\eta) - \tfrac{g(a_0)+g(a_1)}{2}\right] + (a_1 - \eta)g(a_1) + (\eta - a_0)g(a_0) \\ &\quad - \int_{a_0}^{a_1} g(s)ds - \int_{a_0}^{a_1} K(\eta, s)\tau(s)ds \\ &= \delta(a_1 - a_0)\left[g(\eta) - \tfrac{g(a_0)+g(a_1)}{2}\right] + (a_1 - \eta)g(a_1) + (\eta - a_0)g(a_0) \\ &\quad - \int_{a_0}^{a_1} g(s)ds - \left[\int_{a_0}^{\eta}\left(s - \eta + \delta\tfrac{a_1-a_0}{2}\right)\tau(s)ds + \int_{\eta}^{a_1}\left(s - \eta - \delta\tfrac{a_1-a_0}{2}\right)\tau(s)ds\right], \end{aligned}$$

so we have

$$\begin{aligned} &\Bigg|\delta(a_1 - a_0)\left[g(\eta) - \tfrac{g(a_0)+g(a_1)}{2}\right] + (a_1 - \eta)g(a_1) + (\eta - a_0)g(a_0) \\ &- \int_{a_0}^{a_1} g(s)ds - \left[\int_{a_0}^{\eta}\left(s - \eta + \delta\tfrac{a_1-a_0}{2}\right)\tau(s)ds + \int_{\eta}^{a_1}\left(s - \eta - \delta\tfrac{a_1-a_0}{2}\right)\tau(s)ds\right]\Bigg| = \\ &= \left|\int_{a_0}^{a_1} K(\eta, s)\left(g'(t) - \tau(s)\right)ds\right| \\ &\leq \int_{a_0}^{a_1} |K(\eta, s)|\left(\tau(s) - g'(s)\right)ds \end{aligned}$$

$$\le \max_{s\in[a_0,a_1]} |K(\eta,s)| \int_{a_0}^{a_1} (\tau(s) - g'(s))\, ds$$

$$= \max\left\{ \left|\eta - a_0 + \delta\tfrac{a_1-a_0}{2}\right|, \delta\tfrac{a_1-a_0}{2}, \left|a_1 - \eta - \delta\tfrac{a_1-a_0}{2}\right| \right\}$$

$$\times \left(\int_{a_0}^{a_1} \tau(s)ds - g(a_1) + g(a_0) \right). \tag{2.20}$$

Rearrangement of (2.20) gives the inequality (2.18). □

Remark 2.15. The inequality established in (2.18) is the special case of Theorem 3 presented in [13] and [14]. □

Corollary 2.16. *Suppose that all the assumptions of Theorem 2.14 hold and if we substitute* $\eta = \frac{a_0+a_1}{2}$ *in* (2.18)*, then we will get*

(2.21)

$$m_{19}(\delta) \le \delta\left[g\left(\tfrac{a_0+a_1}{2}\right) - \tfrac{g(a_0)+g(a_1)}{2}\right] + \tfrac{g(a_0)+g(a_1)}{2} - \int_{a_0}^{a_1} g(s)ds \le M_{19}(\delta),$$

where

$$m_{19}(\delta) =$$

$$= \tfrac{1}{a_1-a_0}\left[\int_{a_0}^{a_1} \left(s - \tfrac{a_0+a_1}{2}\right)\tau(s)ds + \delta\tfrac{a_1-a_0}{2}\left(\int_{a_0}^{\frac{a_0+a_1}{2}} \tau(s)ds - \int_{\frac{a_0+a_1}{2}}^{a_1} \tau(s)ds\right)\right.$$

$$\left. - \max\left\{\delta\tfrac{a_1-a_0}{2}, (1-\delta)\tfrac{a_1-a_0}{2}, (\delta-1)\tfrac{a_1-a_0}{2}\right\}\left(\int_{a_0}^{a_1} \tau(s)ds - g(a_1) + g(a_0)\right)\right]$$

and

$$M_{19}(\delta) =$$

$$= \tfrac{1}{a_1-a_0}\left[\int_{a_0}^{a_1} \left(s - \tfrac{a_0+a_1}{2}\right)\tau(s)ds + \delta\tfrac{a_1-a_0}{2}\left(\int_{a_0}^{\frac{a_0+a_1}{2}} \tau(s)ds - \int_{\frac{a_0+a_1}{2}}^{a_1} \tau(s)ds\right)\right.$$

$$\left. + \max\left\{\delta\tfrac{a_1-a_0}{2}, (1-\delta)\tfrac{a_1-a_0}{2}, (\delta-1)\tfrac{a_1-a_0}{2}\right\}\left(\int_{a_0}^{a_1} \tau(s)ds - g(a_1) + g(a_0)\right)\right].$$

Remark 2.17. If $\gamma(\eta) \le g'(\eta) \le \tau(\eta)$ for any $\eta \in [a_0, a_1]$ and $\gamma, \tau \in C[a_0, a_1]$, and if we choose $\delta = 1$, then the error of nonstandard quadrature can be bounded as

$$m_{20} \le \tfrac{1}{2}\left[-g(a_0) + 2g\left(\tfrac{a_0+a_1}{2}\right) + g(a_1)\right] - \tfrac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \le M_{20}, \tag{2.22}$$

where

$$m_{20} = \tfrac{1}{a_1-a_0}\left[\int_{a_0}^{\frac{a_0+a_1}{2}} (s-a_0)\,\gamma(s)ds + \int_{\frac{a_0+a_1}{2}}^{a_1} (s-a_1)\gamma(s)ds + \tfrac{(a_1-a_0)}{2}\int_{a_0}^{a_1} \gamma(s)ds\right]$$

and

$$M_{20} = \tfrac{1}{a_1-a_0}\left[\int_{a_0}^{\frac{a_0+a_1}{2}} (s-a_0)\,\tau(s)ds + \int_{\frac{a_0+a_1}{2}}^{a_1} (s-a_1)\tau(s)ds + \tfrac{(a_1-a_0)}{2}\int_{a_0}^{a_1} \tau(s)ds\right]$$

which is the Corollary 3 obtained in [13] and the Corollary 4 obtained in [14]. □

Proof. To prove (2.22) we should use the results of both Corollary 2.13 and Corollary 2.16 simultaneously. First by substituting $\delta = 1$ in (2.17), we have

$$\frac{1}{a_1-a_0}\left[\int_{a_0}^{a_1}(s-a_0)\,\gamma(s)ds + \frac{a_1-a_0}{2}\left(\int_{a_0}^{\frac{a_0+a_1}{2}}\gamma(s)ds + \int_{\frac{a_0+a_1}{2}}^{a_1}\gamma(s)ds\right)\right] \le$$

(2.23)

$$\le \tfrac{1}{2}\left[-g(a_0) + 2g\left(\tfrac{a_0+a_1}{2}\right) + g(a_1)\right] - \frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds$$

provided that $\gamma(\eta) \le g'(\eta)$, $\forall \eta \in [a_0, a_1]$.

On the other hand, by assuming $\delta = 1$ in (2.21), we obtain

(2.24)

$$\tfrac{1}{2}\left[-g(a_0) + 2g\left(\tfrac{a_0+a_1}{2}\right) + g(a_1)\right] - \frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \le$$

$$\le \frac{1}{a_1-a_0}\left[\int_{a_0}^{a_1}(s-a_0)\,\tau(s)ds + \frac{a_1-a_0}{2}\left(\int_{a_0}^{\frac{a_0+a_1}{2}}\tau(s)ds + \int_{\frac{a_0+a_1}{2}}^{a_1}\tau(s)ds\right)\right]$$

provided that $g'(\eta) \le \tau(\eta)$ $\forall \eta \in [a_0, a_1]$.

Now by combining the above two results (2.23) and (2.24), the inequality (2.22) is derived. □

Remark 2.18. If $\gamma(\eta) \le g'(\eta) \le \tau(\eta)$ for any $\eta \in [a_0, a_1]$ and $\gamma, \tau \in C[a_0, a_1]$, and if we put $\delta = 1$, then the error of nonstandard quadrature can be bounded as

$$m_{21} \le \tfrac{1}{2}\left[g(a_0) + 2g\left(\tfrac{a_0+a_1}{2}\right) - g(a_1)\right] - \frac{1}{a_1-a_0}\int_{a_0}^{a_1} g(s)ds \le M_{21} \tag{2.25}$$

where

$m_{21} =$

$$= \frac{1}{a_1-a_0}\left[\int_{a_0}^{\frac{a_0+a_1}{2}}(s-a_0)\,\tau(s)ds + \int_{\frac{a_0+a_1}{2}}^{a_1}(s-a_1)\tau(s)ds - \frac{(a_1-a_0)}{2}\int_{a_0}^{a_1}\tau(s)ds\right]$$

and

$M_{21} =$

$$= \frac{1}{a_1-a_0}\left[\int_{a_0}^{\frac{a_0+a_1}{2}}(s-a_0)\,\gamma(s)ds + \int_{\frac{a_0+a_1}{2}}^{a_1}(s-a_1)\gamma(s)ds - \frac{(a_1-a_0)}{2}\int_{a_0}^{a_1}\gamma(s)ds\right]$$

which is the Corollary 4 of [13] and Corollary 5 of [14]. □

Proof. Proof of (2.25) is similar to that of Remark 2.17, if one replaces $\delta = 1$ in (2.17) and (2.21), respectively, and combines them together. □

3. APPLICATIONS TO NUMERICAL QUADRATURE RULES

Let $A_n : a_0 = z_0 < z_1 < \cdots < z_n = a_1$ be a partition of the interval $[a_0, a_1]$ and let $\Delta z_k = z_{k+1} - z_k, k \in \{0, 1, 2, \cdots, n-1\}$. Then

$$\int_{a_0}^{a_1} g(s)ds = Q_n(A_n, g) + R_n(A_n, g) \tag{3.1}$$

Consider a general quadrature formula

(3.2)

$$Q_n(A_n, g) :=$$
$$:= \sum_{k=0}^{n-1} \left[\Delta z_k \delta \left\{ g(\eta_k) - \tfrac{g(z_k)+g(z_{k+1})}{2} \right\} + (z_{k+1} - \eta_k) g(z_{k+1}) + (\eta_k - z_k) g(z_k) \right]$$

for all $\delta \in [0, 1]$.

THEOREM 3.1. *Suppose that all the assumptions of Theorem 2.1 hold. Considering (3.1), where $Q_n(.,.)$ is given by formula (3.2) and the remainder $R_n(.,.)$ satisfies the estimates*

$$|R_n(A_n, g)| \le \sup \{ |R_1|, |R_2| \} \tag{3.3}$$

where

$R_1 =$

$$= \int_{-\delta\frac{\Delta z_k}{2}}^{z_{k+1}-\eta_k-\delta\frac{\Delta z_k}{2}} \left(\tfrac{\zeta_k+|\zeta_k|}{2} \gamma \left(\zeta_k + \eta_k + \delta \tfrac{\Delta z_k}{2} \right) + \tfrac{\zeta_k-|\zeta_k|}{2} \tau \left(\zeta_k + \eta_k + \delta \tfrac{\Delta z_k}{2} \right) \right) d\zeta_k$$
$$+ \int_{\eta - z_{k+1} + \delta\frac{\Delta z_k}{2}}^{\delta\frac{\Delta z_k}{2}} \left(\tfrac{\zeta_k+|\zeta_k|}{2} \gamma \left(\zeta_k + \eta_k - \delta \tfrac{\Delta z_k}{2} \right) + \tfrac{\zeta_k-|\zeta_k|}{2} \tau \left(\zeta_k + \eta_k - \delta \tfrac{\Delta z_k}{2} \right) \right) d\zeta_k$$

and

$R_2 =$

$$= \int_{-\delta\frac{\Delta z_k}{2}}^{z_{k+1}-\eta_k-\delta\frac{\Delta z_k}{2}} \left(\tfrac{\zeta_k-|\zeta_k|}{2} \gamma \left(\zeta_k + \eta_k + \delta \tfrac{\Delta z_k}{2} \right) + \tfrac{\zeta_k+|\zeta_k|}{2} \tau \left(\zeta_k + \eta_k + \delta \tfrac{\Delta z_k}{2} \right) \right) d\zeta_k$$
$$+ \int_{\eta - z_{k+1} + \delta\frac{\Delta z_k}{2}}^{\delta\frac{\Delta z_k}{2}} \left(\tfrac{\zeta_k-|\zeta_k|}{2} \gamma \left(\zeta_k + \eta_k - \delta \tfrac{\Delta z_k}{2} \right) + \tfrac{\zeta_k+|\zeta_k|}{2} \tau \left(\zeta_k + \eta_k - \delta \tfrac{\Delta z_k}{2} \right) \right) d\zeta_k$$

for all $\eta_k \in [z_k, z_{k+1}]$.

Proof. Applying inequality (2.1) on the intervals, $[z_k, z_{k+1}]$, we can state that

$R_k(A_k, g) =$

$$= \int_{z_k}^{z_{k+1}} g(s)ds - \Delta z_k \delta \left\{ g(\eta_k) - \tfrac{g(z_k)+g(z_{k+1})}{2} \right\} + (z_{k+1} - \eta_k) g(z_{k+1}) + (\eta_k - z_k) g(z_k)$$

we sum the inequalities presented above over k from 0 to $n-1$. This gives

$$R_n(A_n,g)=\int_{a_0}^{a_1}g(s)ds-\sum_{k=0}^{n-1}\Big[\Delta z_k\delta\Big\{g(\eta_k)-\tfrac{g(z_k)+g(z_{k+1})}{2}\Big\}+(z_{k+1}-\eta_k)g(z_{k+1})+(\eta_k-z_k)g(z_k)\Big]$$

It follows from (2.2) that

$$\begin{aligned}
&|R_n(A_n,g)|=\\
&=\left|\int_{a_0}^{a_1}g(s)ds-\sum_{k=0}^{n-1}\left[\Delta z_k\delta\left\{g(\eta_k)-\tfrac{g(z_k)+g(z_{k+1})}{2}\right\}\right.\right.\\
&\left.\left.+(z_{k+1}-\eta_k)g(z_{k+1})+(\eta_k-z_k)g(z_k)\right]\right|\\
&\le\sup\left\{\left|\int_{-\delta\frac{\Delta z_k}{2}}^{z_{k+1}-\eta_k-\delta\frac{\Delta z_k}{2}}\left(\tfrac{\zeta_k+|\zeta_k|}{2}\gamma\left(\zeta_k+\eta_k+\delta\tfrac{\Delta z_k}{2}\right)\right.\right.\right.\\
&\left.+\ \tfrac{\zeta_k-|\zeta_k|}{2}\tau\left(\zeta_k+\eta_k+\delta\tfrac{\Delta z_k}{2}\right)\right)d\zeta_k\\
&+\int_{\eta-z_{k+1}+\delta\frac{\Delta z_k}{2}}^{\delta\frac{\Delta z_k}{2}}\left(\tfrac{\zeta_k+|\zeta_k|}{2}\gamma\left(\zeta_k+\eta_k-\delta\tfrac{\Delta z_k}{2}\right)\right.\\
&\left.\left.+\ \tfrac{\zeta_k-|\zeta_k|}{2}\tau\left(\zeta_k+\eta_k-\delta\tfrac{\Delta z_k}{2}\right)\right)d\zeta_k\right|,\\
&\left|\int_{-\delta\frac{\Delta z_k}{2}}^{z_{k+1}-\eta_k-\delta\frac{\Delta z_k}{2}}\left(\tfrac{\zeta_k-|\zeta_k|}{2}\gamma\left(\zeta_k+\eta_k+\delta\tfrac{\Delta z_k}{2}\right)\right.\right.\\
&\left.+\ \tfrac{\zeta_k+|\zeta_k|}{2}\tau\left(\zeta_k+\eta_k+\delta\tfrac{\Delta z_k}{2}\right)\right)d\zeta_k\\
&+\int_{\eta-z_{k+1}+\delta\frac{\Delta z_k}{2}}^{\delta\frac{\Delta z_k}{2}}\left(\tfrac{\zeta_k-|\zeta_k|}{2}\gamma\left(\zeta_k+\eta_k-\delta\tfrac{\Delta z_k}{2}\right)\right.\\
&\left.\left.\left.+\ \tfrac{\zeta_k+|\zeta_k|}{2}\tau\left(\zeta_k+\eta_k-\delta\tfrac{\Delta z_k}{2}\right)\right)d\zeta_k\right|\right\}
\end{aligned}$$

□

THEOREM 3.2. *Let ϕ be defined as in Theorem 2.11. Then (3.1) holds where $Q_n(A_n,g)$ is given by formula (3.2) and the remainder $R_n(A_n,g)$ satisfies the estimates*

$$|R_n(A_n,\phi)|\le\sup\{|R_3|,|R_4|\}\tag{3.4}$$

where

$$R_3=\left[\int_{z_k}^{z_{k+1}}(s-\eta_k)\gamma(s)ds+\delta\tfrac{\Delta z_k}{2}\left(\int_{z_k}^{\eta_k}\gamma(s)ds-\int_{\eta_k}^{z_{k+1}}\gamma(s)ds\right)\right.$$
$$-\max\left\{\delta\tfrac{\Delta z_k}{2},\left|\left(\eta_k-z_k+\delta\tfrac{\Delta z_k}{2}\right)\right|,\left|\left(z_{k+1}-\eta_k-\delta\tfrac{\Delta z_k}{2}\right)\right|\right\}\times$$

$$\times\left(g(z_{k+1})-g(z_k)-\int_{z_k}^{z_{k+1}}\gamma(s)ds\right)\Bigg]$$

and

$$R_4=\Bigg[\int_{z_k}^{z_{k+1}}(s-\eta_k)\gamma(s)ds+\delta\tfrac{\Delta z_k}{2}\left(\int_{z_k}^{\eta_k}\gamma(s)ds-\int_{\eta_k}^{z_{k+1}}\gamma(s)ds\right)$$
$$+\max\left\{\delta\tfrac{\Delta z_k}{2},\left|\left(\eta_k-z_k+\delta\tfrac{\Delta z_k}{2}\right)\right|,\left|\left(z_{k+1}-\eta_k-\delta\tfrac{\Delta z_k}{2}\right)\right|\right\}$$
$$\times\left(g(z_{k+1})-g(z_k)-\int_{z_k}^{z_{k+1}}\gamma(s)ds\right)\Bigg]$$

for all $\eta_k\in[z_k,z_{k+1}]$.

Proof. Applying inequality (2.15) on the intervals, $[z_k,z_{k+1}]$, we can state that

$$R_k(A_k,g)=\int_{z_k}^{z_{k+1}}g(s)ds-\Delta z_k\delta\left\{g(\eta_k)-\tfrac{g(z_k)+g(z_{k+1})}{2}\right\}$$
$$+(z_{k+1}-\eta_k)g(z_{k+1})+(\eta_k-z_k)g(z_k)$$

we sum the inequalities presented above over k from 0 to $n-1$. This gives

$$R_n(A_n,g)=\int_{a_0}^{a_1}g(s)ds-\sum_{k=0}^{n-1}\Big[\Delta z_k\delta\left\{g(\eta_k)-\tfrac{g(z_k)+g(z_{k+1})}{2}\right\}$$
$$+(z_{k+1}-\eta_k)g(z_{k+1})+(\eta_k-z_k)g(z_k)\Big]$$

It follows from (2.3) that

$$|R_n(A_n,g)|=$$
$$=\Bigg|\int_{a_0}^{a_1}g(s)ds-\sum_{k=0}^{n-1}\Big[\Delta z_k\delta\left\{g(\eta_k)-\tfrac{g(z_k)+g(z_{k+1})}{2}\right\}$$
$$+(z_{k+1}-\eta_k)g(z_{k+1})+(\eta_k-z_k)g(z_k)\Big]\Bigg|$$
$$\le\sup\Bigg\{\Bigg|\Bigg[\int_{z_k}^{z_{k+1}}(s-\eta_k)\gamma(s)ds+\delta\tfrac{\Delta z_k}{2}\left(\int_{z_k}^{\eta_k}\gamma(s)ds-\int_{\eta_k}^{z_{k+1}}\gamma(s)ds\right)$$
$$-\max\left\{\delta\tfrac{\Delta z_k}{2},\left|\left(\eta_k-z_k+\delta\tfrac{\Delta z_k}{2}\right)\right|,\left|\left(z_{k+1}-\eta_k-\delta\tfrac{\Delta z_k}{2}\right)\right|\right\}$$
$$\times\left(g(z_{k+1})-g(z_k)-\int_{z_k}^{z_{k+1}}\gamma(s)ds\right)\Bigg]\Bigg|,$$
$$\Bigg|\Bigg[\int_{z_k}^{z_{k+1}}(s-\eta_k)\gamma(s)ds+\delta\tfrac{\Delta z_k}{2}\left(\int_{z_k}^{\eta_k}\gamma(s)ds-\int_{\eta_k}^{z_{k+1}}\gamma(s)ds\right)$$
$$+\max\left\{\delta\tfrac{\Delta z_k}{2},\left|\left(\eta_k-z_k+\delta\tfrac{\Delta z_k}{2}\right)\right|,\left|\left(z_{k+1}-\eta_k-\delta\tfrac{\Delta z_k}{2}\right)\right|\right\}$$

$$\times\left(g(z_{k+1})-g(z_k)-\int_{z_k}^{z_{k+1}}\gamma(s)ds\right)\Big]\Big|\Big\}\qquad\square$$

THEOREM 3.3. *Suppose that all the assumptions of Theorem 2.14 hold. Considering (3.1) where $Q_n(A_n,g)$ is given by formula (3.2) and the remainder $R_n(A_n,g)$ satisfies the estimates*

$$|R_n(A_n,g)|\le\sup\{|R_5|,|R_6|\}\tag{3.5}$$

where

$$\begin{aligned}R_5=&\Big[\int_{z_k}^{z_{k+1}}(s-\eta_k)\tau(s)ds+\delta\tfrac{\Delta z_k}{2}\left(\int_{z_k}^{\eta_k}\tau(s)ds-\int_{\eta_k}^{z_{k+1}}\tau(s)ds\right)\\&-\max\left\{\delta\tfrac{\Delta z_k}{2},\left|\left(\eta_k-z_k+\delta\tfrac{\Delta z_k}{2}\right)\right|,\left|\left(z_{k+1}-\eta_k-\delta\tfrac{\Delta z_k}{2}\right)\right|\right\}\\&\times\left(\int_{z_k}^{z_{k+1}}\tau(s)-g(z_{k+1})-g(z_k)ds\right)\Big]\end{aligned}$$

and

$$\begin{aligned}R_6=&\Big[\int_{z_k}^{z_{k+1}}(s-\eta_k)\tau(s)ds+\delta\tfrac{\Delta z_k}{2}\left(\int_{z_k}^{\eta_k}\tau(s)ds-\int_{\eta_k}^{z_{k+1}}\tau(s)ds\right)\\&+\max\left\{\delta\tfrac{\Delta z_k}{2},\left|\left(\eta_k-z_k+\delta\tfrac{\Delta z_k}{2}\right)\right|,\left|\left(z_{k+1}-\eta_k-\delta\tfrac{\Delta z_k}{2}\right)\right|\right\}\\&\times\left(\int_{z_k}^{z_{k+1}}\tau(s)-g(z_{k+1})-g(z_k)ds\right)\Big]\end{aligned}$$

Proof. Applying inequality (2.19) on the intervals, $[z_k,z_{k+1}]$, we can state that

$$\begin{aligned}R_k(A_k,g)=&\int_{z_k}^{z_{k+1}}g(s)ds-\Delta z_k\delta\left\{g(\eta_k)-\tfrac{g(z_k)+g(z_{k+1})}{2}\right\}\\&+(z_{k+1}-\eta_k)g(z_{k+1})+(\eta_k-z_k)g(z_k)\end{aligned}$$

we sum the inequalities presented above over k from 0 to $n-1$. This gives

$$\begin{aligned}R_n(A_n,g)=&\int_{a_0}^{a_1}g(s)ds-\sum_{k=0}^{n-1}\Big[\Delta z_k\delta\left\{g(\eta_k)-\tfrac{g(z_k)+g(z_{k+1})}{2}\right\}\\&+(z_{k+1}-\eta_k)g(z_{k+1})+(\eta_k-z_k)g(z_k)\Big]\end{aligned}$$

It follows from (2.3) that

$$\begin{aligned}&|R_n(A_n,g)|=\\&=\Big|\int_{a_0}^{a_1}g(s)ds-\sum_{k=0}^{n-1}\Big[\Delta z_k\delta\left\{g(\eta_k)-\tfrac{g(z_k)+g(z_{k+1})}{2}\right\}\\&+(z_{k+1}-\eta_k)g(z_{k+1})+(\eta_k-z_k)g(z_k)\Big]\Big|\le\end{aligned}$$

$$\leq \sup\Bigg\{\Bigg|\Bigg[\int_{z_k}^{z_{k+1}}(s-\eta_k)\tau(s)ds+\delta\tfrac{\Delta z_k}{2}\left(\int_{z_k}^{\eta_k}\tau(s)ds-\int_{\eta_k}^{z_{k+1}}\tau(s)ds\right)$$
$$-\max\left\{\delta\tfrac{\Delta z_k}{2},\left|\left(\eta_k-z_k+\delta\tfrac{\Delta z_k}{2}\right)\right|,\left|\left(z_{k+1}-\eta_k-\delta\tfrac{\Delta z_k}{2}\right)\right|\right\}$$
$$\times\left(\int_{z_k}^{z_{k+1}}\tau(s)-g(z_{k+1})-g(z_k)ds\right)\Bigg]\Bigg|,$$
$$\Bigg|\Bigg[\int_{z_k}^{z_{k+1}}(s-\eta_k)\tau(s)ds+\delta\tfrac{\Delta z_k}{2}\left(\int_{z_k}^{\eta_k}\tau(s)ds\ -\int_{\eta_k}^{z_{k+1}}\tau(s)ds\right)$$
$$+\max\left\{\delta\tfrac{\Delta z_k}{2},\left|\left(\eta_k-z_k+\delta\frac{\Delta z_k}{2}\right)\right|,\left|\left(z_{k+1}-\eta_k-\delta\tfrac{\Delta z_k}{2}\right)\right|\right\}$$
$$\times\left(\int_{z_k}^{z_{k+1}}\tau(s)-g(z_{k+1})-g(z_k)ds\right)\Bigg]\Bigg|\Bigg\}\qquad\square$$

4. CONCLUSION

Inspired by the work of [13] and [14], we generalized Ostrowski inequality to obtained explicit error bounds for standard and nonstandard numerical quadrature formulae. By using appropriate substitution in our main results we get various established results given in [12], [13], [14] and [21] as our special cases. In the last section we have discussed its applications in Numerical quadrature rules.

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